PROCEEDINGS

OF

THE ROYAL SOCIETY

OF

EDINBURGH.

VOL. XXVIII.

NOVEMBER 1907 TO JULY 1908.

EDINBURGH:
PRINTED BY NEILL AND COMPANY, LIMITED.
MDCCCCVIII.

CONTENTS.

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. XXVIII. 1907-8.

THE 125TH SESSION.

GENERAL STATUTORY MEETING.

Monday, 28th October 1907.

The following Council were elected :—

President.

THE RIGHT HON. LORD KELVIN, G.C.V.O., F.R.S.

Vice-Presidents.

ROBERT MUNRO, M.A., M.D., LL.D.
Professor ANDREW GRAY, M.A., LL.D., F.R.S.
RAMSAY H. TRAQUAIR, M.D., LL.D., F.R.S.
Professor A. CRUM BROWN, M.D., LL.D., F.R.S.
Professor J. C. EWART, M.D., F.R.S.
JOHN HORNE, LL.D., F.R.S., F.G.S.

General Secretary—Professor GEORGE CHRYSTAL, LL.D.

Secretaries to Ordinary Meetings.

Professor D. J. CUNNINGHAM, M.D., LL.D., F.R.S.
CARGILL G. KNOTT, D.Sc.

Treasurer—JAMES CURRIE, M.A.

Curator of Library and Museum—JOHN S. BLACK, M.A., LL.D.

Ordinary Members of Council.

B. N. PEACH, LL.D., F.R.S., F.G.S.
JAMES J. DOBBIE, M.A., D.Sc., F.R.S.
Professor GEORGE A. GIBSON, M.A., LL.D.
Professor J. ARTHUR THOMSON, M.A.
Professor E. A SCHÄFER, LL.D., F.R.S.
The Hon. Lord M'LAREN, LL.D.
Professor F. O. BOWER, M.A., D.Sc., F.R.S.
Professor T. HUDSON BEARE, M.Inst.C.E.
Professor F. W. DYSON, M.A., F.R.S., Astronomer Royal for Scotland.
Professor D'ARCY W. THOMPSON, C.B.
O. CHARNOCK BRADLEY, M.D., D.Sc.
CHARLES TWEEDIE, M.A., B.Sc.

I.—An Experimental Investigation of the Temperature Changes occurring in Fresh-Water Lochs. By E. M. Wedderburn, M.A., LL.B.

(Read June 3, 1907. MS. received July 8, 1907.)

Introductory.

In a previous communication published in the *Transactions* of the Society (vol. xlv., part ii., p. 407) I attempted to discuss the temperature observations made in Loch Ness during the years 1903 and 1904. Some of the conclusions which I arrived at have not been generally accepted, and in particular limnologists have been slow to acknowledge the existence of the temperature seiche first described by Mr Watson, and which I consider was fully borne out by the observations published in my previous communication.* In order to get some ocular demonstration of the possibility of such a phenomenon, I had recourse to laboratory experiments, and it is the description of these experiments which is the main object of the present communication. Besides demonstrating the nature of the temperature seiche, the experiments also throw light on the formation of what has been called by German and Austrian writers the *Sprungschicht*, and which I now propose to call in English the "discontinuity layer," which, with the word *Sprungschicht*, has the merit of being descriptive.

Current systems were also studied experimentally. There is, of course, always a great doubt as to how far small-scale experiments in a case of this kind are applicable, and I do not pretend to say that accurate deductions can be drawn, or quantitative results obtained, from the experiments which have been made. They were, however, very suggestive of ideas, and for that, if for no other reason, they merit description.

Before proceeding to a description, I wish in the first place to state briefly the cycle of temperature changes occurring in fresh-water lochs, in order that it may be understood what the phenomena are which require to be experimentally produced.

It has been recognised generally that there are three distinct phases in the cycle of temperature changes in those lochs which, following Forel's classification, are of the Tropical type—that is, in those lochs the tempera-

* See *Pet. Geogr. Mitt.*, L.B., p. 170.

ture of whose waters at no time falls below 39° F., the maximum density point of fresh water; and it is with such lochs that I am at present concerned. Taking Loch Ness as the type of such lochs, the three phases of the temperature cycle are: (1) December to April, (2) May to July, (3) August to November. During the first phase the water in the loch is all of uniform, or nearly uniform, temperature. During the second phase there is a gradation of temperature from top to bottom; but the temperature gradient (or the rate of change of temperature with depth) is most considerable near the surface, and falls away towards the bottom of the lake.

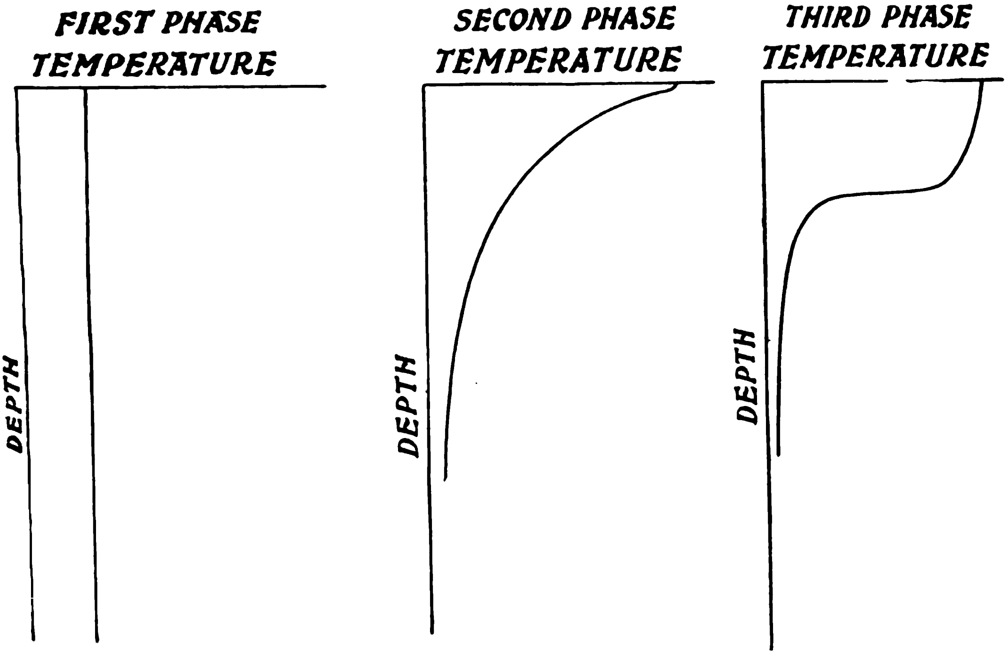


Fig. 1.

In the third phase the discontinuity layer has made its appearance, with a layer of water above of nearly uniform temperature, and below another layer also of nearly uniform temperature, but colder than the upper layer, and consequently of greater density. These three phases are illustrated by figs. 1 and 2. Fig. 1 represents the three phases by means of the usual temperature-depth diagrams. The first phase is represented by a vertical straight line; the second by a smooth curve, showing a rapid fall in temperature near the surface, and a more gradual change towards the bottom; while in the third phase the first portion of the curve is nearly vertical; then there is a discontinuity in the curve—a very rapid change of temperature—followed by another part of the curve also nearly vertical. Fig. 2 illustrates the same phenomena by means of isothermal lines drawn in a diagram representing a longitudinal cross-section of a loch—for convenience the cross-section being taken as rectangular. In the first phase no isotherms can be drawn, as the water is all of uniform temperature. In the second phase the isotherms are closest together towards the surface, and the distance between the successive isotherms

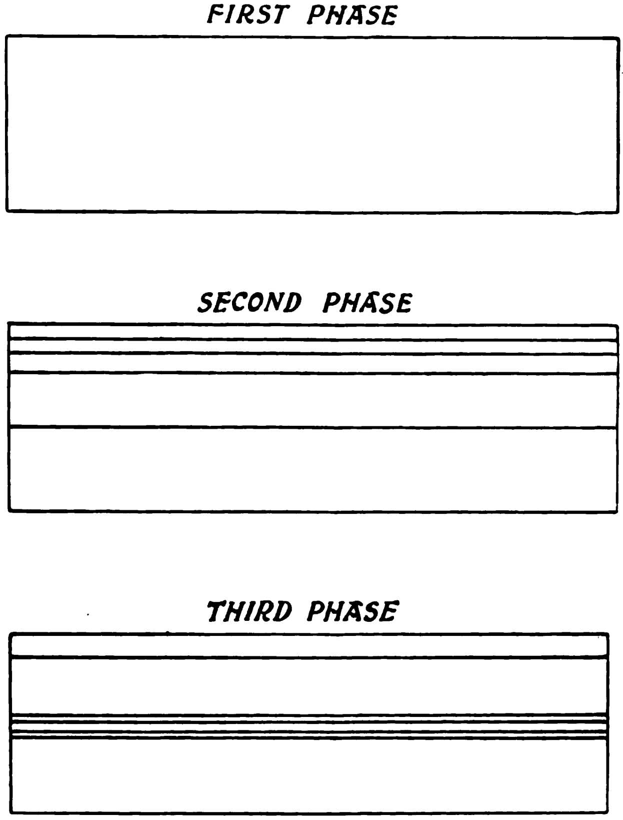


Fig. 2.

gradually increases with their depth. During the course of this phase the loch is gaining in heat, and successive isotherms make their appearance at the surface. All the isotherms are gradually sinking, new isotherms appearing at the surface. In the third phase there is a bunch of isotherms forming the discontinuity layer. Above and below the discontinuity layer the isotherms are widely separated. The discontinuity layer gradually sinks deeper and deeper down, while isotherms separate themselves from the bunch at the discontinuity layer and, rising into shallower water, eventually disappear at the surface. Thus the upper layer of water is gradually cooling and increasing in depth until the whole basin again reaches the first phase of the cycle and becomes of uniform temperature.

These are the three typical phases to which, in an experimental investigation, it is necessary to approximate, and in what follows I wish to explain the methods used.

Apparatus and Methods.

The apparatus * at my disposal consisted of a glass trough 152 cm. long, 10·5 cm. wide, and 12·5 cm. deep. This is the same trough as was used by Messrs White and Watson for their Seiche experiments.† Use was also made of the parabolic bottom which they employed. A continuous blast of air could be driven along the trough by means of an electrically driven rotary fan. The top of the trough was covered over for nearly its whole length, and the trough was as a rule filled to within about 2 inches of the top. The wind-current was directed along the channel between the cover of the trough and the surface of the water. The arrangement used is shown in fig. 3.

It was not found possible to experiment with water of varying temperature. The temperature gradient in a loch (or the rate of change of temperature with depth) is small. If the temperature gradient in the experimental tank were made the same as in a natural basin, very small differences of temperature would require to be experimented with. Where the temperature gradient is made large, conduction and convection currents become of very much greater importance than they are in a natural loch; and as the depth to which the disturbance of surface-waves is felt is relatively much greater in an experimental trough than in a natural basin, the equalising effect of surface disturbances is also much greater. If the gradient in the experimental trough is made comparable to the natural gradient, the range of temperature is very small—so small that the experiments would not have been possible.

The temperature changes occurring in lochs are mainly due to the difference in density of water at various temperatures, and if in experimenting the differences in temperature are very small, the differences in density will be too small to make experiments depending on these differences practicable. I therefore fell back on the device of imitating the differences in temperature by differences in density. In this way it is easy to exag-

* The apparatus which I used belongs to the Natural Philosophy Department of the University, and was put at my disposal by Professor MacGregor.

† *Proc. Roy. Soc. Ed.*, vol. xxvi., part iii., p. 142.

gerate the differences in density which in a loch are due to temperature, and so to make the experiments more manageable, and the effect of conduction is thus eliminated. For the most part, brine solutions were used—a dense solution representing the coldest water in the loch, and water of less salinity representing the warmer and lighter layers.

To represent the first temperature phase, when all the water in the loch is cold and of uniform temperature and density, the trough was filled to within about 2 inches of the top with brine; and, as there is nearly always

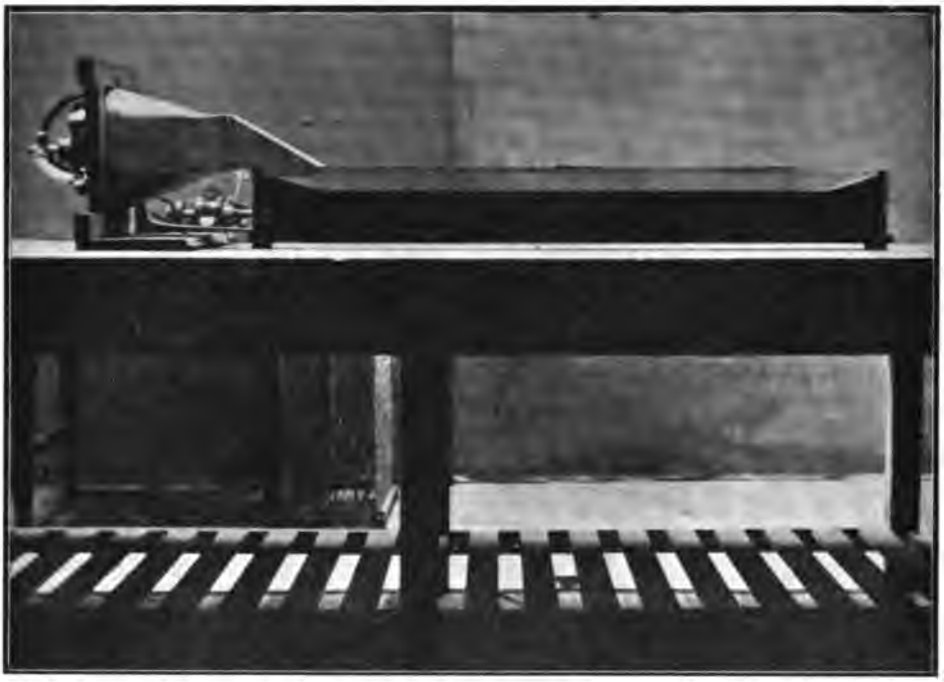


Fig. 3.

a wind blowing at some part of a loch, a current of wind was driven along the surface of the brine.

To imitate the gradual gain of heat in a loch, fresh water was very carefully and gradually allowed to flow over the surface of the brine in such a way that, but for the action of the wind-current, there would have been little mixing, and a separate layer of fresh water would have formed on the surface of the brine. A syphon overflow was provided to compensate for the inflow of fresh water. The agitating influence of the wind-current, however, mixes this inflow of fresh water with the brine, and gradually the distribution of density becomes similar to the distribution of density due to temperature in the second period of the temperature cycle.

Another method was also employed to arrive at the density distribution in the second phase, but it is not imitative of the natural conditions in a loch. A quantity of some soluble salt was put in the bottom of the trough, and fresh water carefully poured over it in such a way as to prevent mixing. The trough was then left for two or three days. The salt gradually dissolved and diffused through the water, and after the lapse of a considerable time gave a density distribution such as was desired. The process was quickened by occasionally letting a wind-current blow along the surface, and also by starting ordinary seiches. For this experiment potassium bichromate was used, and was found very convenient to work with.

The next problem was to pass from the second phase to the third phase experimentally. The change takes place at the time when the loch is just beginning to lose heat. The wind-current was continued blowing along the surface of the liquid in the trough. At first no change in the density distribution was noticeable; but gradually the currents in the liquid produced by the wind-current along the surface changed in character. To make these currents visible, a small quantity of black stain was introduced, and the motion of this black stain easily showed the trends of the currents. More will be said about these currents later, and they are only mentioned here as evidencing the progress of some change which eventually results in the formation of a discontinuity layer. The liquid gradually separates out into two distinct layers of different density, and the discontinuity becomes well marked. The transition from the second to the third phase has then been accomplished.

This experimental imitation of the manner in which the discontinuity layer is produced is, I consider, one of the most remarkable results of these experiments. It indicates as a cause of the formation of the discontinuity layer that, when the surface water ceases to be of considerably less density than the lower layers, the return current is more localised, and takes place in shallower water. While the loch is gaining in temperature there must be on the surface a layer of water considerably warmer than the layers below. It is this warm layer which is carried along by the wind and accumulates at the end of the loch, the return current being, I imagine, slow and distributed through the whole depth of the loch. But when the surface loses instead of gains heat, the tendency for the water driven along by the wind will be to sink until it reaches a layer of water of equal density, where the return current will be set up. The mixing action of the waves, also, must have some effect in forming a layer at the surface of uniform temperature.*

* See also p. 11.

To complete the cycle there only remains the transition from the third phase back to the first phase, and this transition is also attainable experimentally. For with the continuance of the wind-current the discontinuity layer in the experimental trough, as in a natural basin, gradually sinks deeper. Owing to the currents induced at the discontinuity layer by the wind-current, there is a gradual mixing of the lower with the upper layer. The difference in density between them diminishes, and the deepening of the discontinuity layer gradually results in the complete mixing of the whole liquid, and finally the whole liquid in the trough again becomes of uniform density.

The Temperature Seiche.

The third phase in the temperature cycle is accompanied by the temperature seiche, and it was primarily to demonstrate the possibility of its existence that the present investigation was undertaken. An oscillation of the liquid below the discontinuity layer in the experiments already described was easily obtained. While the wind-current was kept blowing along the surface there was noticeable a considerable transference of the surface water to the lee * end of the trough, the surface layer at that end being considerably deeper than at the end of the trough from which the wind was blowing. If the wind-current was suddenly cut off, a seiche in the lower liquid followed, with a period depending on the difference in density between the upper and lower layers. The period for the ordinary seiche in the trough was about 3 seconds, while the period of the "temperature seiche" varied from 10 to 20 seconds in most of the experiments. The seiche very rapidly died down, not making more than two or three oscillations.

Even while the wind-current was blowing a small "temperature seiche" was usually observable, possibly due to inequalities in the wind-current.

Another phenomenon accompanied the cessation of the wind-current, and it may be called a "density bore." A solitary wave was formed at the discontinuity layer at the end of the trough from which the wind was blowing, and slowly travelled the length of the trough, when it was reflected, but with great loss of amplitude. There may be a corresponding bore in lochs, and if there is, it might account for many apparent irregularities in temperature; for the passage of such a bore would explain a sudden rise in temperature of short duration, and at a considerable depth. I have not,

* The term "windward" is used for "leeward," and *vice versa*, on pp. 416 and 418 of my paper on Lake Temperatures in *Trans. R.S.E.*, vol. xlv., part ii., p. 409.

however, direct evidence of the existence of this bore in lakes: the fact that most of the observations in Loch Ness were made at one end of the loch makes them unsuitable from this and from other points of view. It

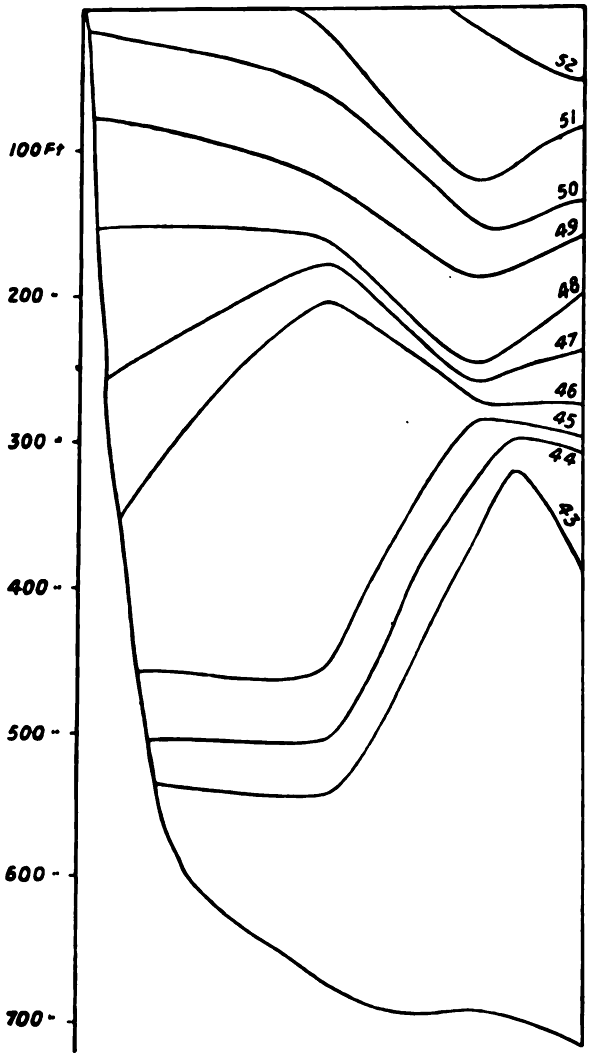


Fig. 4.—Loch Ness, 29th August 1903.

would be of great interest to have a series of observations taken near the centre of the lake.*

* It may be, however, that the temperature distribution given in fig. 4 shows the beginning of a temperature bore. The observations from which the diagram is drawn were taken shortly after the cessation of a heavy wind, and the observers may have lighted on this bore. This diagram is further referred to on p. 20.

To make the temperature seiche more evident, a layer of paraffin resting on a layer of fresh water was also used. Owing to the low density of the paraffin, a "temperature seiche" of greater amplitude could be produced, and the sharpness of the boundary between the paraffin and the water made the phenomenon more easily observed.

Currents produced by Wind.

Although the experiments which have been described were undertaken primarily with a view to demonstrating the nature of the temperature seiche, they directed my attention to the currents which take place in a loch as a result of the surface current directly produced by the wind. Surface water is driven along by the wind, and a slight transference is also due to the waves. The place of the water so transferred must be filled, and a "return current" is set up. Where the wind-current does not cover the whole length or breadth of the loch, this return current takes place largely on the surface, and the circulation produced is purely a surface circulation. But where, as in a narrow loch, the wind-current is over the whole surface, a return current takes place through the loch, and a *vertical* circulation takes place.

From the observations made in the experimental trough, I incline to the view that, during the first period of the temperature cycle, when the loch is of uniform temperature, the return current is distributed all through the loch, and is consequently very slow. In the experimental trough, with a liquid of uniform density, colouring matter was introduced at the surface and the wind-current started. There followed a gradual mixing of the colouring matter all through the liquid, showing that in the experimental trough at least the effects of the currents produced by the wind were not confined to the surface.

The conditions in an actual loch are very different, but there, too, any observations available go to show that the effects of wind at this period are felt to great depths.

Fig. 5 shows the temperature distribution over part of Loch Ness on 17th October 1903, towards the end of the third period, and when there were no great differences of temperature. There had been a S.W. wind blowing for some days previous, which on the 17th October changed to a N.E. wind. This accounts for the direction of the upper isotherms. But the direction of the lower isotherms shows well the great depth to which the currents induced by wind are felt. Reference may also be made to

Sir John Murray's observations in Loch Ness on 26th April 1887, mentioned in my previous communication (*Trans.*, vol. xlv., p. 417).

During the second phase, my opinion is that the return current is still appreciable to great depths, but that it is not so evenly distributed through the loch, being stronger near the surface than at the bottom. This is, however, a mere opinion, in support of which I have no evidence to adduce. But this assumption makes it easier to understand the genesis of the discontinuity layer. For if the return current falls off at the bottom and becomes stronger towards the surface, finally this and the other influences noted on page 7 will cause a gradual separation of the upper and the lower waters, and the formation of the discontinuity layer.

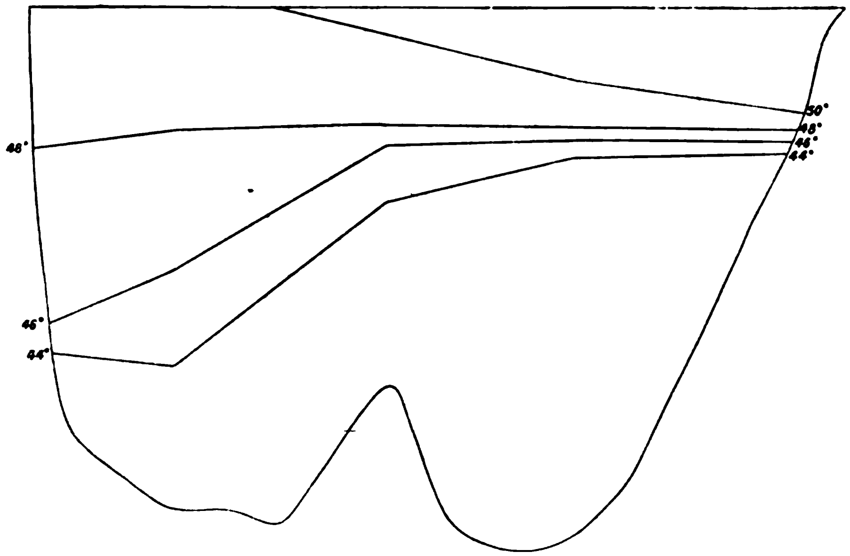


Fig. 5.—17th October 1903.

During the third phase, when the discontinuity layer is distinct, the return current takes place chiefly along this layer. This is easily illustrated experimentally. Starting with a layer of fresh water on the top of brine and with a current of wind blowing, if a coloured liquid * is introduced at the end of the trough towards which the wind is blowing, it will be seen at once to be carried along the surface of separation between the water and the brine, leaving a long streak behind it. This return current does not extend to the brine at all, but, though at its commencement it is narrow and comparatively swift, it gradually widens out as it reaches the other

* The colouring liquid I used in these experiments to make currents evident was black alcohol stain, formed by a mixture of alcohol and lamp-black.

end of the trough and rises to the surface. I endeavoured to ol photograph of this experiment, but was unsuccessful, as the return ‹ very quickly mixed the colouring liquid all through the upper la water.

By means of the colouring matter, it could also be observed at point the return current began. There was always at the end of the t towards which the wind was blowing a quantity of more or less p water outside the ordinary current systems of the trough, and beyon passive water the return current began. The point at which the ı current began could be varied by removing part of the covering fro trough, and so making the wind-current less strong at the part of the tı which was uncovered.

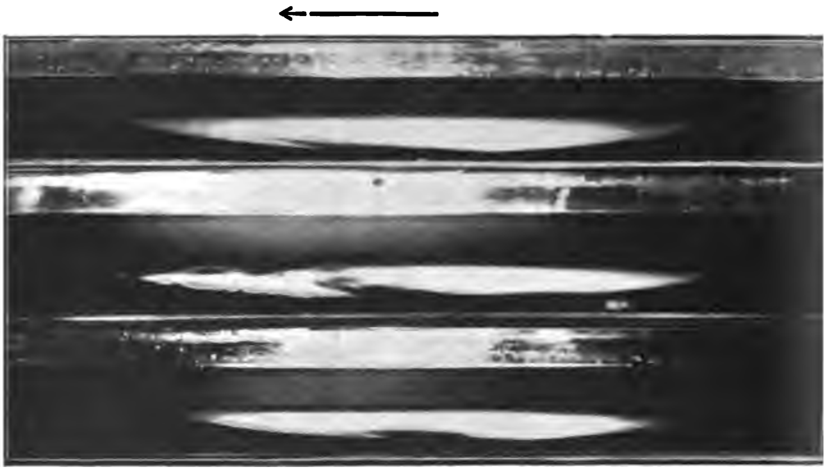


Fig. 6.

But this return current at the surface of separation induced anotl return current in the lower layer. As it travelled along the surface of t brine, it carried some of the brine with it, exactly as the wind-current the surface had carried some of the surface water with it. Consequent a secondary return current was set up in the lower layer, and in the san direction as the wind. This return current could be made evident expeı mentally by introducing a coloured liquid at the bottom of the trough, an the direction in which this liquid moved showed the direction of the secondar return current. Fig. 6 shows three photographs to illustrate the secondar return current. A layer of fresh water coloured black was introduce over uncoloured brine. Black stain was then introduced into the botton of the trough by means of a pipette, and a current of air driven along th upper surface. The arrow on the photograph shows the direction of the wind, and it will be seen that the black stain at the bottom of the trough has in each case tailed out in the same direction as the wind, showing the existence of a current at the bottom of the trough *in the same direction as the wind.* The photographs were taken at night, a limelight lantern being used to illuminate the water. An exposure of about 7 minutes with rapid plates was given in each case. The first two photographs were taken after the wind-current had been stopped, but the third photograph was taken while the wind was blowing. In the third experiment the water of the upper layer was not coloured very darkly, but in the photograph there appears a broad dark line at the boundary between the two layers. This is probably an optical effect produced by the return current, which raises small waves at the surface of the lower liquid. The effect of these waves would be to reflect some of the light falling on them, and so to give a dark line in the photograph.*

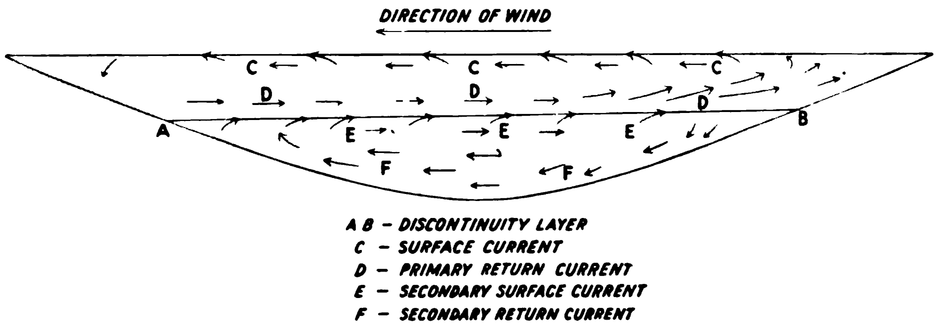


FIG. 7.

Figs. 8–13 show, by means of isotherms drawn at intervals of 2 degrees, the temperature distribution in Loch Ness on 13th and 17th September 1904, at a time when the discontinuity between the upper and lower layers of water was fairly well marked. The gradual oscillation of the discontinuity layer in the three consecutive diagrams for these days due to the temperature seiche can be seen. On the 13th September, the temperature distribution was due to a prevalence of N.E. winds, and it will be observed that the isotherms are closer at the end of the loch towards which the wind is blowing, and that they diverge from that end. On the 17th September, however, the temperature distribution is due to a prevalence of S.W. winds, and the fan arrangement of the isotherms, diverging from the end of the loch towards which the wind is blowing, is much more evident. The local conditions at the two ends of the loch are, I think, quite sufficient

* Fig. 7 shows diagrammatically the current-system which is indicated by these experiments.

to account for the fact that the isotherms are not so close at Fort Augustus (the south-west end) on the 13th as they are at the north-east end of the loch on the 17th; for at the south-west end of the loch there are large rivers entering, and at the north-east end there is the sole outflow of the loch. Also, the loch narrows very much at this end, which must accentuate the effect of the wind blowing the surface water to that end.

The experiments which have been described illustrate well the fact that the isotherms diverge from the end of the loch towards which the wind blows. The discontinuity between the upper and lower layer was always very pronounced at this end, and least pronounced at the end from which the wind was blowing. Where the return current began, the discontinuity was sharp; but in its course it gradually drew with it, and became mixed with, some of the denser water of the lower layer. The result was that the return current gradually widened out and became denser than the water of the upper layer, though less dense than the bottom layer. The consequence of this was that, at the end of the loch from which the wind was blowing, the change of density with depth was gradual, and there was no well-marked discontinuity between the upper and lower layers.

Another point was illustrated by the experiments. It was frequently noticed that the lower isotherms sloped in a different direction from the upper isotherms. On first thought it was natural to suppose that the wind, in driving the surface water to one end of the loch, would have produced a tilt of the isotherms all in one direction. This is very rarely the case, but such a distribution has occurred after a very heavy wind. As a rule, however, the isotherms arrange themselves like a fan, the upper isotherms sloping upwards and the lower isotherms sloping downwards. A density distribution corresponding to this was observable in the experimental trough when the difference in density between the two layers was not too great. The explanation is very simple. The return current which takes place at the layer of discontinuity acts at the surface of the lower layer in the same manner as the wind-current acts at the surface of the upper layer. It produces a slope in the isotherms of the lower layer in the same way as the wind produces a slope in the isotherms of the upper layer; and as the wind-current and the return current are in opposite directions, so are the slopes of the isotherms in the upper and lower layers in opposite directions.

Note added 8th July.

It is a question to what depth the secondary return current of the lower layer is appreciable. The secondary return current is of course very much slower than the ordinary return current by which it is induced. From the photographs of the experiments shown in fig. 5, it is seen that in the

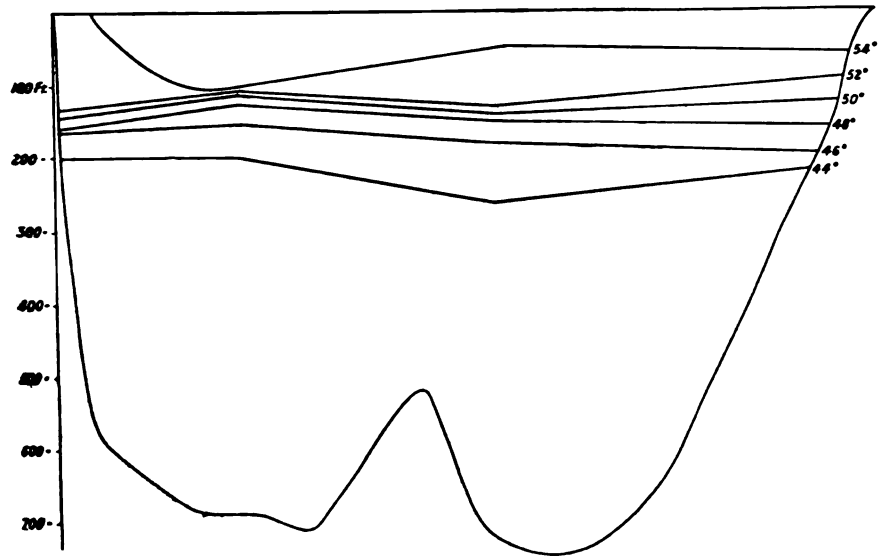


FIG. 8.—Loch Ness, 6 a.m., 13th September 1904.

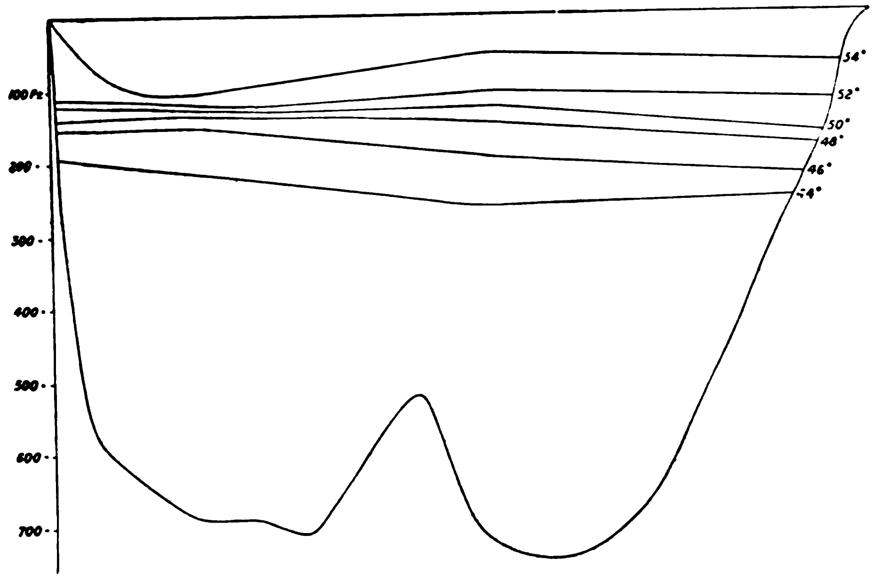


FIG. 9.—Loch Ness, 12 noon, 13th September 1904.

experimental trough this secondary return current is felt right to the bottom of the trough, and my opinion is that in deep lochs also the secondary return currents are felt to great depths. They are, I think, extremely slow. The current at the surface of discontinuity is slow, and the secondary return current induced by it must be very much slower if it is distributed over a considerable depth, as I think it is. To give a rough idea of the velocity of this current, assume that the rate of the ordinary return current is one mile per hour, which must be considerably over the mark. Assume also that this return current, which, it must be remembered, is considered as taking place above the discontinuity, induces a current 10 feet deep below the discontinuity, and of an average rate of half a mile per hour. Assume further that the secondary return current is distributed through a depth of 300 feet, and it will be seen that its average velocity is only one-sixtieth of

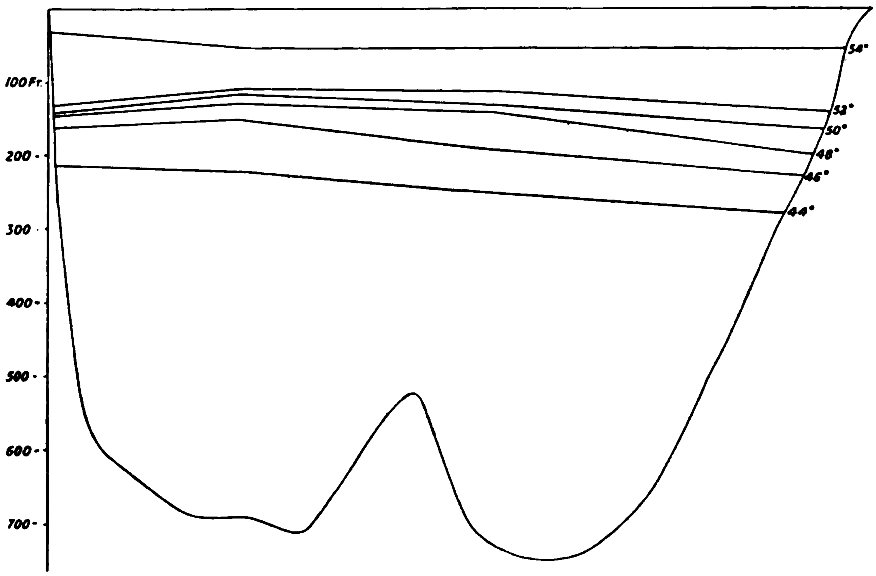


FIG. 10.—Loch Ness, 6 p.m., 13th September 1904.

a mile per hour, or that it takes two and a half days for the return current to move one mile; and as this is the average velocity of the current, its velocity at great depths may be much slower.

These figures are only taken for the sake of example, and I have no idea as to their quantitative accuracy. They show, however, that an assertion that the secondary return current is felt to great depths does not mean that rapid changes take place in temperature at these depths. Changes, however, do take place, and to explain these changes it is necessary to assume some system of currents.

For instance, it was found by the Lake Survey in Loch Morar * that

* *Geographical Journal*, July 1904. p. 74.

even at the depth of 1000 feet there was a variation in temperature during the year of over a degree. In Loch Ness also variations in the temperature

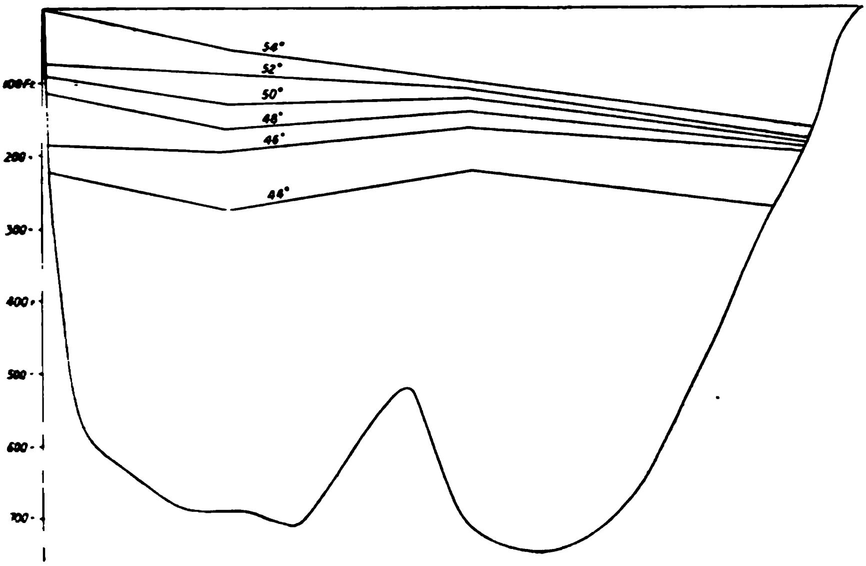


FIG. 11.—Loch Ness, 6 a.m., 17th September 1904.

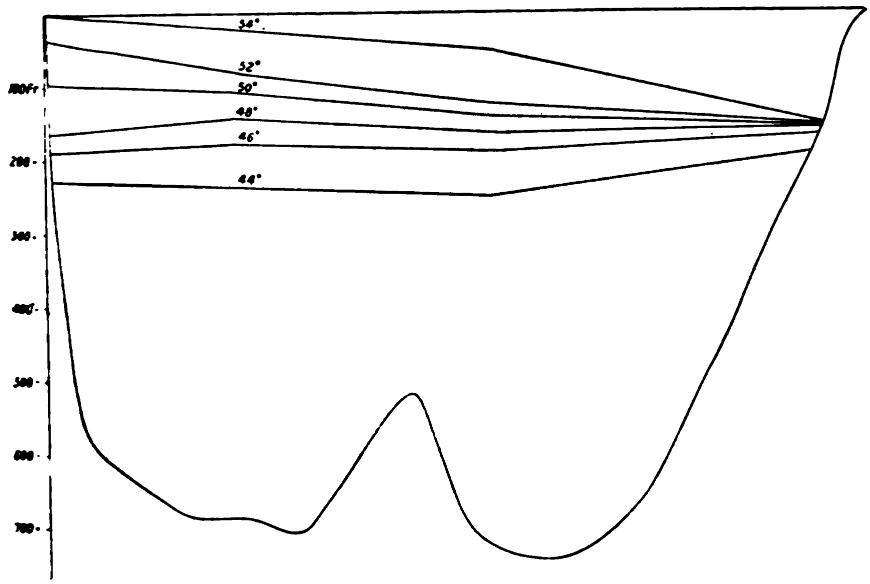


FIG. 12.—Loch Ness, 12 noon, 17th September 1904.

at great depths were observed, such as one would expect if there were currents reaching to the bottom of the loch. I may instance one striking case. Fig. 4 shows the arrangement of the isotherms in Loch Ness on 29th August, and fig. 14 their arrangement on 31st August 1903.

The isotherms in fig. 14 below 44° are drawn for every tenth of a degree; above that, the isotherms are drawn for every degree. It will be seen that there is a pretty sharp discontinuity at about 48°. The isotherms above that do not present any great peculiarity, nor are the isotherms immediately below the discontinuity layer other than would be expected. But the isotherms below 44° are very much disturbed, and the deeper the isotherm the greater its slope. This is certainly not what would be

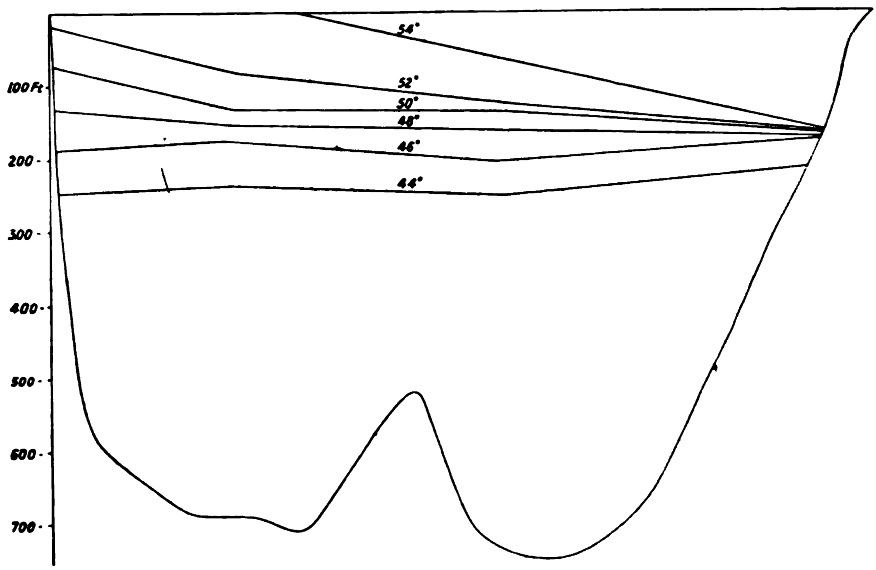


FIG. 13.—Loch Ness, 6 p.m., 17th September 1904.

expected, and I think it shows that the cause of this disturbance of the lower isotherms is not directly to be found on the surface, but that the temperature distribution must be explained by some system of deep currents. Unfortunately, the meteorological observations for this period are very scant. On the 29th, 30th, and 31st August the winds were variable. The observations on which fig. 14 is based were taken on a Monday, and unfortunately no observations are available for the previous day. It is accordingly difficult to say what has produced the distribution shown. There was a strong S.W. wind blowing from the 25th to the 28th August, as shown from the records of the meteorological station at Fort Augustus. This wind may not have been general along the loch, for the observations from the meteorological station at Inverness do not record high winds at this time.

A series of temperatures was taken along the loch on 25th August, and at that date the temperature distribution was quite normal. But the strong wind rapidly lowered the surface temperature at Fort Augustus. On the 26th the surface temperature was 55·3°, on the 27th, 52·5°, and on the 28th it had fallen to 46·2°. On the 28th it was evident that there was a very strong return current rising to the surface close into the shore at Fort Augustus. Observations were made in 7 feet of water, and it was found

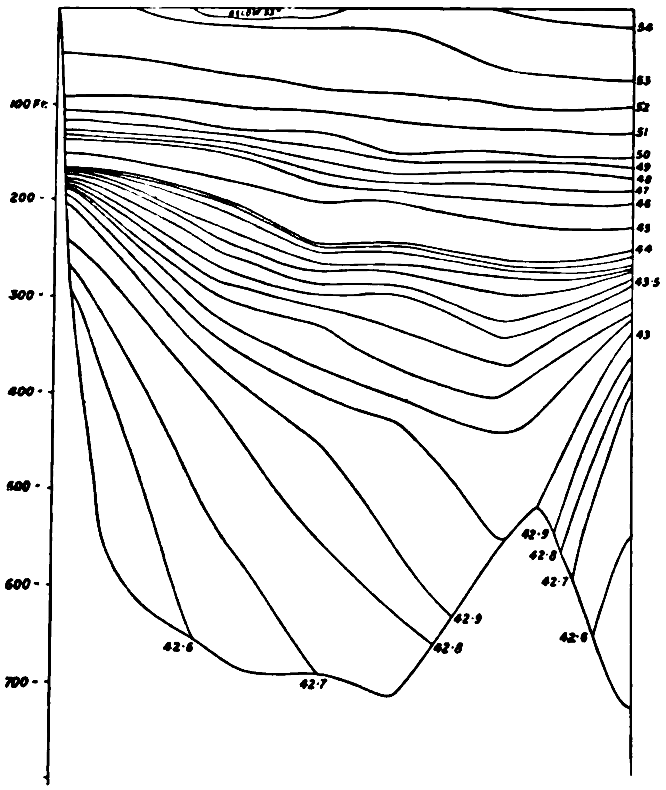


FIG. 14.—Loch Ness, 31st August 1903.

that the temperature at the surface was as low as 43·9°, and that at the depth of 7 feet the temperature was 46·6°, or nearly 3 degrees warmer. There was no such inversion of temperature further out in 50 feet of water. This must, I think, have been due to the return current bringing with it cold water from a considerable depth. The discontinuity, as will be seen from the figure, was sharpest at the isotherms for 46° or 48°, and as the surface temperature was just about 46° it is probable that the return current was strongly felt at the point where the Fort Augustus observations were regularly taken, which is what one would expect during a heavy wind.

The distribution shown in the figure must be another consequence. It shows that there is a much greater quantity of water between 43° and 44° than usual, and compared with figs. 8 to 13 it shows that the water below 43° is very much less than usual. There has, then, been a great mixing of the water formerly below 43° with the water above that temperature, which can be satisfactorily explained on the assumption that there is a secondary return current which gradually carries water from just below the discontinuity layer to greater depths, thus raising the temperature of water at these depths. It is found that, during the autumn or third phase of the temperature cycle, the lowest layers gradually rise in temperature. The example given is an extreme case, and we find that, whereas before the storm of the 25th to the 28th the temperature at a depth of 700 feet was 42·1°, observations taken thereafter show that the temperature at these depths was about half a degree higher, and all the water of lower temperature than about 42·5° had disappeared. On 29th August, just after the wind had fallen, a series of observations was taken for 7 miles up the loch, with the result shown in fig. 4. It shows at a depth of over 300 feet a great body of water at a temperature between 45° and 46°. It will be remembered that on the previous day the surface temperature, on account of the return current, was only 46·2°, as the discontinuity layer was just above this temperature. This body of water must then, I think, be the accumulation of water caused by the primary return current; and, as mentioned on page 9, there may be illustrated here the temperature bore which the experiments lead one to expect. At all events, the observations taken two days later, on the 31st, show practically no trace of this body of water of a temperature between 45° and 46°, and it is contrary to all other evidence to say that this body of water has in so short a time become mixed up with the surrounding water. It may be that the distribution shown in fig. 14 is caused by the return of the temperature bore at a greater depth. The rate at which the bore would travel is quite agreeable to this explanation. However that may be, the observations are, I think, sufficient to warrant an assertion that surface winds induce movements at the bottom of even our deepest lochs.

(*Issued separately December* 24, 1907.)

II.—Notes on some Oligochæts found on the Scottish Loch Survey. By C. H. Martin, B.A. (Magdalen College, Oxford). *Communicated by* Sir John Murray, K.C.B. (With Plates I., II.)

(MS. received April 3, 1907. Read May 20, 1907.)

These notes deal mainly—

(1) With the structure of the genital organs of *Stylodrilus Gabreteæ* (Vejd.), and with its possible identity with *Bathynomus Lemani* (Grube).

(2) A new species of oligochæt (*Stylaria Lomondi*).

I owe my opportunity of working on these forms to Sir John Murray, whom I must again thank for his great kindness.

The Genital Organs of Stylodrilus Gabreteæ.

Stylodrilus Gabreteæ (Vejd.) is one of the most characteristic deep-water oligochæts occurring in Scottish lakes. In nearly every haul from a depth of more than a hundred feet, if the bottom is composed of soft mud, one or two specimens of this species, with its cocoons (generally empty), will almost certainly be found.

It occurs sporadically on muddy bottoms in more shallow waters, and on two occasions in Loch Lomond, near Tarbet, I found it in about four feet of water.

The anatomy of the immature form has been described by Vejdovsky in his great work upon oligochæts, but as I do not know any account of the anatomy of the sexual organs in the *mature* form of any species of this genus, I have thought it well to give some description of them.

The genus is readily identified amongst the Lumbriculids by the absence of the characteristic blind cæca to the blood-vessels, and by the presence of long non-retractile penes.

I find that, as regards the external anatomy, I cannot add anything material to Vejdovsky's description, "Lebhaft rosaroth von 3–4 cms. lang mit einfach und gegabelten borsten. Der Kopflappen konisch zugespitzt langer als das mund Segment. Die Penis-röhren beinahe der lange als die Querachse des Körpers beitragt."

Although I had found Stylodrilus commonly in Loch Tay in 1905, it was not until the beginning of September 1906 that I found sexuall mature forms and cocoons containing embryos.

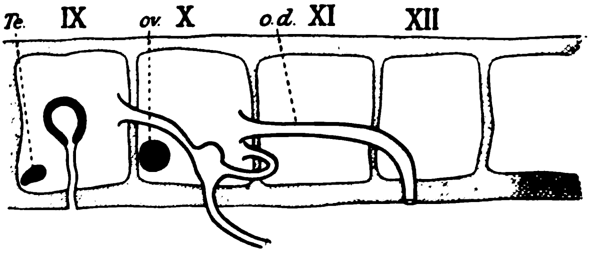


FIG. 1.—Vejdovsky.

As regards the general arrangement of the sexual organs, the tables in Vejdovsky's and Beddard's monographs are mutually discrepant, and neither agree with the arrangement that I find (text figures 1, 2, 3).

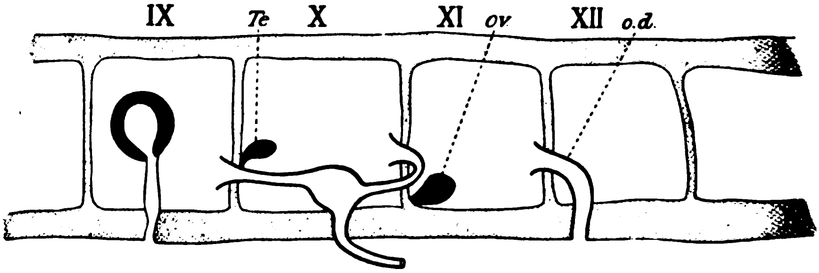


FIG. 2.—Beddard.

It will be seen that, while there is no doubt the male aperture is on segment X and the female aperture on segment XII, Vejdovsky shows only one pair of testes, which he places in segment IX and the ovaries in

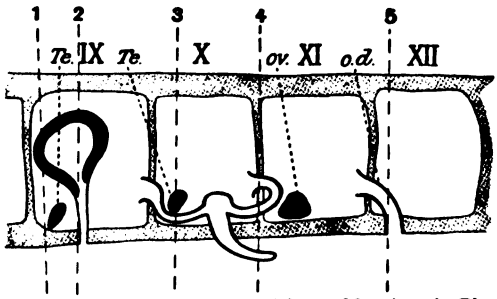


FIG. 3.—Figures in italics show Positions of Sections in Plate I. 1–5.

segment X. The oviduct in his arrangement runs through portions of three segments to open on segment XII. Beddard places the testes in segment X and the ovary in segment XI, whereas in the form which I have examined the arrangement is as follows :—

There are two pairs of testes, the anterior in IX, the posterior in X. On each side there are two vasa deferentia, the anterior piercing the septum 9/10 and running back to the glandular atrium in segment X, while the posterior opens by a funnel into segment X, and running back through the septum 10/11, turns on itself, and again piercing septum 10/11, runs forward to the atrium.

There is a single pair of ovaries in segment XI, and a short oviduct which pierces the septum 11/12 to open to the exterior in the anterior region of segment XII.

The Male Genital Apparatus.

The testes are four in number, the anterior lying in segment IX on either side of the nerve cord (Pl. I. fig. 1), the posterior lying in a corresponding position *directly upon* the anterior vasa deferentia in segment X. (fig. 3). The greater part of the maturation of the sperm appears to take place in the seminal vesicles, which are very large. These consist of paired sacs, the anterior of which lie in segment VIII and the posterior in segments X, XI, XII, XIII.

The vasa deferentia open by large funnels into their appropriate segments, slightly dorsal to the gut. There is a slight difference as regards the aperture, which in the anterior opens directly into the seminal vesicles, whilst the posterior open into the body cavity.

The atria are rounded glandular structures, composed of the following layers from within outwards,—

(1) a single layer of large cells with flattened basal nuclei and a clear periphery;
(2) a layer of circular fibres;
(3) an irregular mass of clear cells, with large nuclei and granular contents.

The atria opens directly into the penes, which are long tubes directed backwards, and about equal in length to the transverse diameter of the worm.

The Female Generative Apparatus.

The ovaries lie in the anterior part of segment XI.

Vejdovsky and Beddard distinguish three main types of egg formation in oligochæts, and they agree in placing the Lumbriculidæ amongst those forms in which—

(*a*) any cell in the ovary may become an ovum, and
(*b*) no share is taken in its development by the surrounding cells.

I do not believe that either of these two statements is literally true as regards Stylodrilus. Here the nuclei of the cells immediately surrounding a growing ovum have a peculiarly shrunken appearance, and structures (Pl. IV. fig. 7) may sometimes be found within the plasma of the egg cells which seem very like the degenerate nuclei such as are figured by Obst in the developing egg of *Helix Pomatia.*

The egg sac is single and dorsal. It begins in segment XII, and may extend back between the paired vesicula seminales into segment XVII.

The oviducts are short tubes, commencing by wide funnels in segment XI and opening to the exterior on segment XII.

The cocoon is a common and quite characteristic product of a deep-water haul in a Scottish loch. It is roughly spherical, but the two poles are prolonged into long tubes, through which the young worm escapes (Pl. IV. fig. 6).

The cocoon itself is composed of a yellow chitinous substance which is very resistant to chemical reagents, and can probably remain for years practically unchanged in the soft mud of a lake.

The spermathecæ are large globular bodies lying dorsal to the gut, close up against the dissepiment 8/9, and opening to the exterior on segment IX by thick-walled ducts. I have never been able to find in these forms the cystal which Claparède found in *Stylodrilus Heringianus.*

Forel, in his description of Lake Geneva (tome iii.), observes that the most common oligochæt from deep water is *Bathynomus Lemani* (Grube), with its cocoons.

This genus has had a somewhat chequered history, and has always been a complete puzzle to the monographers of this group (Vejdovsky, Vaillant, Beddard). It was first described (without figures) by Grube in the *Jahresbericht der Schlesischen Gesellschaft für Vaterlandische Cultur* (1878).

In this paper Grube remarks that of the six genera given by Forel as characteristic inhabitants of the deep water of Lake Geneva, he could only distinguish three, viz. a Tubifex, Sænuris, and a third, *Bathynomus Lemani.* It is rather interesting to note that the two former genera are constantly found in association with Stylodrilus in the deep water of all the Scotch lochs I have examined. The chief points in Grube's genus are the four bundles of chætæ in each segment, which can only be recognised as bifurcate under a high power. The length was about 20 mm. in immature spirit specimens, possessing 40–60 segments, of which the anterior 2–8 were very short.

He remarks, further, that "ruckengefass dass jederseits eine reihe aeste absendet macht sich weniger bemerkbar.

"An der Bauchwand des etwa IX segment, scheinen kurze blind Schlauche zu munden" (possibly the penes).

In a footnote to another paper read before the same Society in 1879 Grube remarks that during a second stay of two days on Lake Geneva he found on a new specimen that the dorsal vessel "besitzt auch noch paarige kurzere blind endende aeste," and he changed the name from Bathynomus to Bythonomus.

It is rather remarkable that he should have overlooked in his first paper these usually very conspicuous structures, which are so characteristic of all Lumbriculids except Stylodrilus.

Since this time no one has, as far as I am aware, been able to identify this form.

Michaelsen has since revived the name for a genus, of which a characteristic feature appears to be the relation of the gonads. But as Grube never described the gonads in his account of this form, it seems somewhat difficult to understand on what grounds this identification is based.

I think that the most feasible assumption is, that the form which Grube found so common in Lake Geneva, and which he described in his first paper, must have been a Stylodrilus, and that it was a form of some other Lumbriculid genus in which he observed the blind contractile vessels of his second account.

Stylaria Lomondi.

This worm was found on three occasions in August 1906 near Tarbert, Loch Lomond, on a soft muddy bottom at a depth of 12–20 feet.

At this season chains of two zooids were found, also some single individuals in a sexually mature condition. From its naked-eye appearance this worm might readily be mistaken for a rather stout *Stylaria lacustris* (Linn.), but directly it is examined under a low power, several important differences are noted. The prostomium is well developed, and is prolonged into a proboscis, which, however, is never as long as in *Stylaria lacustris*, and there are two dorsal eyes.

The most distinctive feature is, however, furnished by the structure and arrangement of the chætae. The four anterior segments, as in *Stylaria lacustris*, have only ventral bundles composed of 6–7 sigmoid notched chætæ (fig. 4).

The dorsal bundles are all alike, and contain 12–14 very fine capilliform chætæ, which possess a very characteristic fan-shaped arrangement. The chætæ are slightly larger than the transverse diameter of the body: unfortunately those figured are from a stained preparation, in which the arrangement is slightly distorted. As regards the internal anatomy, the only specimen of which I possess sections was in a sexual condition, although the ova were not ripe.

The testes are in segment V; the vas deferens opens into a glandular atrium in segment VI. Paired receptacula seminis were present in V.

I am not quite clear as to the arrangement of the vesicular seminules, but a dorsal median sac was present in segment IV.

Paired ovaries were found on septum 5/6. The structure of the gut and nephridia appeared to correspond with that found in *Stylaria proboscidea.*

The circulatory system was well developed, and the dorsal vessel was present in the posterior segments (fig. 2), a character which, according to Vejdovsky, differentiates Stylaria from the other Naids.

Most recent systematists place *Stylaria lacustris* (Linn.) in a distinct genus (Vejdovsky, Bourne), but Beddard considers it congeneric with *Nais elinguis.* It appears to me that it would be best, with Vejdovsky, to make one genus, Stylaria, including *Stylaria lacustris*, *Lomondi* and the somewhat doubtful genus *Ripistes* or *Pterostylarides.*

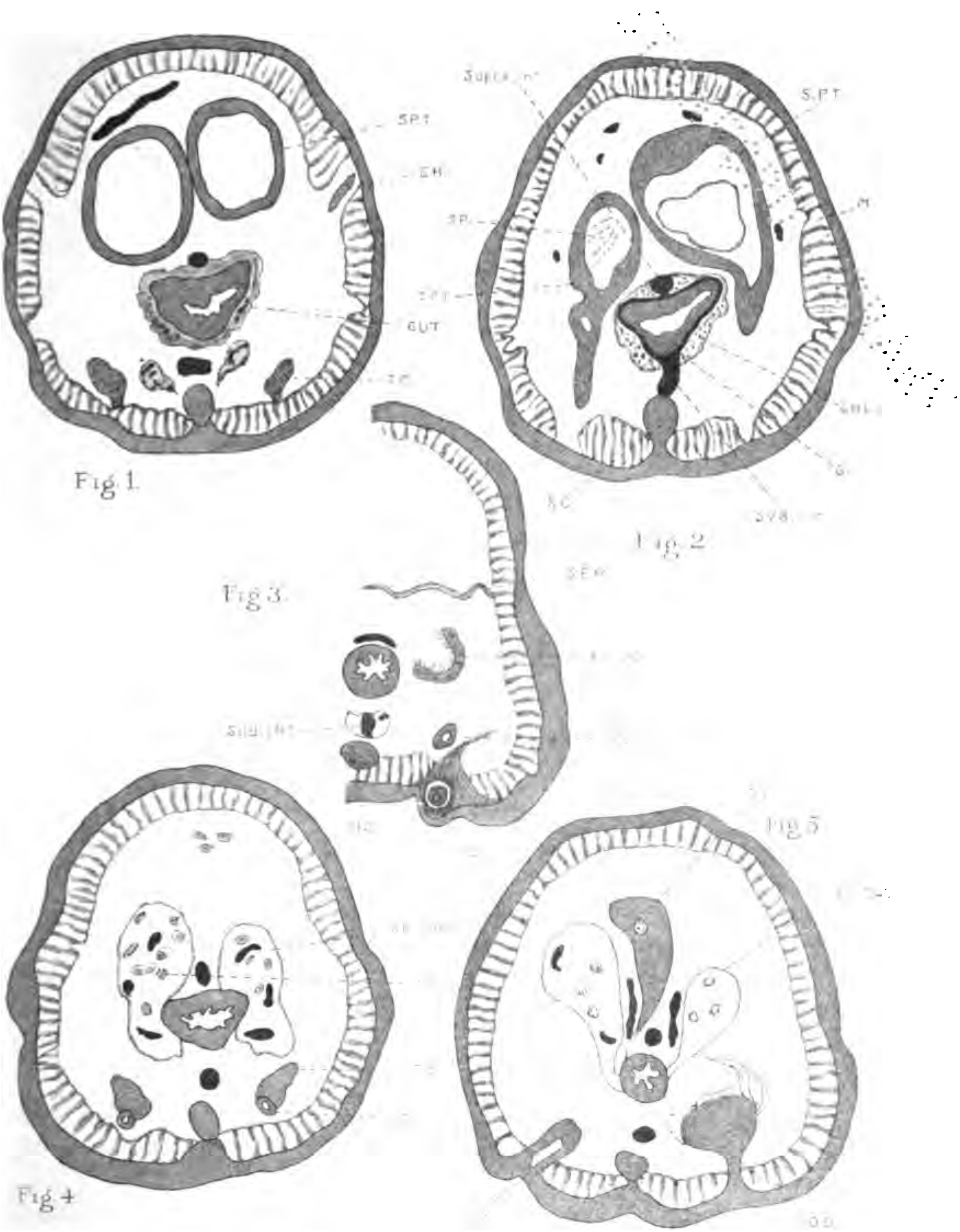
The only character which separates Pterostylarides from Stylaria, according to Bourne, is "the peculiar character of the cephalization," by which he means the possible absence of ventral chætæ in segments IV, V.

Bourne himself admits that "the idea that the region between the second and third seta-bearing segments does represent two segments is an assumption"; and even if it existed, I cannot see how the term cephalization is applicable to the absence of seta on two intermediate segments of the anterior region of a worm.

Diagnosis—Stylaria Lomondi.

Rather stout forms about 10 mm. long, with a short proboscis and eyes. Fine capilliform setæ (12–16), arranged in a fan-shaped manner in the dorsal bundles, beginning in segment V. Ventral bundles throughout the animal's length, 6–8 sigmoid forked chætæ.

MARTIN. Plate I.



M'Farlane & Erskine, Lith. Edin'

MARTIN.

Plate II.

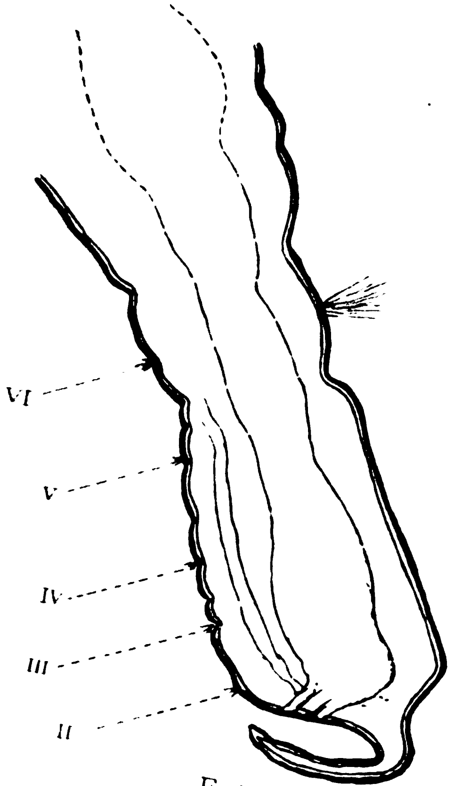


Fig. 1

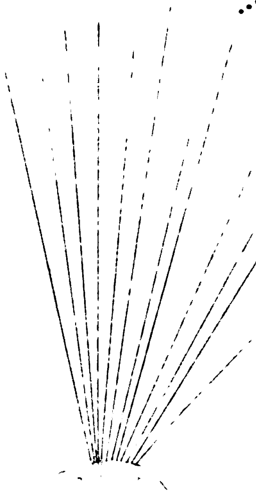


Fig. 3.

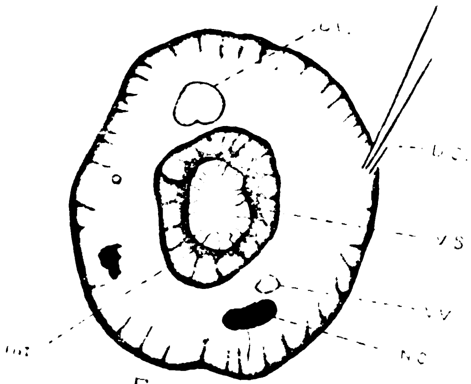


Fig. 2.

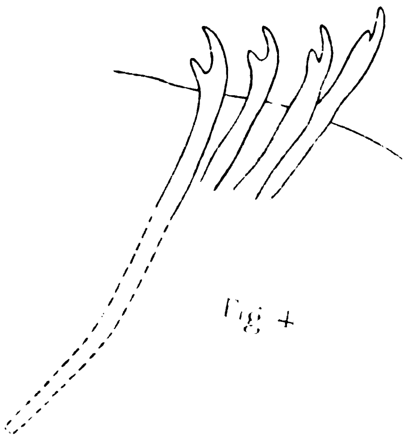


Fig. 4

Proc. Roy. Soc. Edin.   MARTIN.   Plate II.   Vol. XXVIII.

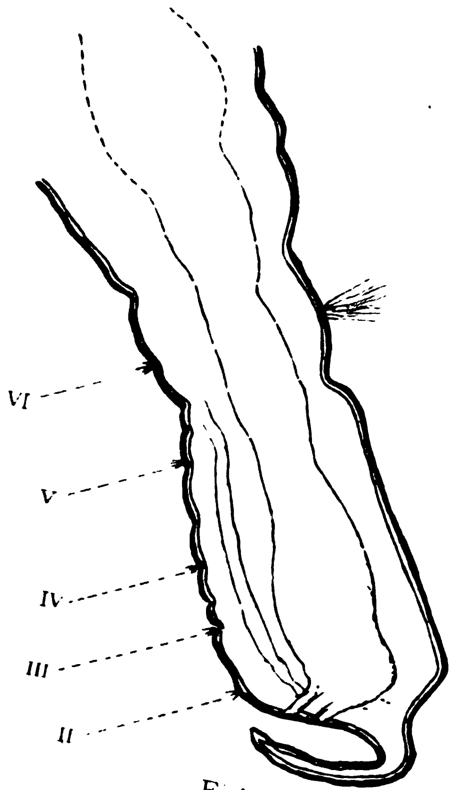


Fig. 1.

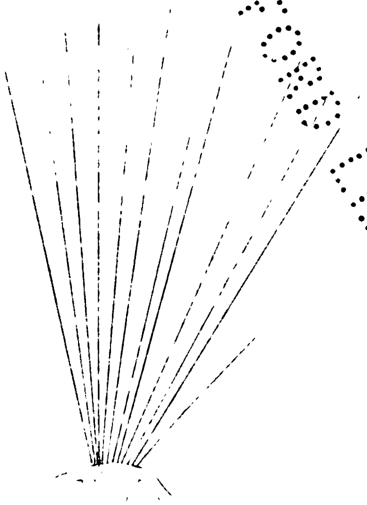


Fig. 3.

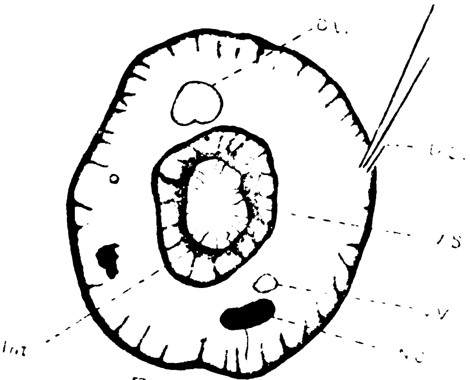


Fig. 2.

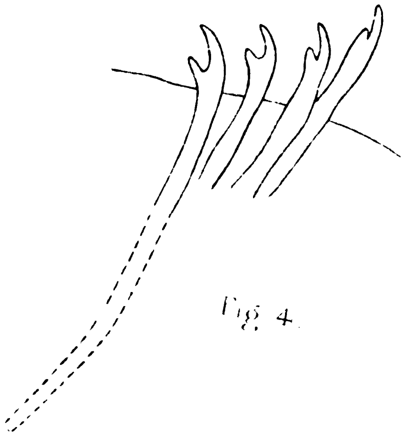


Fig. 4.

M'Farlane & Erskine Lith. Edin.

MARTIN. Plate II.

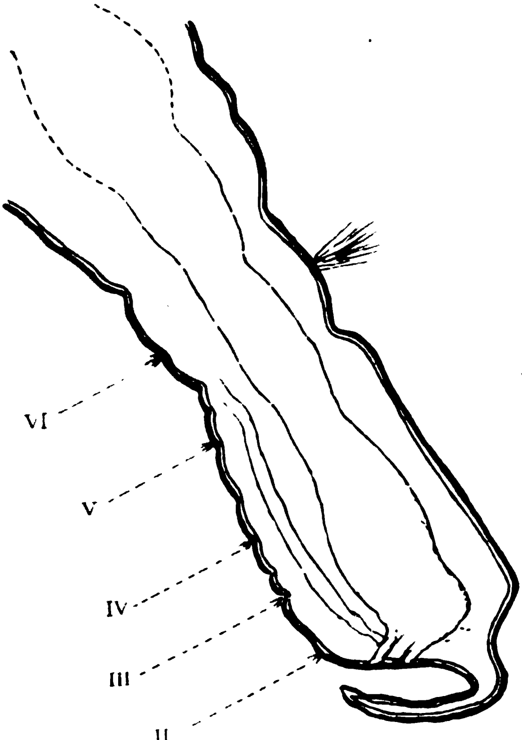


Fig. 1.

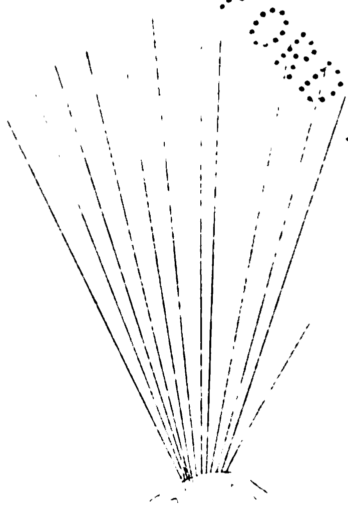


Fig. 3.

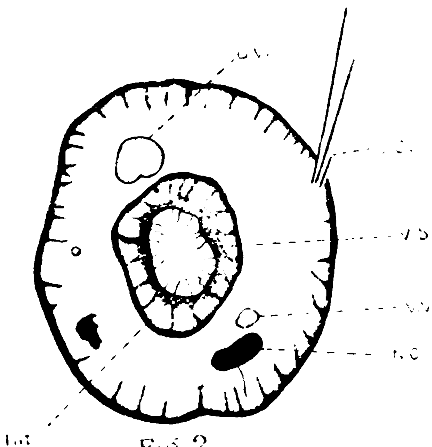


Fig. 2.

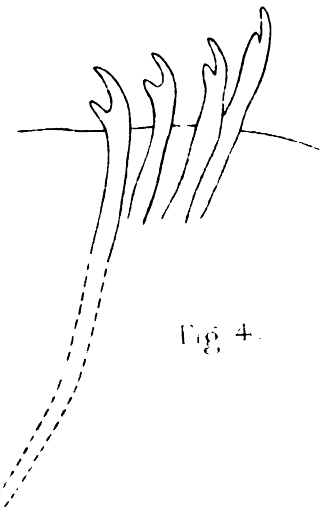


Fig. 4.

M'Farlane & Erskine Lith Edin^r

LITERATURE.

(1) Benham, "Aquatic Oligochæts," *Q.J.M.S.*, xxxiii.
(2) Beddard, *Monograph of Oligochæts.*
(3) Bretscher, *Revue Suisse de Zoologie*, 1889.
(4) Claparède, *Mém. Soc. Physiol. Genève*, xvi.
(5) Grube, "Flora v. Fauna Schweitzerseen," *J. B. Schles. Ges.*, 1878, pp. 115–117; 1879, p. 228.
(6) Vejdovsky, *System v. Morphologie des Oligochæts*, Prag, 1884.
(7) Beddard, *Monograph of Oligochæts.*
(8) Bourne, "On Naidiform Oligochæts," *Q.J.M.S.*, vol. xxxii., 1891.
(9) Vejdovsky, *Monograph des Oligochæts.*

DESCRIPTION OF PLATES.

Plate I.—*Stylodrilus Gabreteæ.*

Fig. 1. Section through the anterior testes and spermathecæ. A + 6 comp. oc
Fig. 2. Section through the ducts of the spermathecæ. A + 2 oc.
Fig. 3. Section through aperture, vas deferens, and penis.
Fig. 4. Section through posterior testes.
Fig. 5. Section through oviduct.
Fig. 6. Cocoon.
Fig. 7. Section through the base of a ripening egg. 4 oc. + 2 mm.

(*Figs.* 6 *and* 7 *will be found on Plate IV., page* 34.)

Ch. chæta.
Chl. chloragogen cells.
Fu. v. d. funnel of vas deferens.
G. gut.
M. muscle plates.
O. oviduct.
Ov. ovum.
Pe. penis.
Sp. sperm.
Sp. S. sperm sac.
Sp. T. spermatheca.
Supra Int. supra-intestinal blood-vessel.
Sub. Int. sub-intestinal.
Sep. septum.
Te. testis.
V. D. vas deferens.

Plate II.—*Stylaria Lomondi.*

Fig. 1. Head end, stained specimens. A + 2 oc.
Fig. 2. Transverse section through posterior segments. A + 2 oc.
Fig. 3. Dorsal bundle. 6 comp. + 4 mm.
Fig. 4. Ventral chætæ. 6 comp. + 2 mm.

D.V. dorsal vessel.
D.C. dorsal chætæ.
Int. intestine.
N.C. nerve cord.
V.S. visceral sinus.
V V. ventral vessel.

(*Issued separately December* 24, 1907.)

III.—Notes on some Turbellaria from Scottish Lochs. By C. H. Martin, B.A. (Magdalen College, Oxford). *Communicated by* Sir John Murray, K.C.B. (With Plates III., IV.)

(MS. received April 3, 1907. Read May 20, 1907.)

In this paper I wish to refer to some of the more interesting Turbellaria found during the months of July, August, and September 1905 at Ardeonaig on Loch Tay, and during August and the early part of September 1906 at Tarbet on Loch Lomond.

I also wish to describe in more detail a newly discovered Kalyptorhynch, and to put forward some observations on the origin of the nematocysts in some *Microstoma lineare.*

The opportunity of working on these forms occurred in connection with the work of the Scottish Lake Survey under Sir John Murray, to whom, for his unfailing kindness, I owe a very deep debt of gratitude.

Polycystis Goettei.

During the early summer of last year (1906) I found large numbers of small Proboscidea in a shallow pool on the Deri near Abergavenny, and later I found a few examples on the shores of Loch Lomond.

I had already completed my drawings and descriptions of this form when I found a full account of the animal by E. Bresslau in the *Zoologischer Anzeiger* for July 1906. As, however, there is as yet no account of this animal in English, and as my account differs from his in some important points, particularly as regards the water-vascular system, I have decided to publish it.

In those places in which it occurs, *Polycystis Goettei* is a very abundant form, swimming actively amongst water weeds, in association with *Prorhyncus stagnalis, Vortex truncatus,* and *Gyrator notops.*

Its food appears to consist mainly of copepods. In general appearance it closely resembles a small *Polycystis nægelii*: the body (Pl. III. fig. 1) is about 2 mm. long and more or less cylindrical, though the shape undergoes great changes, depending upon the extent of muscular contraction. In extreme cases the whole anterior end of the body can be invaginated so that the eyes appear to be near the posterior end.

At the anterior end there is a large protrusible proboscis with four long retractor muscles; behind this lies the brain, on the dorsal surface of which are placed the two eyes.

The mouth lies on the ventral surface, at about one-third of the length of the animal from the anterior end.

The pharynx rosulatus in young forms leads into a sac-like gut, which occupies most of the body, but in the sexually mature forms, as in *Gyrator notops*, the gut breaks down to form a loose packing around the gonads. In the form examined by Bresslau no such degeneration of the gut was found.

The common sexual aperture lies a little more than half way down the body, and near the posterior end is the aperture of a contractile bladder into which the paired water-vascular ducts open.

The proboscis, pharynx, gut, and brain do not offer any features of particular interest, but it is necessary to describe in some detail the gonads and water-vascular system, as in these organs I find that my account differs rather materially from that of Bresslau. The testes are more or less oval bodies lying dorsal to the uterus; they pass into short vasa deferentia, which unite and open into a common reservoir which lies close to the secrete reservoir, a little to the right and anteriorly of the common genital aperture (Pl. III. fig. 4).

The duct from the secrete reservoir passes through a short covered chitinous tube before joining the sperm duct, and opening into genital atrium (Pl. IV. figs. 1–2).

The female genital apparatus consists of paired ovaries (Pl. III. fig. 2) lying on either side of the genital aperture. The ovaries open into a duct which passes forward, receiving the opening of the uterus and the ductus seminalis, to the common genital aperture. I cannot find a distinct bursa seminalis in the position figured in Bresslau's diagram, in which also the entire male apparatus is placed on the animal's left side.

The cocoon (IV. 5) is a brown oval structure, concavo-convex in transverse section, and ending at one pole in a drop of viscid substance. Each cocoon contains a single embryo. The yolk glands are bilaterally arranged, irregularly branched structures, each gland consisting of two long anterior branches and two long and one short posterior branch, which unite about half way down the body into a common transverse duct opening near the base of the uterus.

The reserve stuff in the yolk gland appears to consist mainly of fat, which blackens readily in osmic preparations.

Apparently this substance undergoes some chemical change, as the cocoon does not appear to contain so much fat.

The water-vascular system (Pl. III. fig. 1) consists of two longitudinal trunks, which can be readily seen both in the living animal and in sections.

I know of no Turbellarian in which the main structure of the water-vascular system is so obvious. The two main ducts can be followed from the region of the eyes to the extreme posterior end, where they double forward upon themselves to open by means of two short transverse canals into the bladder.

The wall of the bladder, as can be readily seen in transverse sections, contains parallel bands of muscle fibre (Pl. III. fig. 3).

Bresslau states that the bladder is "nichts anders als eine einfache Epidermis einstülpung." And he therefore considers that it offers a marked contrast to the end-bladder of the Plagiostomidæ, which is formed by the union of the two ducts. It seems to me that the wall of the bladder is too well marked off from the epidermis (1) by the absence of rhabdites, (2) by the presence of muscle bands, for this view to be accepted, and it appears to me to mark off *Polycystis Goettei* very distinctly from all other Kalyptorhyncha.

The Scottish form appears to agree in all essential points with the *Polycystis Goettei* of Bresslau.

Whether, however, having regard:

(1) to the extremely aberrant condition of its water-vascular system,

(2) to the possible absence of a bursa seminalis,

it would not be better to place *Polycystis Goettei* in a special genus, seems to me to be still a moot question.

Microstoma lineare.

This form was found in all the Scottish lochs which I have personally examined.

The anatomy of the budding form has long been well known, and the only points to which I wish to refer in this note are—

(1) The nematocysts.

(2) The gonads.

It has long been known that the nematocysts of *Microstoma lineare* are very like those of Hydra; and Von Siebold, in his description of these organs, remarks that they are structures "welche denen der Hydra auf einen haar gleichen sollten."

Von Graff believed that the nematocysts of Microstoma differed from

t

those of Hydra not only in their smaller size but in the possession of four instead of three barbs.

In the skin of Microstoma from the shores of the Scottish lochs in places where Hydra is abundant, I have found the three forms of nematocyst which are characteristic of Hydra. In some blind Microstoma from deep water where Hydra was absent there was no nematocyst.

This blind form without nematocysts was first described by Du Plessis in 1897; and later, Zacharias raised it to specific dignity under the name of *Microstoma inermis*. I do not believe that this species is a good one, for two reasons.

In the first place, in nearly all Turbellaria which inhabit both deep and shallow water one finds this degeneration of the eyes (e.g. *Automolos Morgiensis*) in the deep-water forms.

In the second place, the nematocysts of the Microstoma in littoral waters are derived from the Hydra on which it feeds, in much the same way in which Grosvenor found that the Æolids derive their nematocysts from their cœlenterate prey.

In transverse sections through a Microstoma which has recently fed upon Hydra, nematocysts are found in three places:—

(1) In the gut.
(2) Surrounded by mesenchyme cells in the space between the gut and the ectoderm.
(3) Under the ectoderm.

I hope to show in a further paper the mechanism by which this transference is accomplished; and further, that all true nematocysts with a thread capable of expulsion, found in Rhabdocoels, are obtained in a similar manner.

Gonads.

During the greater part of the summer Microstoma reproduces entirely by budding, and it is only in September that I have first found sexual forms on Loch Tay and Loch Lomond.

In both years male forms were common long before I could find any females. But in two cases, in sections of fully developed male individuals, I found the testes degenerating, and bodies which looked very like ova in the mid-ventral line. I hope to be able to get more material this year, and finally decide how far Microstoma can be described as protandrous.

Bothrioplana Bohemica.

This was first described by Braun in 1881, and it has usually been recognised that it occupies a more or less intermediate position between the Rhabdocoels and Triclads. Vedjovsky has also given a very full description of the anatomy of this form in the same paper.

Late in August 1906 I found a small pool near the hotel at Tarbet, Loch Lomond. This pool is of some interest, as it appears only to exist in very wet weather, and after a fortnight's fine weather at the beginning of September it was so dry that it was impossible to get any more material. The pool was about twenty yards long, with an average depth of about a foot, but the two ends of the pool were almost separated by a shallow. At one end of the pool large numbers of *Vortex truncatus* were found. At the other end large numbers of *Prorhyncus curvi-stylis* (Braun), *Opistoma Schulzeanum*, and *Bothrioplana Bohemica* (Vejdovsky) were found.

Opistoma Schulzeanum is one of the more rare Rhabdocoels, but its anatomy has recently been thoroughly described by Vejdovsky.

It may be interesting to note that, whilst Vejdovsky concluded from his observation that Opistoma did not live through the summer months, becoming sexually mature in the spring, sexual forms were abundant in Loch Lomond in August.

Prorhyncus curvi-stylis.

This animal was first found by M. Braun in a small ditch near Dorpat, and he has given a very careful description of its anatomy in the *Turbellaria Livlands.* It is readily distinguished from *Prorhyncus stagnalis* by the hooked copulatory organ and its size. The form found near Loch Lomond appeared to agree in all essentials with the form described by Braun, except for the absence of eyes in the Scotch form.

Automolos Morgiensis.

Automolos Morgiensis is one of the most common Turbellaria in the deeper waters of all the Scottish lochs that I have examined. Its cocoon, which I figure, is also fairly common (Pl. IV., 5). It occurs more rarely in shallow waters, but only in those places where the bottom is composed of a soft mud, *e.g.* twice in four feet of water, August 1907, in the estuary of a small stream, Tarbet, Loch Lomond. It was first discovered by G. du Plessis in the deep water of Lake Geneva, and for a long time doubt existed as to its true position.

Its discoverer placed it amongst the Alloiocoels, under the name of

Monotus Morgiensis, but Von Graff believed that its true position was amongst the Mesostomida, and therefore changed its name to *Otomesostoma Morgiensis*. The first thorough description of its anatomy was given by M. Braun, who recognised its true position, and gave it the name of *Automolos Morgiensis* in a paper upon the Turbellaria of Livland, published in 1884. In a quite recent paper in the *Zeitschrift für Wissenschaftliche Zoologie* for December 1906, Nils von Hofstein has published a somewhat lengthy paper upon this form, in which he endeavours to show (1) that the worm cannot be regarded as an Automolos, but must be placed in a separate genus amongst the Alloiocoela, for which he revives the old name Otomesostomum of Graff; and (2), a point of the utmost general importance, that all the ova are fertilised in the germarium long before they have reached maturity.

As regards the first point, the chief differences between Otomesostoma and Automolos are—

(1) In Otomesostoma the oviducts are short and transverse, whereas in Automolos the ovaries are anterior to the pharynx and long.
(2) There is no uterus in Otomesostoma.

Unfortunately, the anatomy of Automolos does not seem to be very well known, so that the question cannot at present be settled.

As regards the second point, the only proof Hofstein gives that the eggs are fertilised at such an early stage rests upon two drawings of sections of oocytes, in both of which darkly staining bodies lie near the nuclei.

These bodies, which I also find in my preparations, do present a certain superficial resemblance to spermatozoa, but I think they must be identical with the *vitellogen Schicht* or *dotter Kern* such as has been well figured by Bambecke in *Pholcus phalangioides*.

Gurwitch, in his *Morphologie und Biologie der Zelle* (Fischer: Jena, 1904) remarks that—

"Nach den übereinstimmenden Ergebnissen der neuerer Forscher besitzen die jungen Oocyten in ihrer sog. vitellogenen Schicht, einer sichelförmige Plasmaanhäufung welche das Idiozoma der oocyten umgibt, und Kappenförmig dem Keimbläschen anliegt, die eigentliche primäre Matrix der Dotterbildung."

As a matter of fact, this structure is best defined in the very young oocytes, at a later stage it appears to undergo a granular disintegration, and at the time when the egg is ready to enter the uterus nothing can be seen of it (Pl. IV. 3, 4).

LITERATURE.

(1) Bresslau, "Eine neue Art der Marinen Turbellariengattung Polycystis aus dem Susswasser," *Zool. Anz.*, Bd. xxx., 1906.

(2) Du Plessis, *Zool. Anz.*, xviii.

(3) Von Graff, (1) *Monographie der Turbellarien Rhabdocoelide*, 1882; (2) "Marine Turbellarien Orotavas und der Küsten Europas," *Zeit. f. Wiss. Zool.*, Bd. lxxxiii., 1905.

(4) Hallez, "Gyrator Notops," *Zoologie Descriptive*, Paris, 1900.

(5) Braun, *Die Turbellarien des Livlands*, Dorpat, 1885.

(6) Vejdovsky, *Zeit. für Wiss. Zool.*, Bd. lx., p. 1895.

(7) Von Graff, *Monographie der Turbellarien*, 1882; Braun's *Thierreich: Turbellaria.*

(8) Fuhrmann, "Der Turbellarien der Umgebung von Basel," *Revue Suisse de Zoologie*, 1894.

(9) Schulze, *Archiv für Naturge.*, 1849.

(10) Von Siebold, *Lehrbuch der vergleichende Anatomie*, 1845.

(11) Zacharias, *Biol. Stat. Plon.*, 1894, 1902.

(12) Braun, *Die Rhabdocoeliden Turbellarien Livland*, 1885.

(13) Von Graff, *Turbellaria*, 1882.

(14) Jensen, *Turbellaria ad litora Norvegia occidentalia*, 1878.

(15) Du Plessis, G., "Mesostomum Morgiense," *Bull. Soc. Vaud.*, tom. xiv., 1876.

FIGURES.

Plate III.—*Polycystis Goetti.*

Fig. 1. General diagram outline. $\frac{1}{6}$ Zeiss + 2 oc.

Fig. 2. Ovaries opening into female atrium. 4 mm. + 6 comp. oc.

Fig. 3. Bladder of excretory system. Zeiss 2 mm. apo. + 6 comp. oc.

Fig. 4. Transverse section through middle region of the body. Reduced $\frac{1}{2}$. 4 mm. + 6 comp. oc.

Fig. 5. Transverse section through posterior region. Reduced $\frac{1}{2}$. 4 mm. + 6 comp. oc.

Plate IV.

Fig. 1. Transverse section through secrete reservoir and sperm reservoir. 2 mm. + 6 comp. oc. Reduced $\frac{1}{2}$.

Fig. 2. Following section to 6, showing secrete reservoir passing by chitinous tube into the ductus seminales. 2 mm. + 6 comp. oc. Reduced $\frac{1}{2}$.

Fig. 3. Early oocyte in the ovary of Monotus. 2 mm. + 6 oc.

Fig. 4. Late stage.

Fig. 5. Cocoon. A + 6 oc.

op. bl. opening of bladder.
bl. bladder.
coc. cocoon.
ect. ectoderm.
lat. tr. lateral trunk, water-vascular system.
ov. ovary.
ph. pharynx.
pr. proboscis.
sec. res. secrete reservoir.
sp. res. sperm reservoir.
sp. spermatozoa.
ut. uterus.
v. d. vas deferens.

(*Issued separately December* 24, 1907.)

MARTIN. Plate III.

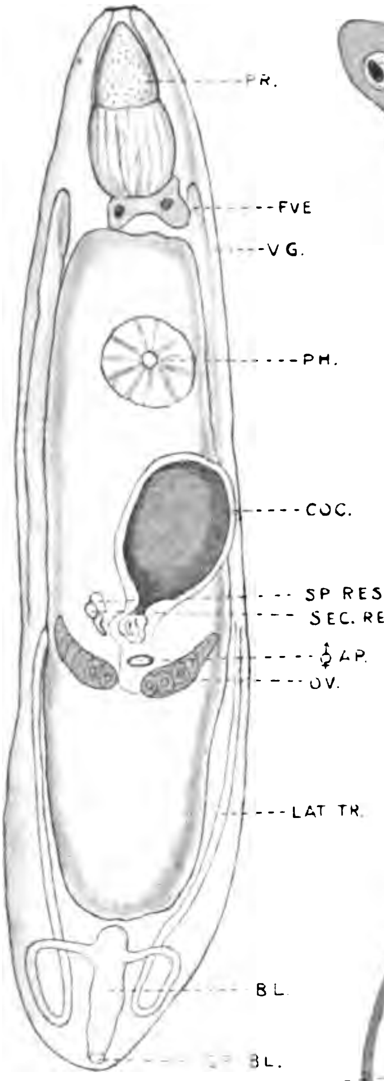


Fig. 4.

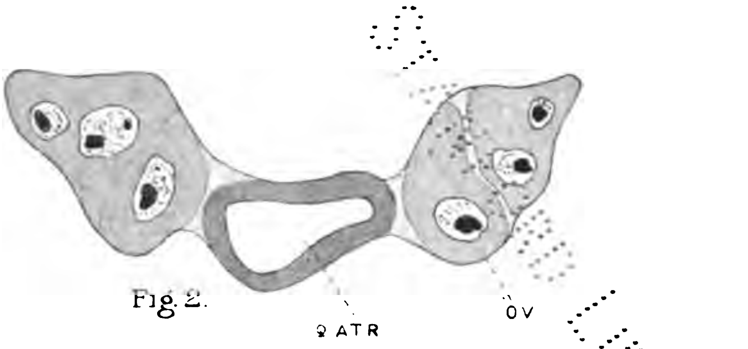


Fig. 2.

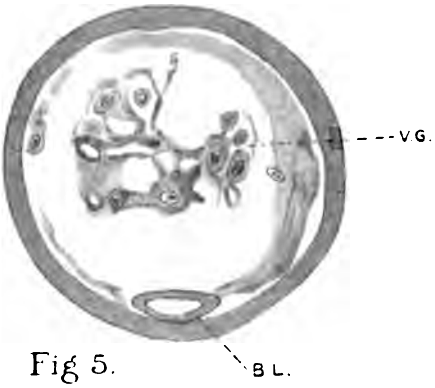


Fig 5.

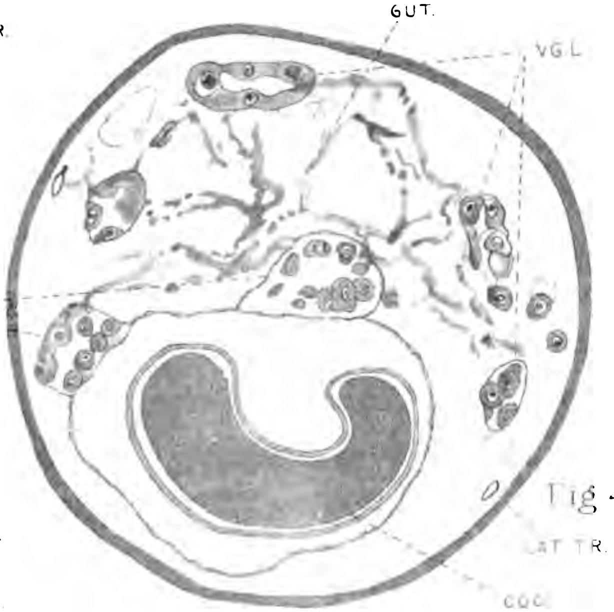


Fig 4.

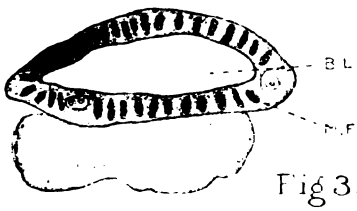


Fig 3

M'Farlane & Erskine Lith. Edin.

MARTIN. Plate IV.

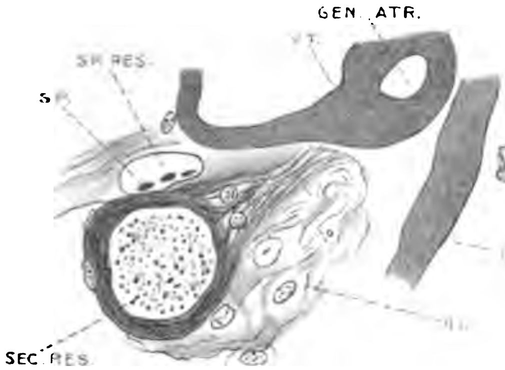


Fig. 1.

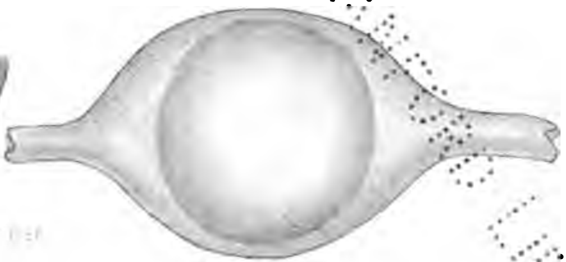


Fig. 6.

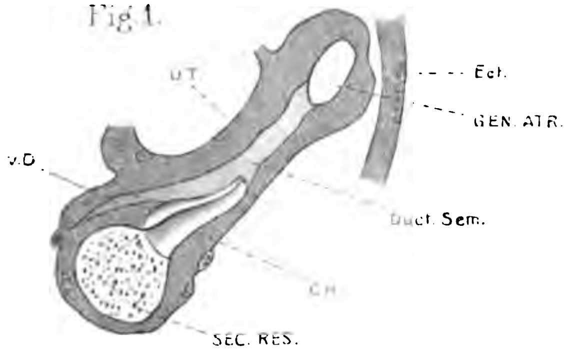


Fig. 2.

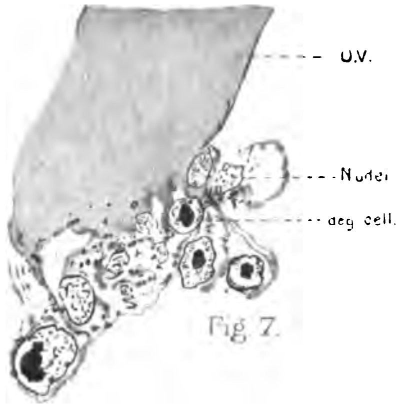


Fig. 7.

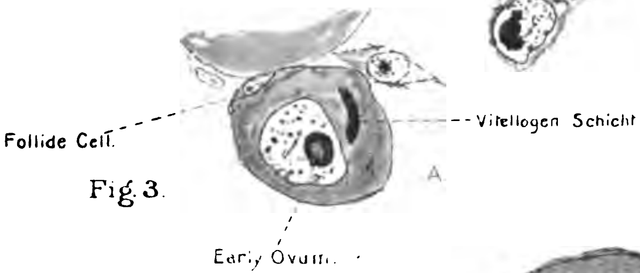


Fig. 3.

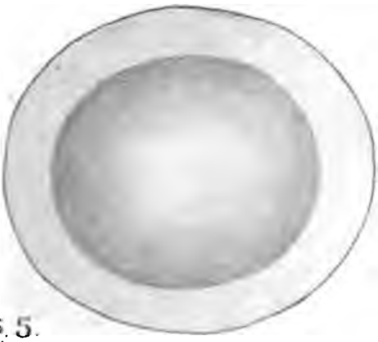


Fig. 5.

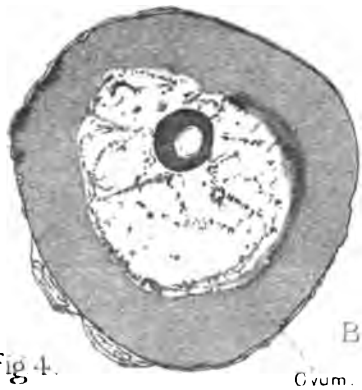


Fig. 4.

M'Farlane & Erskine, Lith. Edin.

IV.—An Account of a Brachydactylous Family. By H. Drinkwater, M.D. (Edin.), M.R.C.S. (Lond.), etc. *Communicated by* Professor D. J. Cunningham, F.R.S.

(MS. received August 10, 1907. Read November 4, 1907.)

Abnormalities of the digits occur under a great variety of forms, and with such frequency that *most medical* men interested in the subject are more or less familiar with them.

In studying such cases one cannot fail to be struck with the fact that, although the tendency to abnormality may be inherited, the inheritance is not always "true," *i.e.* the precise form is not accurately reproduced in the offspring. Especially, perhaps, is this the case where the abnormality consists in a deficiency of digits. One individual of a family may have four fingers, another three, and another two, or even one, and that one very imperfect. Such cases have often been recorded (see *B.M.J.*, 1886, vol. ii. p. 976).

To students of heredity such cases are not nearly so interesting as those abnormalities which are practically identical in successive generations; for the latter show more clearly the influence of heredity in reproducing the variation from the normal type.

It was early in the present year that I first noticed the brachydactylous condition of the hands of one of my patients—a married woman. There appeared to be an entire absence of the middle phalanx from the first, ring, and little fingers, and from all the toes, from the second to the fifth inclusive, whilst the middle phalanx of the middle finger and the first phalanx of the thumb and big toe were extremely short in each instance. She told me that several of her relations showed the same peculiarity; they had short fingers and were "double-jointed," like herself, and in fact that all the members of her family had *either* quite normal digits *or* they were all short, both in hands and feet. Here then was a family well worth studying from the point of view of heredity.

This abnormality does not seem to have been recorded as occurring in this country, and I was not aware that it had been recorded elsewhere until some weeks after my investigations commenced. Then Mr Bateson forwarded me a copy of a paper by William C. Farabee,* in which

* In March 1905. Peabody Institute, Harvard.

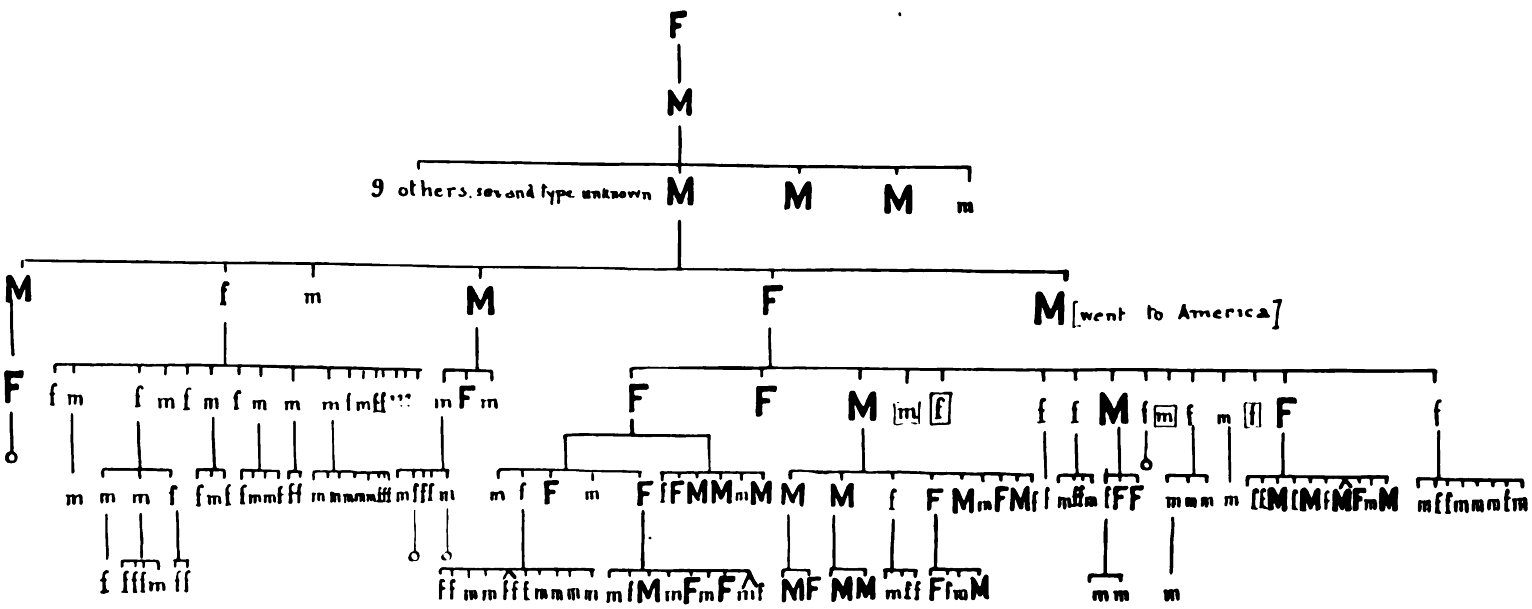
some apparently identical cases seem to have been observed by him *in America*. His cases, in fact, are so similar to mine that I am inclined to think they must be descended from an abnormal member of the English family, four generations back, who migrated to America, but of whom no tidings have since been received by his relations in this country.

My conclusions *as to the precise nature of the abnormality* differ from those of Farabee, who thought the essential feature was the ABSENCE of the middle phalanx. I have seen every surviving abnormal individual in this country, twenty-five in number, as well as most of the normal ones, and as a result of my investigations I am able to confirm my patient's statement with regard to the extent of the peculiarity, namely, that whenever it occurs, *all* the digits of *both hands* and *both feet* are abnormal, and the abnormality is, in all cases, *practically identical* with that exhibited by my patient, though varying somewhat in degree in some of her relatives. As the result of a great deal of correspondence, and inquiries made locally, it has been possible to construct a *genealogical chart showing seven generations through which the abnormality has occurred*, and to make this chart complete as to the *number* of individuals in all the last five generations (fig. 1). The third generation is also complete as to numbers, but incomplete as to the sex and exact proportion of abnormals and normals, but only *one* individual can now be traced in the first, and one in the second, generation. The chart includes 174 individuals, of whom 107 are living. I have also taken a good number of measurements of 86 of these individuals. They are given in an appendix to this paper.

Views of heredity held by many students of biological science at the present day differ very considerably from those held up to the close of the last century, owing to the belated discovery in 1900 of the work of *Gregor Mendel*, in which he gave an account of some extremely interesting and important experiments carried on by him in the sixties. Mendel propounds new theories based on his observed facts, and these theories, whether true or false, will serve as good working hypotheses for some time to come. My own observations were undertaken primarily with the object of recording *facts*, and without any bias resulting from the study of Mendel's published work. However, one can hardly fail to recognise the support that these facts give to Mendel's laws of inheritance.

One of Mendel's laws is as follows:—Among the descendants of two parents, one of whom is normal and the other is in possession of a

BRACHYDACTYLY



Present in all the digits of both HANDS and both FEET in those members of the family shown in Capital Letters

References: M = abnormal male F = abnormal female m = normal male f = normal female □ = died in infancy; abnormality uncertain.

o = married but no children.

Fig. 1.

"dominant" abnormality or characteristic, the variation of the abnormal parent will be reproduced (in the offspring) in approximately 50 per cent. Such is the case in this family. The total number of descendants from the abnormal parents, beginning at the fourth generation (for in the earlier ones the respective numbers are not known), is seventy-five, and of these thirty-nine are abnormals—a result corresponding with what we should expect from Mendel's law.

Another point in which this family agrees with Mendel's observations is the fact that *the children of normal parents are all normal.* The peculiarity is solely transmitted by a brachydactylous parent, and when it once disappears, there is no tendency for it to crop up again in any of the descendants. This is shown by the decendants of the female, number 17, amongst whom not a single abnormal individual has appeared in three successive generations.

This abnormality thus differs in a marked manner from certain hereditary diseases, which are known to appear in the children of a parent in whom it was latent.

There are twenty-five abnormals living at the present time in England and Wales. I have been successful in obtaining photographs or radiographs (or both) of all except the oldest surviving member of the family and four very young children. The old woman lives in a rural district, seven miles from a town, and she promised to send me a photograph of her hand, but up to the present it has not yet arrived. The young children have been photographed, but the results are unsatisfactory: they could not be made to keep their hands still for a sufficient time to give the desired results.

The hands have been photographed and X-rayed more carefully and completely than the feet, as the abnormality seems more striking in the hands.

I am greatly indebted to Dr List of the North Stafford Infirmary and to Drs Humphrey and Geoffrey Williams for their kindness in procuring the radiographs for me.

The hands and feet, as already stated, are abnormal in each affected individual, and the feet are, if anything, more abnormal than the hands, at least as regards the digits. The middle phalanx is practically or virtually —though not actually—absent from each finger and toe. The metacarpal bones are short, and otherwise abnormal, but the metatarsus is scarcely, if at all, affected. Nor is the variation limited to the hands and feet, for *all* the individuals, with the exception of young children and perhaps *one* female, are below the average stature, as is shown by a reference to the table of measurements. It will be well to study each part in turn in the following order: hands, feet, stature, etc.

Hands: External Aspect.

Length.—The most conspicuous feature is the shortness, especially of the fingers: these are only slightly more than *half* the normal length, sometimes even less than half, whilst the hand *looks* abnormally broad. The

No. 98.

Fig. 2.

middle finger, measured on the palmar surface from the base or metacarpo-phalangeal crease to the tip, is normally as long as the width of the palm at the knuckles: in these people it is approximately half this, the average length being $1\frac{5}{16}$ inches (figs. 2 and 3). Sometimes it is shorter than the first finger* (Nos. 98 (fig. 2), 93, and 101). The extreme shortness in

* The numbers refer to the genealogical chart, in which the members are numbered consecutively from above downwards and from left to right. The first in the 4th, 5th, 6th, and last line are respectively Nos. 16, 22 58, and 134.

No. 98 is due chiefly to the metacarpal bone. The hand, measured from the carpal end of the radius to the tip of the first finger, has an average length of 5½ inches in the men and $5\frac{5}{16}$ in the women.

Markings—The skin creases are peculiar. Each finger shows only one crease, corresponding to the space between the first and third phalanx: it is *single*, like the one present in ordinary fingers opposite the second joint. That opposite the first joint is double in most normal hands.

Whilst it would be difficult to tell that there are only two phalanges

No. 97.

No. 96.

Fig. 3.

in each finger from an inspection of their dorsal aspect, the palmar view shows clearly that such is the case; and this is confirmed by radiography in the majority of instances. Farabee does not draw attention to the palmar aspect of his cases.

The palm shows two peculiar lines. The most characteristic one is a line running straight across the hand transversely. It is well shown in Nos. 43, 45, 60, 94, and especially in 96 (fig. 3) and 163 (fig. 4). It appears to be a union of the palmist's lines of "heart" and "head."

The second line starts from the middle of this transverse line and runs to the space between the first and second fingers. (This is seen in Nos. 88

(fig. 5), 90, 93, 99, 101, 102, 156, and 163.) The skin is loose and the whole hand is soft and flabby.

The palm is very compressible, owing to the wide intermetacarpal spaces. They are all "double-jointed," *i.e.* the fingers can easily be flexed dorsally by slight pressure applied direct to their palmar surfaces.

A slight lateral pressure makes the palm half an inch narrower, without altering the plane of the metacarpals. Radiographs B and C, No. 90, are both taken from the same hand. B shows the ordinary condition, C is compressed by a single turn of a thin bandage encircling it in such a way as to

No. 163.

FIG. 4.

keep it flat at the same time that it is compressed laterally. The narrowing, therefore, is real, and is not due to a transverse folding of the hand.

Several individuals can make their finger-joints "crack" by ordinary flexion.

On flexing and extending a finger at the basal joint (metacarpo-phalangeal joint) *slowly*, one can feel that the opposing surfaces are not uniformly curved as in the normal hand. At certain points the phalanx seems to slide over a ridge. This is what one would anticipate from an examination of some of the radiographs (Nos. 45, 50 (fig. 8), 88, 101, etc.). The ring finger in several instances is bent at the middle, so that the tip points towards the middle finger (Nos. 45, 88, 90, 95, 96, 98, 101).

Free perspiration occurs readily on slight excitement: for instance, it was generally noticed when taking the radiographs that the envelope containing the plate was covered with moisture at the part on which the hand had rested. The skin of the back (dorsum) of the hand is very coarsely reticulated—in fact No. 88 (fig. 6) almost looks like a photograph of an etching. The mouths of the sweat glands are conspicuous. The nails are well formed in every individual.

Strength of Grip.—I am unable at present to give any figures showing the actual strength of the hands. It is quite certain that it is *considerably* below the normal average, although the men think themselves equal to

No. 88.

Fig. 5.

others in this respect. Their fingers are too short to grasp all the fingers of my own hand at once, but on getting them to grasp two fingers, the firmest grip of the men was not at all painful, although my hand is peculiarly sensitive to pressure. The people *complain* of being unable to do three things as a consequence of the shortness of their fingers.

First.—Not one is able to play the piano or any musical instrument where the normal length of finger is requisite.

Second.—Their grasp of objects is smaller than normal.

Third.—They cannot do netting.

Of course it is obvious that the individuals whose fingers are short, stumpy, and below the average strength *must* be handicapped in competition with the general population, and this is probably shown to *some* extent by the fact that most of the men and women are engaged in occupations where there is no great demand for manual dexterity. I am not at liberty to mention the actual work in which they are engaged, as it might serve to draw attention to the people themselves, and this would cause offence. I think it only right to record my deepest gratitude to the men, and especially to the women, for allowing the photographs and radiographs to be taken. Several of them only consented on receiving my

No. 88.

Fig. 6.

promise not to publish an account in any form that would draw attention to them individually. Should any medical man, who is acquainted with any member of the family, meet with this account, it is to be hoped he will not discuss the subject with them, or do anything calculated to cause them annoyance. For the same reason I am not at liberty to locate those who live outside my own county. They are all English, and even those who live in North Wales are of English descent.

That the brachydactylous members of this family are handicapped by their abnormality is pretty conclusively shown by their social position compared with their normal relatives. The latter are farmers, butchers, dairymen, grocers, and housekeepers, ladies' companions, etc., whilst the abnormals, without a single exception, are labourers or employed in the ranks of unskilled labour.

The Feet.—Here the one main peculiarity is the shortness of the toes—each one apparently having only two phalanges. They are also broad and

No. 120.

Fig. 7.

straight, with very little tendency to the extreme flexion of many "ordinary" toes. One boy said he could not run as fast as other boys of the same height, and this he attributed to the shortness of his feet, but none of the others made the same complaint; in fact, three brothers have played "forward" in the leading football club of their town.

The photograph of the plantar aspect of the foot of No. 120 (fig. 7) shows the whole length of the toes, my thumb retracting the skin of the sole in order to obliterate the crease between it and the bases of the toes. The toes are almost square in shape.

Facts revealed by Radiography.—The X-rays show that the middle phalanx is not really absent. In Nos. 43, 50, and 110 (fig. 8) it is true that only two bones are visible in each finger. As No. 43 was one of the first cases examined, I was inclined to agree with Farabee that each digit contains only two phalanges, the first and the third, and that the second one was absent. It is, however, readily seen in these same figures that the terminal phalanx differs in shape from the normal phalanx. This difference is clearly manifest on comparing the radiographs of normal and abnormal hands with one another.

In Nos. 45, 56, 88, 90, 94, 95, 96, and 98 the middle phalanx is visible as a separate bone in the middle finger. Here it has a cubical shape corresponding precisely with the base of the terminal phalanx in each of the other fingers.

This cubical basal portion is the second (middle) phalanx that has become ankylosed to the terminal phalanx, which normally is triangular or pyramidal in shape. *The pyramidal distal portion of these bones corresponds to the ungual phalanx and the basal cubical portion to the middle phalanx.*

The middle phalanx is, in a few instances, separate in *two* fingers (the middle and ring) in Nos. 94, 96, 98, 101. It does not exist as a separate bone in either the index or little finger in a single adult.

What has happened to the middle phalanx? It varies in two respects from the normal:—

(1) It is very short.

(2) It generally becomes ankylosed to the base of the terminal phalanx.

In several individuals a distinct but functionally useless joint can be felt next the terminal phalanx in the middle finger, and less often in the ring finger. The fact that the middle phalanx is abortive but not completely absent is proved conclusively by an examination of Nos. 111 (fig. 8) and 156, which are those of young children, and in which the middle phalanx is seen before ankylosis has occurred.

The epiphyses at the bases of the phalanges are ossified from separate centres, but the discs shown in these figures at the bases of the phalanges are *not* epiphyses, but are the rudimentary second phalanges.

It is thus clear that there is no real absence of the second phalanx in any individual, but merely a rudimentary condition, and that at a certain stage of development there is a union of this with the terminal phalanx.

The essential feature of the abnormality apparently consists in an

No. 110. No. 50. No. 111.

absence of the "epiphysis" at the base of the second phalanx. It is possible that the epiphysis is also missing, in some instances, from the third phalanx, and that the two phalanges (second and third) consist at first of a single piece of cartilage.

There does not appear to be any tendency to revert to the normal type, for Nos. 118 (sixth generation) and 163 (seventh generation) will, when adult, be quite as abnormal as any member of the fifth generation, and the hand (110, fig. 9) is quite as abnormal as that of the father, No. 50.

The index and fourth fingers seem more aberrant than the second and third, as they never show the middle phalanx as a separate bone in the adult.

Functionally, the fingers are all reduced to the bi-phalangeal condition, and thus come to resemble thumbs. One normal member of the family greatly offended one of the brachydactylous men by remarking, on one occasion, that "his fingers were all thumbs," a statement not far from the truth. The chief change in the *thumb* consists in a shortening of the first phalanx, which is reduced to a cube. There is no attempt at ankylosis, however. An examination of this bone in Nos. 93, 111, 118 (fig. 9), 156, and 163 fails to detect any sign of an epiphysis. Does not this support the theory that the so-called first phalanx of the thumb is homologous with the second phalanx of the fingers? and, moreover, that the metacarpal bone of the thumb is in reality the first phalanx, for *here* the epiphysis is clearly seen at the *base*, whereas in the other metacarpals the epiphysis is at the distal end of the bones.

The metacarpals vary more or less in different individuals, but, as a rule, where ossification is complete, they are abnormal. Where ossification is not yet complete, they appear merely somewhat shortened, as is well seen in No. 118 (fig. 9).

The head of each metacarpal is distinctly nodulated in many cases, as in Nos. 43, 45, 50 (fifth generation), 88, 95, 96, 99, and 101 (sixth generation). The middle one is the shortest of the four in Nos. 45, 99, and 101. The fifth shows a distinct tendency to thicken, especially in 45 and 96, as well as to become more curved. The abnormal shape of the metacarpals is more marked in the man (No. 45) and *in his descendants* than in his sister (No. 43) and her descendants.

Sesamoid.—Sesamoid bones are well shown in several cases.

Spaces.—The intermetacarpal spaces are generally increased in width, especially between the *heads* of the bones. This accounts for the compressibility of the palm laterally. Ossification is further advanced in the epiphysis for the head of the metacarpal bones and the bases of the first phalanges in No. 163 (child æt. 3), than in its cousin's hands (æt. 4).

No. 118.

FIG. 9.

FEET.

The feet are slightly shorter than normal, the shortness being due solely to the abortive middle phalanx and the somewhat stunted growth of the first.

The metatarsus is practically normal, thus differing from the metacarpus. The greatest variation is in the first phalanx of the big toe, this bone being so far shortened as to become cubical, and in Nos. 18 and 99 the shortening goes even further.

The middle phalanx in the other toes has become ankylosed to the terminal one, and this union is more general or occurs at an earlier stage in the foot than in the hand, though the hand as a whole is more abnormal.

In No. 118 there is an absence of the basal epiphysis of the first "phalanx" of the big toe, though it is conspicuous in the first phalanx of each of the other toes, so that this bone appears really to be homologous with the second phalanx of the other toes.

There does not seem to be a single instance of a resemblance to the "midparent" of Galton, but, on the contrary, the inheritance is always "exclusive" and never blended, and probably never particulate.

The thigh and leg are all slightly shorter than normal, though they are practically proportioned to the stature.

Thus shortness is *much* more marked in the hands, particularly the fingers, than in any other part of the body.

Measurements.—The following measurements have been taken, the number giving the *averages in inches*:—

Middle finger.

(1) ADULTS:—

(*a*) *Males.* Normals, $3\frac{6}{16}$: excess $= 1\frac{7}{16}$.
Abnormals, $1\frac{5}{16}$.

(*b*) *Females* $= 1\frac{2}{16}$ and 3 excess $= 1\frac{4}{16}$.

(2) ADOLESCENTS:—

Normals (male and female), 3 inches: excess $= 1\frac{6}{16}$.
Abnormals (male and female), $1\frac{0}{16}$.

The middle finger of the normals is thus seen to be *nearly* twice as long as that of the abnormals.

The hand.—The figures are:—

(1) *Adult male*, $7\frac{1}{3}$ and $5\frac{1}{4}$: excess about $1\frac{1}{2}$ inches.
Adult female, $6\frac{1}{10}$ and $5\frac{5}{16}$: excess $= 1\frac{5}{16}$.

(2) *Adolescents*, $6\frac{1}{4}$ and 5: excess $= \frac{3}{4}$.

Length of middle finger compared to width of hand:—

Males. Abnormals, $\frac{14}{32}$.

Normals, $\frac{28\frac{1}{2}}{32}$.

Females. Abnormals, $\frac{14}{30}$.

The following table shows the principal measurements, giving the average in inches:—

MEASUREMENTS: AVERAGE IN INCHES.

	1 Middle Finger.	2 Hand.	3 Ratio of 1 to 2.	4 Radius.	5 Humerus.	6 Tibia.	7 Femur.	8 Height.	9 Span.	10 Reach.
ADULTS.										
m	$3\frac{6}{16}$	$7\frac{1}{4}$	$\frac{28\frac{1}{2}}{32}$	$10\frac{1}{2}$	$12\frac{6}{8}$	$15\frac{3}{16}$	$17\frac{3}{4}$	$69\frac{1}{2}$	$70\frac{3}{4}$	$88\frac{3}{8}$
M	$1\frac{15}{16}$	$5\frac{1}{2}$	$\frac{14}{32}$	$8\frac{3}{4}$	$10\frac{5}{8}$	13	15	61	$59\frac{1}{4}$	$75\frac{1}{2}$
Excess of normals	$1\frac{7}{16}$	2	...	$1\frac{3}{4}$	$2\frac{1}{8}$	$2\frac{3}{16}$	$2\frac{3}{4}$	$8\frac{1}{2}$	$11\frac{1}{2}$	$12\frac{7}{8}$
f	3	$6\frac{10}{16}$		9	$10\frac{3}{8}$	...	...	$63\frac{1}{4}$	...	...
F	$1\frac{3}{4}$	$5\frac{5}{16}$	$\frac{14}{30}$	$8\frac{3}{8}$	10	$12\frac{5}{8}$	$16\frac{1}{4}$	$58\frac{1}{4}$	$55\frac{3}{8}$	...
Excess	$1\frac{1}{4}$	$1\frac{5}{16}$	...	$\frac{5}{8}$	$\frac{3}{8}$	...	...	$4\frac{3}{4}$	...	...
ADOLESCENTS. From 14–21 years.			Width of Hand.			Tibia and Femur.				
m+f	3	$6\frac{1}{4}$	$3\frac{3}{8}$	$8\frac{7}{8}$	$10\frac{6}{8}$	$31\frac{1}{4}$		$60\frac{1}{4}$	$62\frac{1}{2}$	74
M+F	$1\frac{5}{8}$	5	$3\frac{5}{8}$	$7\frac{5}{8}$	$9\frac{5}{8}$	$28\frac{1}{2}$		$57\frac{3}{4}$	$51\frac{3}{4}$	67
Excess	$1\frac{3}{8}$	$1\frac{1}{4}$	$-\frac{2}{8}$	$1\frac{1}{4}$	$1\frac{1}{8}$	$2\frac{3}{4}$		$2\frac{1}{2}$	$10\frac{3}{4}$	7
CHILDREN. From 1–3 years.										
m+f	...	...	...	...	...	...		$30\frac{1}{2}$	...	...
M+F	...	...	...	...	...	...		$31\frac{1}{4}$	...	...
Excess	...	...	...	...	...	...		$-\frac{3}{4}$	...	...

m = normal male. M = abnormal male.
f = „ female. F = „ female.

Here we see the remarkable fact that the brachydactylous children, up to 3 years old, are on the average $\frac{3}{4}$ of an inch *taller* than the normal children.

The *tallest normal male* is 5 *feet* $10\frac{1}{2}$ *inches in height.*
„ abnormal „ „ 5 „ $3\frac{1}{2}$ „ „
„ *normal female* „ 5 „ 9 „ „
„ abnormal „ „ 5 „ 2 „ „

Many normal members of the family have been measured in order to make comparisons with the abnormals. It has not been possible, however, to get a complete series of the measurements of the normals.

For comparison they have been placed in three groups. The *first group* contains all adults over 21 years of age. This is divided into two sub-groups—(*a*) males, (*b*) females.

The *second group* contains males and females from 14 to 21 years of age.

The *third group* contains children from 1 to 3 years, with an average age of 2 years both in the normals and abnormals.

Fig. 10 shows, in diagrammatic form, the average abnormal adult

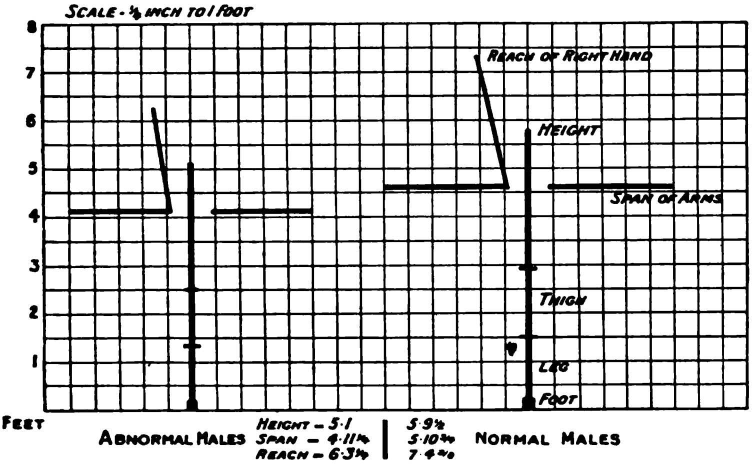


Fig. 10.

male in comparison with the average normal adult male as to *height*, span of outstretched arms, reach of right hand, thigh, and leg.

Thus it is seen that the measurements of normals is in excess of the abnormals in every particular in Classes I. and II., the greatest proportional difference being in the length of the finger and in the span of the outstretched arms—the latter including the sum of the differences of both upper extremities. The femur seems to be shortened in about the same proportion as the tibia, taking into consideration the extra normal lengths.

Class III.

The young children present a striking and unexpected contrast to the rule that holds good in adults and youths. The abnormal children of an average age of 2 years are actually ¾ inch taller than the normal of the same age, so that the arrest of growth occurs in some later period of life, probably beginning soon after this age is passed. There are, however, too few cases between 3 and 14 years from which to take an average of any value.

The shortness of the hands is well shown in Nos. 96 and 97 (fig. 3) and 56 and 109 (fig. 11), where normal and abnormal hands are represented together for the sake of comparison.

No. 109.

No. 56.

Fig. 11.

The comparative stature is shown in Nos. 102 and 115 (two cousins). The smaller (abnormal) boy is 2 years *older* than the taller normal one and 6½ inches shorter.

Symmetry.—In every instance the hands are exactly symmetrical, as shown both by photography and radiography; and I believe the same rule holds true with regard to the feet, though *both* feet were not examined in all cases.

It is a general opinion among the affected mothers that their brachydactylous children are "finer" at birth than the children of normal mothers.

This is probably true, for the affected children included in Table III. were certainly the bigger and had the more robust appearance.

A particularly fine specimen is No. 164, which has a height of 2 feet 4½ inches, although only 9 months old. This is, of course, exceptional, and one must not draw general conclusions from single instances. Still, from an inspection of the young children, one would certainly never suspect that they would eventually be any shorter than their normal relatives.

The general health of the abnormals is excellent, most of them declaring that they have "never been ill," and I failed to detect signs of illness in a single individual, with the exception of the oldest man, who said his "wind was a bit short in going up hill," and had prevented him working for six or seven years.

On the other hand, several of the "normals" are decidedly delicate One woman is extremely anæmic, from what cause I did not ascertain; another woman suffers from spasmodic asthma; a man is tuberculous, with discharging cervicle abscesses (here the phthisis is probably derived from the father, who married into the family, and died of "consumption"); another woman is cyanotic, and probably suffers from mitral disease.

Fecundity.—The whole family is very prolific, the number of children averaging nearly eleven in nine families, the actual numbers being 17, 15, 13, 11, 10, 9, 8, 8, and 6 (=97).

The abnormals are more prolific than the normals. This is shown by comparing the two sisters, Nos. 17 and 20:—

Children of Mrs W., normal (No. 17)	17
Grandchildren	21
Great grandchildren	7
Total	45
Children of Mrs R., abnormal (No. 20)	15
Grandchildren	50
Great grandchildren	34
Total	99

Thus one woman, who was normal, has had 45 descendants; the other woman, her sister, who was brachydactylous, has had 99 descendants.

Of the descendants of No. 17 (normal woman), 9 have married; these have left altogether 28 children and grandchildren.

Of the descendants of No. 20 (abnormal), 18 have married, leaving

84 descendants. This equals 42 children from 9 parents (average), which can be compared with the 9 parents descended from No. 17. Thus the relative fertility of normals and abnormals is as 28 to 42 or as 4 to 6.

Relative fecundity of men and women (abnormals):—

Six married women have had 49 children, an average of $8\frac{1}{6}$.

Eight married men have had 39 children, an average of $4\frac{7}{8}$.

The women are clearly much more prolific than the men (amongst the abnormals).

Marriage.—Taking individuals over 23 years of age, and beginning at the fifth generation. The following never married:—

(1) Dead.—Two normals, both descendants of normal parents. Both were women.

(2) Living.—Fourteen normals (9 female and 5 male).

They are all descended from normal parents. There is not a single *ab*normal individual, either male or female, over 23 years of age, who has remained single. There is no record of any intermarriage, either of normals or abnormals.

Childless Marriages.—There are four of these, all living. Two are normal women, one is a normal man, and one an abnormal woman. They have all been married several years. The abnormal woman is the oldest surviving member of the family.

Origin of Species.—I do not wish to tread on debatable ground further than to state my opinion that the existence and perpetuation of individuals such as these brachydactyli, who show a marked variation, reproduced in several successive generations, seems to support the views of those biologists who contend that evolution frequently occurs *per saltum*, and not invariably, as others suppose, by minute and almost imperceptible gradations.

The abnormality of this family is not such as to confer upon the individuals any advantage in the struggle for existence, but rather are they at a disadvantage, such that, in the wild state, the variety would soon be stamped out. It is, however, quite conceivable that an equally well-marked *beneficial* variation might occur, and in that case one would expect them to survive. These brachydactyli, in a wild state, would have succumbed owing to their inability to use effective weapons of offence and defence. But the conditions of modern civilisation are such that many individuals, handicapped at birth, are not only able to survive, but to perpetuate their defect, even though it tend to deterioration of the race. So dominant are the abnormal members of this family, that there is little chance of their peculiarity becoming extinct so long as its possessors continue to marry. They seem to be on the increase, the number of abnormals in the last four generations being as follows:—

In the fourth generation there are 4 cases.
„ fifth „ „ 7 „
„ sixth „ „ 19 „
„ seventh „ „ 9 „

This last number will probably be increased considerably, as there are three parents still in the child-producing age, and eight individuals—not yet having reached it—may still live to be married and have children.

Of the first appearance of this abnormality one can assign no cause. It is, however, highly probable that it appeared suddenly as a "sport."

If such a marked variation as is here recorded is transmitted generation after generation, it seems quite intelligible that smaller variations which are not apparent to the eye, or capable of detection by the scalpel or microscope or the most delicate chemical analysis, may likewise be transmitted and may account for the tendency which certain individuals manifest towards certain diseases. It shows that "nature" is more potent than "nurture."

Mr Bateson, in his preface to *Mendel's Principles of Heredity*, says: "The study of variation and heredity *must* be built of statistical data." And again (p. 7): "No more useful work can be imagined than a systematic determination of the precise 'law of heredity' in members of particular cases." I offer this monograph as such a contribution.

Measurements of Normals.

	Sex.	No. in Chart.	Age.	Hand.	Radius.	Humerus.	Tibia.	Femur.	Height.	Middle Finger.	Width of Hand.	Span.	Reach.
									ft. in.			ft. in.	ft. in.
J. W. . .	m	26	65	7½	10	12½	15½	19¼	5 10½	3½	3¾	5 10	7 7½
W. W. .	m	31	54	...	...	...	...	...	5 9	...	...	...	...
S. W. . .	m	32	53	...	...	...	...	...	5 10	...	...	..	...
R. W. . .	f	33	51	...	...	...	...	...	5 9	...	...	...	...
M. W. . .	f	35	Adult	...	...	...	...	.	5 8	...	...	...	...
A. W. . .	f	36	„	...	...	...	...	...	5 6	...	...	...	...
Mrs S. . .	f	49	48	7	9½	10¾	...	...	5 1¼	...	...	...	...
Mrs J. . .	f	51	44	7¼	9¼	11	...	...	5 1	...	...	...	...
Mrs P. . .	f	53	40	7	9¼	11	...	...	5 2½	...	...	...	...
H. R. . .	m	54	37	7	11¼	13½	17¼	19½	5 10	3½	3¾	6 0	7 7
Mrs W. .	f	57	34	6⅞	9½	10¾	...	...	5 1½	...	...	...	...
Mrs H. .	f	61	30	...	...	...	...	...	5 3	...	...	...	...
C. A. W. .	f	65	32	...	...	...	...	...	5 4	...	...	...	...

Measurement of Normals—*continued.*

	Sex.	No. in Chart.	Age.	Hand.	Radius.	Humerus.	Tibia.	Femur.	Height.	Middle Finger.	Width of Hand.	Span.	Reach.
									ft. in.			ft. in.	ft. in.
R. W. . .	m	66	32	...	...	...	...	...	5 6	...	...	...	...
Miss W. .	f	69	Adult	...	...	...	...	...	5 6	...	...	...	...
Miss W. .	f	70	"	...	...	...	...	...	5 9	...	...	...	...
S. O. W. .	m	71	18	...	...	...	...	...	5 9	...	...	...	...
J. H. W. .	m	72	16	...	...	...	...	...	5 10	...	...	...	...
L. E. W. .	m	74	12	...	...	...	...	...	4 11½	...	...	...	...
H. R. W. .	m	75	9	...	...	...	...	...	4 5	...	...	...	...
H. M. W. .	f	76	7	...	...	...	...	...	4 0	...	...	...	...
R. O. W. .	f	77	4	...	...	...	...	...	3 4	...	...	...	...
I. G. W. .	f	78	1¼	...					2 6	...	...	...	...
Mrs W. .	f	80	Adult	6	18		33		5 1	3	3¼	4 8	6 3
A. E. S. .	f	82	"	...	...	...	...	...	5 0	...	...	...	...
Mrs D. .	f	85	41	6½	8	10⅜	12	17⅜	4 10	3	3⅜	4 11	6 5⅜
S. M. . .	m	93	16½	7	10	11¾	14¾	16¾	5 4	3	3¾	5 5	6 8¾
Mrs P. . .	f	97	26	6	8	9¾	13	15	4 9	2¾	3	4 7¾	6 1¾
W. H. S. .	m	105	24	7	10	13	13½	18	5 8¾	3½	3½	5 9½	7 0
E. S. . .	f	106	19	6¾	9½	11½	14	18	5 3¼	3¼	3	5 6½	6 9
E. S. . .	f	107	16	6⅝	8¾	10¼	14	17¼	5 1¾	3	3¼	5 1½	6 7
J. S. . .	m	108	14	6¼	8¾	9¾	13¾	15⅝	4 10¼	2¾	3⅜	4 11	6 2⅝
Mrs T. .	f	109	23	5¾	8½	10¼	12	17¾	5 1½	...	...	...	...
J. P. . .	m	112	22½	7½	10¾	12½	14½	16½	5 9	3¼	4	5 11½	7 3
J. P. . .	m	113	21	6	9	10¼	13	16¼	5 8¾	2⅞	3½	4 10¼	6 2
W. P. . .	m	114	6	...	...	...	...	...	3 7	...	...	...	...
A. R. . .	m	115	13	6	8½	9¾	11	17	4 10	2½	2⅝	4 9½	6 0
E. H. . .	f	117	14	5½	7⅞	10¼	12	15½	4 6½	...	...	...	...
M. H. . .	f	121	9	4½	6¼	8	10½	12⅜	3 10½	...	...	...	...
J. H. . .	m	124	4	4	5	6¼	7¾	9⅜	3 1½	...	...	...	...
J. W. . .	m	126	16½	6½	8⅝	11¼	13¾	18	4 10¾	...	...	...	...
A. W. . .	f	127	14½	5⅞	9	10	13¼	16	4 10	...	...	...	...
F. W. . .	f	128	12 4/12	5½	7½	10	11½	15	4 5¼	...	...	...	...
J. W. . .	m	129	10⅔	4¼	5½	6¾	8¼	11½	3 6½	...	...	...	...
W. W. . .	m	130	8	4⅝	6	7¾	9	13¼	3 9½	...	...	...	...
E. W. . .	f	132	4	4	5	7	8	10	3 3	...	...	...	...
C. W. . .	m	133	2	...	...	...	6¾	...	2 8½	...	...	...	...
M. H. . .	f	139	2¾	...	...	...	...	...	2 11	...	...	...	...
D. H. . .	f	140	1¾	...	...	...	...	...	2 4	...	...	...	...
W. H. D. .	m	143	17	6	9½	11	14	17½	5 1½	3¼	3⅝	5 3	6 7½
A. D. . .	m	151	2¾	4	5⅝	5¼	6¾	8	2 8¾	1⅝	2¼	...	...
C. D. . .	f	147	13	6⅛	7¾	3⅝	13	16½	4 9½	3¼	2⅞	4 9	6 3
S. E. R. .	f	153	10	5	7	8	10¼	13½	4 0¾	2½	2¾	4 0	5 2
I. J. R. .	m	155	7	4¾	6½	8½	10	12¼	3 8¼	2¼	2⅝	3 9	4 8½
W. J. P. .	m	165	4	3¼	4½	4½	6¾	7¾	2 8	1½	2	...	...
H. P. . .	f	166	3	...	...	...	...	...	2 4½	...	...	...	...
M. J. P. .	f	167	1 4/12	...	...	...	...	...	2 4	...	...	...	...
W. B. . .	m	170	2	3¼	4	4½	5½	7⅝	2 6¾	1¾	2	2 8	3 4
H. W. T. .	m	172	4	4	5	6	7½	8½	3 4¼	...	...	...	...
F. T. . .	m	173	1	...	...	...	...	...	2 3½	...	...	...	...
H. P. . .	m	174	2	...	...	...	...	...	2 9	...	...	...	...

Abnormals.

	Sex.	No. in Chart.	Age.	Hand.	Radius.	Humerus.	Tibia.	Femur.	Height.		Middle Finger.	Width of Hand.	Span.		Reach.	
									ft.	in.			ft.	in.	ft.	in.
Mrs N. .	f	22	64	4¾	9	9¾	...	...	4	6½	1½	3¾	...		...	
Mrs M. .	f	43	59	5¼	7¾	9½	13	16	4	10½	1⅞	3¾	4	6	5	9
W. R. .	m	45	57	5½	8¼	12¼	13¼	12½	5	1	2	4	4	10	6	6
T. R. .	m	50	46	5¾	9	10⅜	13¾	16¼	5	3½	2⅛	4	5	2	6	9
Mrs H. .	f	56	36	5⅜	8¼	10¼	13¼	15¾	5	1	...	...	...		...	
Mrs R. .	f	88	32	5¼	8	10½	11½	17	4	10½	2	3¾	4	7	6	0
E. J. M. .	f	90	22	5½	8½	10½	14½	15¾	5	2	2⅛	3¾	4	10½	5	6
M. M. .	m	94	15	5½	8	10	13¾	16	5	0	1¾	3½	4	6½	5	3
J. R. .	m	95	30	5¼	8⅝	10⅝	13	15[illegible]	4	11	...	...	...		...	
T. R. .	m	96	28	5	8½	10	12½	14½	5	2	1¾	4¼	4	11½	6	4½
Mrs B. .	f	98	25	5¼	9	10	11¼	17	4	9	2	4	4	6	6	5
F. R. .	m	99	21	4¾	8	10¼	12⅛	16	5	1	1⅞	3¾	4	9½	6	4
E. R. .	f	101	18	4½	8	9¾	12¾	18	5	0	1½	3¾	4	4½	6	1
H. R. .	m	102	15	4⅛	6¾	8⅜	11½	14	4	3½	1⅝	3½	4	0	5	4½
E. R. .	f	110	16	5	8	10	12½	16½	4	11½	...	...	...		...	
F. R. .	f	111	6	3½	5¼	6¼	6¼	9½	3	4½	...	...	...		...	
W. H. .	m	118	13	4⅛	7¼	9	11½	13¾	4	7	1¼	3	4	0¼	5	6½
T. H. .	m	120	10	4	7	8	11	13	4	2	1 1/16	2⅝	3	9½	5	3
H. H. .	m	125	2	...	...	...	...	...	2	9¾	...	...	...		...	
M. E. R. .	f	156	5	3½	5	6¾	8½	9	3	5	1⅛	2⅜	3	0½	4	0¾
G. R. .	f	162	2	2¼	...	...	...	...	2	4½	...	...	...		...	
W. R. .	m	163	3	2¾	4	5¼	7	8	2	10½	¾	2	...		...	
T. R. .	m	164	¾	...	...	...	...	...	2	4½	...	...	...		...	
E. B. .	f	168	4½	3	4½	4¾	6⅝	10	2	11¾	1½	3	2	10	3	9
N. B. .	m	171	10wks.	2	3	3½	3	5	1	10½	1⅛	2	1	6	...	

(*Issued separately January* 10, 1908.)

V.—Notolepis Coatsi, Poisson pélagique nouveau recueilli par l'Expédition Antarctique Nationale Ecossaise. Note préliminaire, par Louis Dollo, Sc.D. (Cantab.), For.Mem.G.S., C.M.Z.S., à Bruxelles (Musée). *Présentée par* M. R. H. Traquair, M.D., F.R.S., V.P.R.S.E.

(Read November 21, 1907. Received same date.)

I. Introduction.

Bien que les *Orcades du Sud* aient été découvertes dès 1821, par Powell, sur le *Dove*, et bien que ces îles aient été visitées à nouveau : *

En 1822–23, par Weddell, qui donna le nom au groupe, avec la *Jane* et le *Beaufoy* ;

En 1838, par Dumont d'Urville, avec l'*Astrolabe* et la *Zélée* ;

En 1874, par Dallmann, avec le *Grönland* ; †

En 1893, par Larsen, avec le *Jason* ;

rien n'était connu de la faune ichthyologique de ces parages au moment où ils furent explorés par l'Expédition Antarctique Nationale Ecossaise (1902–1904).

Par contre, la *Scotia* a, non seulement recueilli, et rapporté, de nombreux Poissons des îles dont il s'agit,—et où elle hiverna,—mais une dizaine d'aquarelles, jointes aux collections, permettent de se rendre compte de la coloration des animaux en vie.

Réservant les *Poissons littoraux* des *Orcades du Sud* pour une prochaine communication,—je m'occuperai, aujourd'hui, d'un *Poisson pélagique* capturé dans la Baie de la Scotia (Ile Laurie) et appartenant à la famille des *Paralepidæ*.

Comme ce Poisson est nouveau, je l'appellerai *Notolepis Coatsi*,—en l'honneur de M. James Coats, jun., de Paisley, et de son frère, le Major Andrew Coats, D.S.O., d'Ayr,—à la munificence desquels l'Expédition Antarctique Nationale Ecossaise dût de pouvoir réaliser son programme, sans être arrêtée par les difficultés financières.

* Three of the Staff (R. N. Rudmose Brown, R. C. Mossman, J. H. Harvey Pirie), *The Voyage of the "Scotia,"* p. 72, Edimbourg, 1906.

† A. Schück, "Die Entwickelung unserer Kenntnisse der Länder im Süden von Amerika," *Verhandlungen des Vereins für naturwissenschaftliche Unterhaltung zu Hamburg*, vol. v. p. 133, 1882.

Je satisfais, en cela, au désir, à moi exprimé, par M. W. S. Bruce F.R.S.E., Directeur du Laboratoire Océanographique Ecossais, à Edimbourg, et Leader de l'Expédition.

II. Les Paralepidæ.

I. La première idée de réunir *Paralepis* à *Sudis* en un groupe autonome remonte à Bonaparte (1840), qui en fit une sous-famille (*Paralepidini*) des *Sphyrænidæ.**

II. L'illustre Johannes Müller (1843) transporta ce groupe dans ses *Scopelini*, mais en l'y noyant. †

Cependant, plus de vingt ans après, M. A. Günther (1864), Conservateur honoraire au British Museum, lui rendit son indépendance (*Paralepidina*), tout en le laissant dans les *Scopelidæ.* ‡

III. Enfin, M. Th. Gill (1872), Professeur à l'Université de Washington, établit, pour lui, la famille des *Paralepididæ.* §

IV. Je crois qu'il y a lieu de maintenir cette famille et d'y ranger aujourd'hui :

1. *Sudis*, Rafinesque, 1810. ||
2. *Paralepis*, Cuvier, 1817. ¶
3. *Plagyodus*, Steller, 1831.**
4. *Prymnothonus*, Richardson, 1845.††
5. *Neosudis*, Castelnau, 1873. ‡‡
6. *Lestidium*, Gilbert, 1905. §§
7. *Notolepis*, Dollo, 1907. ||||

Car cet assemblage me paraît homogène et nettement délimité.

* C. L. Bonaparte, "Prodromus Systematis Ichthyologiae," *Nuovi Annali delle Scienze Naturali*, vol. iv. p. 274, 1840. C. L. Bonaparte, *Iconografia della Fauna Italica per le quattro classi degli Animali Vertebrati*, vol. iii. (*Pesci*), p. 152, Rome, 1832–41.

† J. Müller, "Beiträge zur Kenntniss der natürlichen Familien der Fische," *Archiv für Naturgeschichte*, vol. ix. p. 321, 1843.

‡ A. Günther, *Catalogue of the Fishes in the British Museum*, vol. v. p. 418, Londres, 1864.

§ Th. Gill, "Arrangement of the Families of Fishes," *Smithsonian Miscellaneous Collections*, vol. xi. (No. 247), p. 16, 1872.

|| C. S. Rafinesque, *Caratteri di Alcuni Nuovi Generi e Nuove Specie di Animali e Piante della Sicilia, con varie Osservazioni sui medesimi*, p. 60, Palerme, 1810.

¶ G. Cuvier, *Le Règne Animal*, vol. ii. p. 289, Paris, 1817.

** P. Pallas, *Zoographia Rosso-Asiatica*, vol. iii. p. 383, St.-Pétersbourg, 1831.

†† J. Richardson, "Fishes," *Zoology of H.M.S. "Erebus" and "Terror," under the command of Captain Sir James Clark Ross, R.N., F.R.S., during the years 1839 to 1843*, p. 51, Londres, 1844–48.

‡‡ F. de Castelnau, "Contribution to the Ichthyology of Australia (vii. Fishes of New Caledonia)," *Proceedings of the Zoological Society of Victoria*, vol. ii. p. 118, 1873.

§§ C. H. Gilbert, "The Deep-Sea Fishes of the Hawaiian Islands," *Bulletin of the United States Fish Commission*, vol. xxiii. (1903), p. 607, 1905.

|||| Voir plus loin.

Plagyodus a, il est vrai, une énorme Dorsale antérieure, mais *Prymnothonus* a perdu la sienne: variations, en sens contraires, autour d'un type moyen représenté par *Paralepis*.

III. Notolepis Coatsi.

I. *Diagnose.*—Je donnerai, d'abord, les caractères distinctifs de notre Poisson, en plaçant, après chacun d'eux, le nom du ou des genres dont ce caractère le sépare:

Notolepis Coatsi, Dollo, 1907.

1. *Ecailles*, cycloïdes, très minces, transparentes, caduques (*Plagyodus*, *Lestidium*).
2. *Mandibule*, débordant au delà de la mâchoire supérieure en avant (*Prymnothonus*).
3. *Dents:* pas de grandes dents, ni dans les mâchoires, ni sur le palais (*Sudis*, *Plagyodus*, *Neosudis*, *Lestidium*).
4. *Première Dorsale*, courte, dans la moitié postérieure du corps (*Plagyodus*, *Prymnothonus*).
5. *Deuxième Dorsale*, adipeuse, rudimentaire, mais longue et basse, au lieu d'être courte et haute: deux fois aussi longue que la première dorsale (*Sudis*, *Paralepis*, *Plagyodus*, *Neosudis*, *Lestidium*).
6. *Ventrales*, extrêmement réduites (n'ayant que le tiers de la longueur des ventrales d'un *Paralepis sphyrænoides* de même taille), insérées en avant de la première dorsale et situées presqu'entièrement en avant de celle-ci (*Plagyodus*, *Prymnothonus*).
7. *Couleur:* argenté (Aquarelle 80 de l'Expédition Antarctique Nationale Ecossaise).
8. *Longueur totale:* 110 millimètres environ.

Type du Genre et de l'Espèce: Scottish Oceanographical Laboratory, à Edimbourg (Ecosse).

II. *Deuxième Dorsale.*—La deuxième dorsale, longue et basse, éloigne *Notolepis* de *Sudis*, de *Paralepis*, de *Plagyodus* et de *Lestidium*.

Ce caractère me paraît important. Car nous avons affaire ici à un organe atrophié. La deuxième dorsale courte et haute, rudimentaire, ne s'est, évidemment, pas transformée en une deuxième dorsale longue et basse, rudimentaire, ou réciproquement: pourquoi un tel changement, puisqu'il s'agit d'appareils sans usage?

Notolepis, d'une part, *Sudis*, *Paralepis*, *Plagyodus* et *Lestidium*, de l'autre, remontent donc à des formes ancestrales avec deuxième dorsale fonctionnelle d'un modèle différent.

IV. Bionomie du Notolepis Coatsi.

(I.) *Biogéographie.*

Habitat: 60° 44′ S. et 44° 50′ W.
Baie de la Scotia (Ile Laurie).
Orcades du Sud.
Océan Atlantique.
Quadrant Américain.
Station 325.
Scotia.

(II.) *Ethologie.*

1. *Profondeur.*—De 9 à 10 fathoms.
2. *Nature du Fond.*—Sable et cailloux.
3. *Température du Fond.*—30°-31° F.
4. *Température de la Surface.*—29° F.
5. *Mode de Capture.*—Pris à la main, à la Surface, dans un trou percé pour faire descendre la Nasse.
6. *Date de Capture.*—2 Juillet 1903.
7. *Heure de Capture.*—10 heures du matin.
8. *Nombre d'Individus capturés.*—Un seul.

V. Les Paralepidæ et la Bipolarité.

I. *Bipolarité.*—En 1904, j'ai eu l'occasion d'examiner en détail la théorie de la Bipolarité et de donner les raisons pour lesquelles je ne pouvais m'y rallier, ni en principe, ni en fait.*

II. *W. Kükenthal.*—Cependant, tout récemment (1907), M. W. Kükenthal, Professeur à l'Université de Breslau, écrivait encore:

"Die Bipolarität mariner Organismen, also die Aehnlichkeit der polaren Faunen auf Grund verwandtschaftlicher Beziehungen, ist eine Tatsache, die nicht mehr bestritten werden kann." †

Mais, comme mon mémoire a échappé au zoologiste allemand, et comme je n'ai point vu ailleurs, non plus, la réfutation de mes arguments, je me crois autorisé à considérer ceux-ci comme ayant conservé toute leur valeur jusqu'aujourd'hui.

III. *Nouvel Examen.*—Naturellement, je reprendrai la discussion dans

* L. Dollo, "Poissons de l'Expédition Antarctique Belge," *Résultats du Voyage du S.Y. "Belgica" en 1897, 1898, 1899, sous le commandement de A. de Gerlache de Gomery*, pp. 191–207, Anvers, 1904.

† W. Kükenthal, "Die marine Tierwelt des arktischen und antarktischen Gebietes in ihren gegenseitigen Beziehungen," *Veröffentlichungen des Instituts für Meereskunde und des Geographischen Instituts an der Universität Berlin*, fasc. 11, p. 17, 1907.

mon travail définitif sur les Poissons de la *Scotia*, en m'appuyant sur les nouveaux matériaux arctiques et antarctiques recueillis depuis l'expédition de la *Belgica*, de façon à soumettre une dernière fois la théorie au contrôle des faits.

En attendant, je me bornerai à quelques observations.

IV. *Climat Prétertiaire.*—Les auteurs de la théorie de la Bipolarité postulent un Climat Prétertiaire Uniforme pour le globe entier.

Or, un géologue particulièrement qualifié, par dix ans d'études sur la question, pour traiter des anciens climats, M. J. W. Gregory, Professeur à l'Université de Glasgow, déclare, au contraire :

"The evidence of palæontology proves that the climatic zones of the earth have been concentric with the poles as far back as its records go."

"That tropical or sub-tropical conditions once prevailed within the Arctic Circle is affirmed on the reported occurrence there of fossil Coral Reefs and Tropical Vegetation."

"The palæontological evidence at present available does not throw on us the burden of explaining why the Arctic had a tropical climate, for it simply contradicts assertion as a matter of fact." *

Il n'est donc plus permis de dire, sans de nouvelles et solides preuves à l'appui :

"Sehen wir aber hiervon ab, so scheint mir der Grundgedanke der Pfeffer'schen Ausführungen, dass einst eine allgemeine Warmwasserfauna existiert hat, durch die neueren Untersuchungen sich mehr und mehr zu befestigen. Das lehrt auch wieder die circumtropische Verbreitung der Gattungen und bei bathypelagischen Formen auch der Arten der Tiefseefische. Nach den Funden von Korallenriffen in hohen Breiten kann auch keine Frage sein, dass heute tropische Tiere bis in die heute polaren Zonen hinein früher gelebt haben." †

V. *Migrations Polaires.*—Les auteurs de la théorie de la Bipolarité invoquent aussi, soit la Conservation sur place, soit des Migrations polaires centrifuges et des Migrations verticales négatives, pour expliquer les Faunes arctique et antarctique.

Mais ce sont des Migrations polaires centripètes et des Migrations verticales positives, toutes deux relativement récentes, que révèlent essentiellement les recherches exécutées depuis :

* J. W. Gregory, "Climatic Variations, their Extent and Causes," *Comptes rendus du Congrès géologique international (Mexico, 1906)*, pp. 7, 8, 12.

† A. Brauer, "Die Tiefsee-Fische (I. Systematischer Teil)," *Wissenschaftliche Ergebnisse der deutschen Tiefsee-Expedition auf dem Dampfer "Valdivia" 1898–1899*, p. 355, Iéna, 1906.

"L'Antarctide de M. Osborn rend aussi bien compte de la Spécialisation Polaire Centripète des *Nototheniidæ.* Cette Spécialisation correspondrait à une Migration Polaire Centripète. Qui proviendrait, elle-même, de la Régression de l'Antarctide Tertiaire. Laquelle aurait eu pour conséquence de reculer les Rivages de l'Antarctide sous de Hautes Latitudes. D'où de Nouvelles Adaptations des *Nototheniidæ.*" *

"The indigenous species of the Polar Deep . . . . I also recognize these, however, as originally migrated from Arctic coasts. In the lapse of time they have quite broken off the connection with their former home, and now they appear as particular species and genera." †

"Ferner haben mehr und mehr die Untersuchungen, so die Doflein's über *Brachyuren,* Meisenheimer's über *Pteropoden* und diese vorliegende, dasselbe Resultat, dass die heute polaren Formen von den warmen Gebieten aus erst in die kalten eingewandert sind und, nach ihrem Umfange zu schliessen, seit verhältnismässig nicht langer Zeit, zum Teil vom Litoral zum Litoral, und von diesem in die Tiefsee, zum Teil vom Pelagial zum Pelagial und von diesem in die Tiefsee, zum Teil von der Tiefsee der Tropen in die Tiefsee der polaren Gebiete." ‡

VI. *Les Paralepidæ.*—Grâce à l'Expédition de la *Scotia,* nous pouvons comparer actuellement les Paralépides antarctiques et subantarctiques aux Paralépides arctiques et subarctiques, ce qui était impossible en 1904.§

Et voir, ainsi, si la Biogéographie de cette famille est favorable, ou non, à la théorie de la Bipolarité.

Or, nous avons:

1.—A l'intérieur du Cercle Polaire:

Arctique	*Paralepis Krøyeri.* ‖
Antarctique	*Prymnothonus Hookeri.*¶

* L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, p. 223.

† A. S. Jensen, "On Fish-Otoliths in the Bottom-Deposits of the Sea (I. Otoliths of the Gadus-Species deposited in the Polar Deep)," *Meddelelser fra Kommissionen for Havundersøgelser (Fiskeri)*, vol. i. (No. 7,), p. 4, 1905.

‡ A. Brauer, *Die Tiefsee-Fische, etc.*, p. 355.

§ L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, p. 53.

‖ R. Collett, "Om en Del for Norges Fauna nye Fiske, fundne i 1880–1896," *Archiv for Mathematik og Naturvidenskab*, vol. xix. (No. 8), p. 21, 1897. N. Knipowitsch, "Einige Worte uber das Vorkommen von *Lampris pelagicus* (Gunn.) an den nördlichen Küsten Russlands," *Annuaire du Musée zoologique de l'Académie impériale des Sciences de St.-Pétersbourg*, vol. v. p. 245, 1900. Comme l'auteur parle du *Paralepis borealis*, Jordan et Gilbert, il s'agit donc du *Paralepis Krøyeri*, Lütken.

¶ L. Dollo, "*Prymnothonus Hookeri*, Poisson pélagique de l' "Erebus" et de la "Terror" retrouvé par l'Expédition Antarctique Nationale Ecossaise," *Proceedings of the Royal Society of Edinburgh*, vol. xxvii. p. 44, 1907.

2.—Entre 60° de Latitude et le Cercle Polaire:

Arctique	*Paralepis Krøyeri.* * *Paralepis borealis.* †
Antarctique	*Prymnothonus Hookeri.*‡ *Notolepis Coatsi.*§

si on excepte *Plagyodus ferox,* qui est cosmopolite:

Pacifique	Kouriles.‖ Japon. Tasmanie. N. Zélande.	I. Aléoutiennes. I. Vancouver. Californie.
Atlantique	Grœnland. N. Ecosse. N. Angleterre. Floride. Cuba.	Islande. I. Féroé. C. Finisterre. Madère. Canaries. Guinée. ¶

* C. Lütken, "Korte Bidrag til nordisk Ichthyographi [VIII. Nogle nordiske Laxesild (Scopeliner)]," *Videnskabelige Meddelelser fra den naturhistoriske Forening i Kjøbenhavn,* p. 227, 1891. A. S. Jensen, "The Fishes of East-Greenland," *Meddelelser om Grönland,* vol. xxix. p. 272, Copenhague, 1904.

† J. Reinhardt, "Ichthyologiske Bidrag til den grönlandske Fauna," *Det Kongelige Danske Videnskabernes Selskabs naturvidenskabelige og mathematiske Afhandlinger,* vol. vii. pp. 115 et 125, 1838.

‡ A. Günther, "Report on the Pelagic Fishes," *Voyage of H.M.S. "Challenger" during the years 1873–76: Zoology,* vol. xxxi. p. 41, 1889.

§ Voir plus haut.

‖ C'est ici que l'infortuné G. W. Steller, qui accompagnait la deuxième Expédition de Vitus Bering (1733–1749), composée du *St. Peter* et du *St. Paul,* découvrit, en 1742–43, le type du genre *Plagyodus.*

P. Pallas, *Zoographia Rosso-Asiatica, etc.,* p. 383. E. Büchner, "Die Abbildungen der nordischen Seekuh (*Rhytina gigas,* Zimm.)," *Mémoires de l'Académie impériale des Sciences de St.-Pétersbourg,* vol. xxxviii. (No. 7), pp. 2 et 15, 1891.

¶ C'est au large de la Guinée supérieure que fut recueilli, en 1703, le premier spécimen de *Plagyodus,* lors de l'Expédition de William Dampier dans les Mers du Sud, en 1703–4, avec le *St. George* (ayant à bord, outre le Capitaine Dampier, William Funnell, qui donna la première description et la première figure de notre Poisson, en 1707) et le *Cinque Ports Galley* (ayant à bord Alexander Selkirk, le prototype de *Robinson Crusoé*).

A. Günther, "A Contribution to the History of *Plagyodus* (Steller)," *Annals and Magazine of Natural History,* vol. vii. p. 35, 1901.

3.—Dans chacun des deux Hémisphères :

Boréal	*Sudis.** *Paralepis.*† *Lestidium.* ‡
Austral	*Prymnothonus.* § *Neosudis.* ‖ *Notolepis.* ¶

si on excepte *Plagyodus*, qui est cosmopolite.

VII. *Conclusion.*—Tout ce qui précède, étant contraire à la théorie de la Bipolarité, confirme nos conclusions de 1904, par lesquelles nous déclarions ne pouvoir accepter cette théorie.

* Quatre espèces : *S. hyalina* (Méditerranée), *S. intermedius* (Mer des Antilles), *S. ringens* (Californie), *S. Jayakari* (Mer d'Oman).

† Sept espèces : *P. coregonoides* (Manche, Méditerranée, G. du Mexique), *P. sphyrænoides* (Méditerranée, Madère, Canaries), *P. borealis* (Islande, S. Grœnland), *P. Rissoi* (Méditerranée et au large de la Vendée), *P. speciosus* (Méditerranée), *P. coruscans* (Sud I. Vancouver), *P. Krøyeri* (Finmark à Caroline du Sud).

‡ Une espèce : *L. nudum* (Hawaii).

§ Une espèce : *P. Hookeri* (entre 62° 26′ S. et. 71° 50′ S.).

‖ Une espèce : *N. vorax* (N. Calédonie).

¶ Une espèce : *N. Coatsi* (Orcades du Sud).

(*Issued separately January* 10, 1908.)

VI.—The Body-Temperature of Fishes and other Marine Animals.

By **Sutherland Simpson**, M.D., D.Sc. *Communicated by* Professor E. A. Schäfer, F.R.S.

(MS. received November 20, 1907. Read December 2, 1907.)

Introduction.

With regard to their body-temperature living beings are divided into two great classes,—warm-blooded animals and cold-blooded animals. This is a very old distinction, and although the classification is not quite accurate, these terms are fixed in the literature, and if not interpreted too rigorously, they may be conveniently used to indicate, on broad lines, what appears to be a fundamental difference between the two classes. Birds and mammals alone belong to the former division. They are called warm-blooded because their bodies are warm when compared with the medium in which they live, unless under exceptional circumstances. All other animals are cold-blooded, and they are so called because they have the same temperature as the surrounding medium, and this, to the human hand, is cold as a rule. However, so-called warm-blooded animals, when anæsthetised and placed in a cold atmosphere, may have their temperature reduced to 20° C. or even 15° C., and still remain alive; and, on the other hand, many cold-blooded animals living in the tropics may have a temperature of 38° C. or 40° C.—Richet * observed tortoises for several days with a temperature over 39° C., and frogs have been known to live in water at 33° C. to 37° C. Obviously, therefore, it is not scientifically correct to designate a mammal warm-blooded with a temperature of 20° C., and a tortoise cold-blooded with a temperature of 39° C.

John Hunter † was the first to point out that the essential difference between the two classes lies in the fact that in birds and mammals the temperature is constant, and independent of that of the surroundings, whereas in all other animals it is inconstant, and varies with the temperature of the medium in which they live. To indicate this difference Bergmann ‡ proposed the terms *homoiothermal* and *poikilothermal* instead of warm-

* Richet, *Dict. de Physiol.*, vol. iii., p. 108.

† Hunter, *Complete Works* (London, 1837), vol. iii., p. 16.

‡ Bergmann, *Göttingen Studien*, 1847, Abth. i., S. 595.

blooded and cold-blooded. Homoiothermal animals were supposed to have a constant temperature in all atmospheres, and poikilothermal animals an inconstant or variable temperature.

But objections might also be raised to this classification, for in no animal is the temperature absolutely constant,—there are diurnal and other variations, amounting in some instances to 3 or 4 degrees Centigrade, within the limits of health; and again, hibernating animals, such as the marmot and hedgehog, are homoiothermal in the summer and poikilothermal during their winter sleep, while all so-called homoiothermal animals are poikilothermal in the new-born condition.

Strictly speaking, therefore, it would appear that no absolute distinction can be drawn between warm-blooded or homoiothermal animals on the one hand, and cold-blooded or poikilothermal animals on the other. Sutherland,* Macleay † and Martin ‡ have shown that the monotremes and marsupials form intermediate groups which can neither be described as homoiothermal nor poikilothermal; and in a paper on the physiological evolution of the warm-blooded animal Vernon § has endeavoured to prove that even the lowest of the so-called cold-blooded animals have some power of regulating their heat-production and body-temperature, and that as one ascends the series this power increases, until the most perfect regulation or homoiothermism is reached in man. In this relation might be mentioned the interesting fact that anatomical structure and zoological classification form no guide to physiological function. From the zoological point of view reptiles and birds are intimately related, both having sprung from a common ancestor, yet in the functional activity of their tissues and in the intensity of their metabolic processes the one is far removed from the other, reptiles being cold-blooded, and birds having a body-temperature some degrees above that of the higher mammals.

While much work has been done on the production of heat and the regulation of temperature in warm-blooded or homoiothermal animals from the time of Black and Lavoisier onwards, this field has been left to a large extent unexplored in the case of poikilothermal animals, and it was with the view of adding something to our present knowledge of this subject that the observations recorded in the following pages were made.

* Sutherland, *Roy. Soc. Victoria Proc.*, 1896, vol. ix., new series.

† Macleay, *Linn. Soc. N.S. Wales Proc.*, vol. ix., 1st series, p. 1204.

‡ Martin, *Phil. Trans. Roy. Soc.*, B., vol. cxcv., pp. 1–37.

§ Vernon, *Science Progress*, vol. vii., 1898, p. 378.

Temperature of Fishes.

While spending a holiday in the Orkney Islands during the month of September of this year, cod-fishing was being carried on by the local boats in the Pentland Firth, and it occurred to me that this would offer an excellent opportunity of making some observations on the body-temperature of the fish caught. I accordingly obtained permission to join a fishing-smack, and proceeded to sea provided with the necessary requirements.

Fishing in the Pentland Firth is done by the "hand-line," since, on account of the rocky character of the sea-bottom, neither the trawl nor "long-lines" can be employed. The rate of the current with spring tides is from 8 to 10 knots (9 to 11½ miles per hour), and with neap tides from 4 to 5 knots, and it is only for about an hour at the turn of the tide that fishing can be engaged in. The flood tide runs south-east for about six hours, the ebb tide north-west for six hours, and for an interval of about fifteen minutes with neap tides and less with spring tides, between the end of the flood and the beginning of the ebb tide, little or no current is detectable. This is known locally as the "slack of the flood," and the corresponding interval between ebb and flood as the "slack of the ebb."

Our usual fishing-ground lay about the middle of the Firth, from one to three miles west of the island of Swona. Most frequently we fished at the flood "slack." The smack was hove to towards the end of the flood tide; she drifted or "carried" slowly south-east, then remained practically stationary from ten to fifteen minutes, and then began to drift in the opposite direction with the ebb, at an ever-increasing rate, the lines being kept out until the current became too rapid. The whole fishing period did not cover much more than one hour. The crew consisted of five men, each working one "hand-line" with a six-pound leaden sinker and two hooks baited with limpets and lug (*Arenicola*). The depth of water was usually a little over 50 fathoms, and, the cod being a ground-fish, the baits were run to the bottom.

My object was to compare the temperature of the fish with that of the water in which they habitually lived, and it was therefore necessary that I should be able to ascertain accurately the temperature of the sea at or near the bottom. For this purpose I employed Negretti and Zambra's patent deep-sea reversing thermometer, mounted on a frame of the Scottish Marine Station pattern. A detailed description of this instrument will be found in the *Challenger Reports* (1876), vol. i., part 1, pp. 88–95, but it has been considerably modified and improved since that date. It consists essentially of two parts—the thermometer itself, and the frame on which it is mounted, devised to invert the thermometer at any given depth, the temperature being registered at the moment of inversion. When the instrument has been sunk to the desired depth, a weight—the "messenger"—ingeniously made of iron or lead in two pieces, so that it can be placed on the line at any point, is allowed to run down, and this, when it comes in contact with the frame, liberates a spring-catch which holds the top-heavy thermometer in position, and allows it to make half a revolution. The bulb of the thermometer is protected from pressure by being enclosed in an outer glass case partially filled with mercury.

In operation, the depth of the water was first ascertained with a deep-sea lead, the line being marked at intervals of 10 fathoms. The line was then hauled in, the lead detached and replaced by the thermometer frame, which was run down to within a fathom of the bottom. The "messenger" was then slipped on to the line and sent down, so as to liberate the spring catch and allow the thermometer to invert itself. On bringing the thermometer up, the temperature was read off and recorded. Readings as just described were taken near the bottom, and at 10-fathom intervals between that and the surface, twice on most occasions during the "slack."

I was surprised to find that the temperature of the sea was practically uniform throughout, and was inclined to doubt the accuracy of my thermometer, since I had expected that the water would be warmer near the surface than deeper down; but on referring to a paper by H. N. Dickson in the Reports of the Scottish Fishery Board for 1893 I came upon the explanation. While taking soundings, temperature and other observations on H.M.S. *Jackal* in the North Sea–Atlantic channel between the Orkney and Shetland Islands, Dickson found that at stations in the North Sea east of this channel, where the water was comparatively still, the temperature fell steadily from the surface downwards, and at a depth of 50 fathoms there was a difference of several degrees (13°·6 F. or 7°·6 C. in one instance); but at stations in the tideway off Sumburgh Head where the current runs at the rate of 7 knots, "the mixing of the water is so complete that the temperature does not vary 0°·1 C. from surface to bottom in a depth of 50 fathoms." For my work this was a very fortunate circumstance, since no heat could be gained or lost by the fish while being brought from the bottom, as might conceivably have been the case if the temperature near the surface had differed by several degrees from that deeper down.

An accurate Centigrade thermometer, graduated in fifths and readable to tenths, was employed for taking the body-temperature of the fish.

This and the deep-sea reversing thermometer were compared by immersing them together in a tank of sea-water, and the readings were found to correspond.

Immediately the fish was brought out of the water on to the deck, and before being bled, the thermometer was inserted through the cloaca into the rectum to the depth of six inches, and allowed to remain there till the mercury became stationary. In many cases the temperature of the blood was taken. This was effected by severing the gills on one side with a sharp knife and immersing the bulb of the thermometer in the pool of blood which quickly collected. It is the routine practice of the fishermen to bleed each cod in this way as soon as it is brought on board, because it improves the appearance of the fish for market. In some cases a small incision was made through the skin, and the bulb of the thermometer pushed into the thick mass of muscle on the back, behind the pectoral fin, to the depth of two inches; in this way the temperature of the muscle was obtained.

The fish examined varied in weight from 5 lbs. to 25 lbs., but the larger specimens were selected by preference. They were brought on board frequently in such quick succession that all could not be examined. The time taken by a fisherman to haul a 15-lb. or 20-lb. cod from a depth of 50 fathoms varied from one and a half to two minutes, and all the while it was struggling violently to get free.

Observations were made as described above on 90 cod-fish (*Gadus morrhua*), 5 lings (*Molva vulgaris*), 1 tusk or torsk (*Brosmius brosme*), and 1 saithe or coal-fish (*Gadus virens*), on five different days. The results are given in tabular form for each day.

September 12, 1907.

Calm; bright sunshine; air in shade, 14°·2 C.; depth of water, 49 fathoms. Flood "slack," 1.30 to 2.30 p.m.; 71 cod caught, 31 examined.

Water.		Fish.		Difference.
Fathoms.	Temperature.	Number.	Rectal Temperature.	
5	11°·4 C.	6	11°·8 C.	0°·6 C.
10	11·3	10	11·7	0·5
20	11·3	8	11·6	0·4
30	11·2	7	11·5	0·3
40	11·2			
48	11·2	31	Mean, 11·65	0·45

September 13.

Wind, S.W.; light breeze; dull; air, 11° C.; depth of water, 53 fathoms. Flood "slack," 2 to 3 p.m.; 58 cod caught, 26 examined.

Water.		Fish.		Difference.
Fathoms.	Temperature.	Number.	Rectal Temperature.	
0	11°·2 C.	5	11°·8 C.	0°·5 C.
5	11·3	12	11·7	0·4
10	11·2	6	11·6	0·3
20	11·1	3	11·5	0·2
30	11·2			
40	11·2	26	Mean, 11·68	0·38
52	11·3			

September 17.

Wind, W.; light breeze; heavy swell (westerly gale yesterday); dull; air, 13°·2 C.; water, 47 fathoms. Ebb "slack," 11.30 a.m. to 12.30 p.m.; 37 fish caught (25 cod, 5 lings, 1 torsk), 21 examined.

Water.		Fish.		Difference.
Fathoms.	Temperature.	Number.	Rectal Temperature.	
0	11°·4 C.	Cod 1	11°·8 C.	0°·5 C.
10	11·4	Cod 4	11·7	0·4
20	11·3	Cod 3	11·6	0·3
30	11·3			
40	11·3	8	Mean, 11·68	0·38
45	11·3			
		Ling 2	11·9	0·6
		Ling 1	11·7	0·4
		3	Mean, 11·83	0·53
		Torsk 1	11·7	0·4
			Temperature of Blood.	
		Cod 4	11°·8 C.	0·5
		Cod 3	11·7	0·4
		7	Mean, 11·76	0·46
		Ling 2	11·9	0·6

September 19.

Wind, W.; strong breeze; sunshine; air in shade, 12°·4 C.; water, 56 fathoms. Flood "slack," 6 to 7 a.m.; 20 fish (cod) caught, 10 examined.

Water.		Fish.				Difference.		
Fathoms.	Temperature.	Number.	Rectum.	Blood.	Muscle.	Rectum.	Blood.	Muscle.
5	11°·1 C.	2	11°·5 C.	° ...	° ...	0°·4 C.	° ...	° ...
10	11·2	1	11·6	11·7 C.	11·8 C.	0·5	0·6 C.	0·7 C.
40	11·2	1	11·6	11·6	11·6	0·5	0·5	0·5
50	11·1	2	11·5	11·5	11·6	0·4	0·4	0·5
55	11·1	1	11·4	11·5	11·5	0·3	0·4	0·4
		1	11·4	11·6	11·5	0·3	0·5	0·4
		1	11·3	11·5	11·5	0·2	0·4	0·4
		1	11·4	11·4	11·5	0·3	0·3	0·4
		10	Mean, 11·47	11·54	11·57	0·37	0·44	0·47

September 25.

Wind, S.E.; light breeze; hazy; air, 13°·4 C.; water, 52 fathoms. Flood "slack," 11 to 12 (noon); 22 fish caught (21 cod and 1 saithe), 9 examined.

Water.		Fish.				Difference.		
Fathoms.	Temperature.	Number.	Rectum.	Blood.	Muscle.	Rectum.	Blood.	Muscle.
1	11°·6 C.	Cod 2	11°·5 C.	° ...	° ...	0°·4 C.	° ...	° ...
5	11·6	Cod 2	11·5	11·5 C.	11·6 C.	0·4	0·4 C.	0·5 C.
20	11·4	Cod 1	11·5	11·4	11·5	0·4	0·3	0·4
30	11·5	Cod 1	11·5	11·6	11·6	0·4	0·5	0·5
40	11·1	Cod 1	11·4	11·5	11·5	0·3	0·4	0·4
50	11·1	Cod 1	11·3	11·4	11·4	0·2	0·3	0·3
		8	Mean, 11·46	11·48	11·53	0·36	0·38	0·43
		Saithe 1	11·8	...	...	0·7	...	...

Difference between Temperature of Fish and of Water at Sea-bottom.

	Rectum.	Blood.	Muscle.
Cod	0°·6 C. in 6 specimens	0°·6 C. in 1 specimen	0°·7 C. in 1 specimen
	0·5 „ 18 „	0·5 „ 7 specimens	0·5 „ 6 specimens
	0·4 „ 34 „	0·4 „ 10 „	0·4 „ 6 „
	0·3 „ 20 „	0·3 „ 3 „	0·3 „ 1 specimen
	0·2 „ 5 „		
Mean,	0·40 „ 83 „	0·43 „ 21 „	0·46 „ 14 specimens
Ling	0·6 „ 2 „	0·6 „ 2 „	
	0·4 „ 1 „		
Mean,	0·53 „ 3 „		
Torsk	0·4 „ 1 „		
Saithe	0·7 „ 1 „		

In fourteen cod, the rectal, blood, and muscle temperatures were all recorded in the same individual, and the mean difference between these figures and the temperature of the water at the time was, for the rectum, 0°·36 C.; for the blood, 0°·41 C.; and for the muscle, 0°·46 C.

When the body-temperature of the fish is compared with that of the water in which they swim near the sea-bottom, it is found that in every case the former exceeds the latter, but only by a few tenths of a degree. The conditions were as favourable as could be imagined for generating heat in the fish, since, for the space of two minutes immediately before the temperature was observed, they were using their powerful muscles to the utmost in struggling to get free. On most occasions the water became slightly warmer as the surface was approached, but the change was so little that it may be neglected; and even if there had been a difference of several degrees, the fish were drawn up so quickly that any interchange of heat between their bodies and the water would have been inappreciable. For the same reason no heat could have been acquired from the air, which was never more than 3° C. above the temperature of the water, in the short time that the thermometer was being applied. It was not practicable to have the fish weighed while alive, but my impression is that the higher figures were obtained from the larger specimens.

On September 26, I had a "long-line" set in Hoxa Sound, a strait about two miles wide between the islands of Flotta and South Ronaldshay, in the Orkney group, and connecting the Pentland Firth with Scapa Flow. The depth of water was from 20 to 23 fathoms, and the sea-bottom was clean (sand or gravel), which permitted of this method of fishing.

Seventeen haddocks (*Gadus æglefinus*), four flounders (*Pleuronectes flesus*), and two dog-fish (*Scyllium catulus*) were caught. They were hauled up slowly, carefully unhooked, and the haddocks and flounders were transferred alive to a tank of sea-water in the boat. One hour afterwards the temperature of the water and of the fish was recorded. The temperature of the dog-fish was taken in the sea alongside the boat before they were removed from the line. The temperature of the air was 13°·2 C., of the sea at the surface, 11°·8 C., and of the sea-water in the tank, 12°·1 C. at the commencement of the observations, and 12°·3 C. at the end. In the dog-fish and haddocks the thermometer was placed in the rectum, and in the œsophagus in the flounders. A thick rubber glove was used to prevent the heat of the hand being communicated to the body of the fish. The haddocks and flounders weighed from ½ to 2 lbs., the dog-fish about 3 lbs.

Water.	Fish.		
11°·8 C.	Dog-fish	2	11°·8 C.
12·1	Flounders	2	12·1
		2	12·2
	Haddocks	3	12·0
12·3		6	12·1
		4	12·2
		4 (dead)	12·1

In the dog-fish the temperature was the same as that of the water at the surface; and as this sound is always traversed by strong currents, it may be taken that the temperature was uniform from the surface to the bottom, although no observations were made at different depths on this particular occasion. This species is not a ground-fish, but may be found swimming at any depth.

The conditions under which the observations were made on the haddocks and flounders were not entirely satisfactory. The water in the tank, having been drawn from the sea just before the fish were transferred to it, was at first 11°·8 C., but gradually became warmer, and it is not probable that the body-temperature of the fish would follow at once the changes in the temperature of the water. Richet * found that in the water tortoise, when the animal was placed in a medium which was being slowly cooled, its temperature at any moment was distinctly above that of the medium, and *vice versa* if the medium was being warmed; that is to say, the temperature of the animal lagged behind that of the medium. This is the only occasion on which I have found the fish colder than the water in

* Richet, *Dict. de Physiol.*, vol. iii., p. 106.

which they were swimming, and it is not likely that this would have been the case if the latter had undergone no variation.

On September 28 twenty sillocks or podleys (young coal-fish, first year) were caught on the fly at Stangar Head, Orkney, and set free in a large quantity of sea-water which had been allowed to flow into the boat. From this they were transferred later to a large basket which was anchored in the still water of Pan Bay, and at the end of that time the body-temperature was recorded. Ten of them escaped accidentally before any observations were made. The bulb of the thermometer was introduced through the mouth into the stomach, and each of the ten fish examined had the same temperature as the water in which they were swimming, viz. 12°·2 C.

On four occasions (October 12 and 26, and November 2 and 9) a "long-line" was set in the Firth of Forth off Musselburgh, in comparatively shallow water, and in all twenty-one flounders (*Pleuronectes flesus*), two young coal-fish (*Gadus virens*), and one smelt (*Osmerus eperlanus*) were obtained. Samples of sea-water were brought up from the bottom in a Buchanan-Richard reversing water-bottle, and the temperature of these and of the water at the surface was taken. The fish, when brought to the surface, were grasped by the lower jaw with a pair of strong pincers provided with rubber-covered handles to prevent heat being conducted from the hand of the observer, and so held under the water while the thermometer was introduced through the mouth into the stomach. The temperature was read with the fish completely immersed in the water. The results are given below.

October 12, 1907.

Water.		Fish.			Difference.
Surface	11°·6 C.	Flounders	2	11°·6 C.	0°·0 C.
Bottom (5 fathoms)	11·6		1	11·7	0·1
			1	11·8	0·2
			4	Mean, 11·68	0·08

October 26.

Water.		Fish.			Difference.
Surface	10°·4 C.	Flounders	7	10°·4 C.	0°·0 C.
Bottom (5½ fathoms)	10·4		2	10·5	0·1
			9	Mean, 10·42	0·02

November 2.

Water.		Fish.				Difference.
Surface	10°0 C.	Flounders	5		10°2 C.	0°0 C.
Bottom (2 fathoms)	10·2		1		10·4	0·2
			6	Mean,	10·23	0·03
		Coal-fish (immature)	1		10·2	0·0
			1		10·3	0·1
			2	Mean,	10·25	0 05

November 9.

Water.		Fish.				Difference.
Surface	9°8 C.	Flounders	1		10°0 C.	0°0 C.
Bottom (5 fathoms)	10·0		1		10·1	0·1
			2	Mean,	10·05	0·05
		Smelt	1		10·0	0·0

Difference between Temperature of Fish and of Water at Sea-bottom.

Flounders.	Coal-fish (immature).	Smelt.
0°0 C. in 15 specimens	0°0 C. in 1 specimen.	0°0 C. in 1 specimen.
0·1 " 4 "	0·1 " 1 "	
0·2 " 2 "		
Mean, 0·04 " 21 "	0·05 " 2 "	

In these fish, which were small (from ¼ to 2 lbs.), and which had been gently and not forcibly drawn to the surface, the temperature is practically the same as that of the water at the sea-bottom in which they swim.

Body-Temperature of the Crustacea and Echinodermata.

Crustaceans and echinoderms were easily procurable around the shores of Pan Bay, Orkney, and of these I examined the following species in considerable number: the shore crab (*Carcinus mœnas*), the edible crab (*Cancer pagurus*), the lobster (*Homarus vulgaris*), the starfish (*Asterias rubens*), and the sea-urchin (*Echinus esculentus*). The shore crabs were captured in lobster traps, the edible crabs and the lobster were obtained alive from the lobster fishermen of the locality, the sea-urchins were fished up from the sea-bottom with a landing-net at low water, and the starfishes were got from a depth of 20 fathoms on the "long-line" set to catch haddocks and flounders. While still alive, the creatures were transferred to a large wicker basket and completely immersed in the sea for a day or two before the temperature observations were made. Abundance of food was put in along with them. Shore crabs are not pugnacious, and do not attack each other when several are confined together; but it is otherwise with the edible crab or "partan," and in these the movable pincer or chela of each of the great claws was spiked with a small wooden wedge to render them harmless.

When observations were to be made, a boat was rowed out to the basket, which was anchored in the bay, and moored alongside of it. This plan was adopted with a view to secure against any alteration in the temperature of the water during the period (½ to 1 hour) occupied in taking the thermometer records, as it was found that the shallow water of the beach or of an enclosed pool did not usually remain at a constant temperature for such a long interval. In the deep water of the bay there was no detectable change from beginning to end. A thick worsted mitten and an india-rubber glove were worn on the left hand, which was used to seize and hold the crabs, so as to prevent the conduction of heat to the body of the animal. The sea-urchins and starfishes were supported on a ledge in the basket, and were not handled at all except for a few seconds while the thermometer was being introduced.

In the crab the flexed abdominal portion was raised and a small hole made with a penknife through the carapace in the groove on the thoracic sterna, and the bulb of the thermometer was introduced through this opening into the cephalothorax. The crab was held by the two great pincer claws under water while the thermometer was in position, the hand not being in contact with its body at all. The single lobster which I examined was a small specimen, and had lost both the great claws before I received it. A small opening was made on the ventral aspect in the articular membrane between the second and third abdominal segments, and the thermometer was pushed through obliquely forwards into the powerful muscles of this region. In the sea-urchin the thermometer was introduced through the mouth in some cases, the projecting teeth having been pushed aside, and through the anal orifice at the opposite pole in other cases. Similarly, in the starfish it was passed through the aperture on the ventral aspect of the central disc, and thence into the substance of one of the five arms or rays. The thermometer was left in position in every case until the mercury became stationary.

As the temperature of the water varied slightly from day to day, the results of each series of observations are given separately in the following tables:—

September 20, 1907, 2 *to* 3 p.m.

Air, 13°·8 C.; water, 13°·2 C.; 24 shore crabs examined.

Number.	Body-temperature.	Difference.
16	13°·2 C.	0°·0 C.
6	13·3	0·1
2	13·4	0·2
24	Mean, 13·24	0·04

September 23, 10 *to* 11 *a.m.*

Air, 13°·6 C.; water, 12°·2 C.; 19 shore crabs examined.

Number.	Temperature.	Difference.
14	12°·2 C.	0°·0 C.
4	12·3	0·1
1	12·4	0·2
19	Mean, 12·23	0·03

September 23, 3 *to* 4 *p.m.*

Air, 13°·9 C.; water, 12°·4 C.; 6 shore crabs, 10 edible crabs, and 1 lobster examined.

Number.		Temperature.	Difference.
Shore crabs	5	12°·4 C.	0°·0 C.
	1	12·5	0·1
	6	Mean, 12·42	0·02
Edible crabs	4	12·4	0·0
	3	12·5	0·1
	2	12·6	0·2
	1	12·7	0·3
	10	Mean, 12·5	0·1
Lobster	1	12·5	0·1

September 24, 9 *to* 10 *a.m.*

Air, 13°·0 C.; water, 11°·9 C.; 23 sea-urchins examined.

Number.	Temperature.	Difference.
19	11°·9 C.	0°·0 C.
4	12·0	0·1
23	Mean, 11·92	0·02

September 24, 4 *to* 5 *p.m.*

Air, 13°·8 C.; water, 12°·0 C.; 18 edible crabs examined.

Number.	Temperature.	Difference.
6	12°·0 C.	0°·0 C.
5	12·1	0·1
4	12·2	0·2
3	12·3	0·3
18	Mean, 12·12	0·12

September 25, 5 *to* 6 *p.m.*

Air, ——; water, 12°·4 C.; 10 shore crabs and 12 edible crabs examined.

Number.		Temperature.	Difference.
Shore crabs	8	12°·4 C.	0°·0 C.
	1	12·5	0·1
	1	12·6	0·2
	10	Mean, 12·43	0·03
Edible crabs	3	12·4	0·0
	3	12·5	0·1
	5	12·6	0·2
	1	12·7	0·3
	12	Mean, 12·53	0·13

September 27, 10 *to* 11 *a.m.*

Air, 12°·6 C. ; water, 12°·0 C. ; 8 starfishes and 6 sea-urchins examined.

Number.	Temperature.	Difference.
Starfishes 8	12°·0 C.	0°·0 C.
Sea-urchins { 4	12·0	0·0
2	12·1	0·1
6	Mean, 12·03	0·03

September 27, 2 *to* 3 *p.m.*

Air, —— ; water, 12°·1 C. ; 16 sea-urchins examined.

Number.	Temperature.	Difference.
11	12°·1 C.	0°·0 C.
3	12·2	0·1
2	12·3	0·2
16	Mean, 12·14	0·04

Difference between Temperature of Animal and of Water.

Shore Crab.	Edible Crab.	Lobster.	Sea-urchin.	Starfish.
0°·0 C. in 43 individuals	0°·0 C. in 13 indiv.	0°·1 C. in 1 indiv.	0°·0 C. in 34 indiv.	0°·0 C. in 8 indiv.
0·1 „ 12 „	0·1 „ 11 „	...	0·1 „ 9 „	...
0·2 „ 4 „	0·2 „ 11 „	...	0·2 „ 2 „	...
	0·3 „ 5 „	...	...	...
Mean, 0·034 „ 59 „	0·12 „ 40 „	...	0·029 „ 45 „	...

It will be seen at a glance, from an inspection of the above analysis, that in each of the species, with the exception of the lobster, both from the crustacea and the echinodermata the temperature of the animal is the same as that of the water in which it has lived, in the great majority of cases. The mean difference, however, is greater in the crustaceans than in the echinoderms; that is to say, a larger proportion of those examined had an appreciable temperature. This is most evident in the edible crab, and here also the highest temperature was recorded, viz. 0°·3 C. This crab is not so active nor so quick in its movements as the smaller shore crab, but its muscles are far more powerful, particularly those acting upon the chelæ of the great claws. Not one of the eight starfishes showed any excess of temperature over that of their environment, and only comparatively few of the sea-urchins.

Results compared with those of Former Observers.

Most of the references to work on the body-temperature of fishes and marine invertebrates which I have been able to obtain have been got from Gavarret's * book, *Physique Médicale*, written in 1855. From this and a few other sources I have collected the following facts, which, for the sake of brevity, I shall present in tabular form :—

Animal.	Excess of Temperature over that of Environment.	Observer.
Pike	3°·88 C.	Kraft (1750).
Carp	1·94	Hunter (1782).
"	0·93	Broussonnet (1785).
"	3·00	Buniva.
"	0·86	Despretz.
Tench	0·71	"
"	0·50	Becquerel.
Eel	0·93	Broussonnet (1785).
"	0·00	Davy (1816).
Shark	1·30	" "
Bonito	10·00	" "
Pelamis sarda	7·22	"
Trout	1·10	" (1816).
"	0·55	Martine (1740).
Bleak (*Alburnus lucidus*)	0·55	" "
Flying fish	0·20	Davy (1816).
Grey gurnet	0·65	Martins (1843).
Crayfish	0·60	Davy (1816).
Crab	0·00	" "
Starfish (*Asterias rubens*)	0·60	Valentin (1839).
Sea-urchin (*Echinus saxatalis*)	0·40	" "
" (" *brevis spinalis*)	0·50	" "

Many of the above figures were obtained from one observation on a single specimen, and in days when the thermometer was not such an accurate instrument as it is at the present time. Gavarret also gives instances to show that the temperature of fish and other poikilothermic animals may sometimes be actually lower than that of the surrounding medium. According to him, most frequently the body of the animal †

* Gavarret, *Physique Médicale*, Masson, Paris, 1855, pp. 125–141.

† It is difficult to believe that the body of a fish can have a lower temperature than the water which surrounds it, if the temperature of the water is stationary and not slowly rising.

is slightly warmer than the medium, in some cases it is of the same temperature, and in other cases it is even colder than the medium.

I have never, on any occasion, found the body of the animal to be colder than the medium, except when the temperature of the water was slowly rising during the time that the records were being taken. This always happened when the examination was made in a pond of sea-water left behind by the receding tide; the heat communicated by the hand to the limited quantity of water was sufficient to sensibly elevate its temperature several tenths of a degree in less than half an hour. The observations made on some crabs under these conditions were subsequently discarded as being inaccurate. In the deep and open water of the bay the temperature did not vary sensibly, and the animal was not once found to have a temperature below that of the water.

The heat-loss from a fish surrounded by a medium of such high convective power as water must be very considerable and very rapid, more especially when it is not provided with any protecting covering, such as a layer of subcutaneous fat; and there might be a large heat-production without any appreciable rise in temperature. As large fish would lose less heat, proportionately to body-weight, than small ones, having relatively a smaller radiating surface, it is in the former that one would expect to find a temperature difference, rather than in the latter, and such I find to be the case.

The members of the tunny family of fishes show a remarkable peculiarity in this respect. John Davy found, in a specimen of the bonito (*Thynnus pelamis*) captured near the equator, that the thermometer, when inserted into the deep muscles in the thickest part of the fish, recorded 37°·2 C., while the water from which it was taken was only 27°·2 C. This is a fish which rarely exceeds 3 feet in length. It is very active and swims near the surface, living chiefly on flying-fish. Its muscles are of a deep red colour, but I have been unable to find out from any description of this fish whether there exists an extensive deposit of subcutaneous fat or any other provision for the prevention of great heat-loss. The tunny of the Mediterranean (*Thynnus vulgaris*) is a larger species, the adult measuring from 7 feet to 10 feet in length. It is also found in the warm regions of the Atlantic, and a few specimens have been captured in British waters. The Maltese fishermen recognise that this is a warm-blooded fish, and one of the most intelligent of them, when questioned as to the degree of heat, described it to Davy as "much the same, or little less than that of the blood of a pig when flowing from the divided vessels of the neck on being killed."

The figures given in the table on p. 81 are, for the most part, considerably higher than any obtained by me. In the smaller specimens examined (haddock, flounder, smelt, and immature coal-fish), there was practically no difference between the temperature of the fish and that of the water. Of 25 flounders, ranging in weight from ¼ to 2 lbs., 1 showed a temperature difference of 0°·2 C., 5 of 0°·1 C., and in the remaining 19 the temperature of the water and of the fish was the same. Even in the large cod-fish and ling, which were examined under the most favourable conditions for the development of heat, viz. immediately following violent muscular exercise, the excess of temperature never reached 1°·0 C., and seldom exceeded 0°·5 C.

Similarly with regard to the crustaceans and echinoderms which have come under my observation: the records obtained are far lower than any of the figures given by other observers. Valentin has arranged the marine invertebrates in the order of their heat-producing capacity, as measured by the temperature difference between the animal and the medium, and has found that this series runs parallel with the zoological series. Although it is highly probable that this arrangement is more or less artificial, my results agree with it in so far that the mean for the echinoderms is below that for the crustaceans. From 53 echinoderms (45 sea-urchins and 8 starfishes) the mean temperature difference was 0°·025 C., while from 100 crustaceans (59 shore-crabs, 40 edible crabs, and 1 lobster) it was 0°·069 C.

After Valentin.

Polypes	0°·21 C.
Medusæ	0·27
Echinoderms	0·40
Molluscs	0·46
Cephalopods	0·57
Crustaceans	0·60

It would be quite erroneous and misleading to assign to any of the species which I have examined a definite excess of temperature over that of the medium in which they live. The body-temperature is inconstant in a twofold sense: it varies with the temperature of the medium, and the difference between it and that of the medium also varies probably from time to time. The attempt at regulation is of the very feeblest, and in most of the species it can hardly be said to exist at all.

SUMMARY.

The body-temperature of the following fishes, crustaceans, and echinoderms has been examined and compared with the temperature of the water in which they live:—Cod-fish (*Gadus morrhua*), ling (*Molva vulgaris*),

torsk (*Brosmius brosme*), coal-fish or saithe (*Gadus virens*), haddock (*Gadus œgelfinus*), flounder (*Pleuronectes flesus*), smelt (*Osmerus eperlanus*), dog-fish (*Scyllium catulus*), shore crab (*Carcinus mœnas*), edible crab (*Cancer pagurus*), lobster (*Homarus vulgaris*), sea-urchin (*Echinus esculentus*), and starfish (*Asterias rubens*). The minimum, maximum, and mean temperature difference for each species are given in the following table:—

Species.	Number.	Minimum.	Maximum.	Mean.
Cod	90	0°·2 C.	0°·7 C.	0°·4 C.
Ling	5	0·4	0·6	0·56
Torsk	1	0·4	0·4	0·4
Coal-fish (adult)	1	0·7	0·7	0·7
„ (immature)	12	0·0	0·1	0·008
Haddock	17	Uncertain		...
Flounder	25	0·0	0·2	0·028
Smelt	1	0·0	0·0	0·0
Dog-fish	2	0·0	0·0	0·0
Shore crab	59	0·0	0·2	0·034
Edible crab	40	0·0	0·3	0·12
Lobster	1	0·1	0·1	0·1
Sea-urchin	45	0·0	0·2	0·029
Starfish	8	0·0	0·0	0·0

The excess of temperature is most evident in the larger specimens. This is well shown in the case of the coal-fish, where in the adult it was 0°·7 C., and in the great majority (11 out of 12) of the young of the first year, 0°·0 C. The body-weight and the conditions under which the fish are captured probably form the most important factors in determining the temperature difference.

In 14 codfish, where the rectal, blood, and muscle temperatures were recorded in the same individual, it was found to be highest in the muscle and lowest in the rectum, the mean temperature difference being 0°·46 C. for the muscle, 0°·41 C. for the blood, and 0°·36 C. for the rectum.

In conclusion, I desire to express my indebtedness to W. S. Bruce, Esq., LL.D., Director of the Scottish Oceanographical Laboratory, and to Alexander Frazer, Esq., M.A., for the use of apparatus; and also to Mr D. Rosie and his sons, on whose vessel most of the observations were made.

(*Issued separately January* 29, 1908.)

VII.—**Classification of Igneous Rocks according to their Chemical Composition.** By **H. Warth,** D.Sc., Tübingen, late Dep.-Superintendent, Geological Survey of India. *Communicated by* Professor James Geikie, F.R.S.

(MS. received June 22, 1907. Read July 15, 1907.)

In my previous paper (see *Geological Magazine*, March 1906, p. 131) I explained a method of separating igneous rocks into sixteen groups of chemically similar individual rocks. A more minute subdivision appears, however, desirable for the special purpose of identifying and describing particular rocks. I have therefore devised the following further classification with four times as many groups, and I trust that the proximity thus attained between the individual rocks of each group will amply meet all requirements.

The method is still founded upon the fact previously pointed out by Harker,* that any large number of igneous rocks, selected at random from those commonly examined and analysed, yields approximately the same average composition. The average composition of 1000 rocks utilised for the present purpose (see page 150) is practically identical with that of 500 rocks used in the former case. The mean composition of the rocks remaining practically the same, we may feel confident that the classification which answers for 1000 rocks will also answer for any additional specimens, and will in fact be generally applicable.

We will now first describe the method of classification. It will be noticed that in the tables of composition the second place of decimals has been omitted, and that not more than seven basic oxides have been recorded. The rarer bases such as MnO, BaO, SrO were included respectively under FeO and CaO. The following quantities of oxides derived from the general average composition are added together,

$$(1) \qquad Al_2O_3 + (Fe_2O_3 + FeO) + MgO + CaO,$$

the result being found $= 33{\cdot}7$. We now separate all the 1000 rocks into two sets, those which yield individually for the same bases a sum smaller than 33·7, and those which yield a sum larger than 33·7. All the former

* A. Harker, *Geological Magazine*, May 1899.

are marked (*a*), the latter (*b*). There happen to be 498 of the former and 502 of the latter.

We now further calculate for each of these two sets (*a*) and (*b*) the average totals of—

(2) $(Fe_2O_3 + FeO) + MgO + CaO + Na_2O$.

The results are respectively 12·4 and 31·5. These figures enable us to subdivide set (*a*) and set (*b*) again into rocks respectively below the new constant 12·4 and below the new constant 31·5. We thus obtain the following four sets:—

(*a*) (*a*) . . .	249 rocks
(*a*) (*b*) . . .	249 rocks
(*b*) (*a*) . . .	294 rocks
(*b*) (*b*) . . .	208 rocks
Total .	1000 rocks

We now repeat the same process successively by means of the following totals:—

(3) $MgO + CaO + Na_2O + K_2O$

(4) $CaO + Na_2O + K_2O + Al_2O_3$

(5) $Na_2O + K_2O + Al_2O_3 + (Fe_2O_3 + FeO)$

(6) $K_2O + Al_2O_3 + (Fe_2O_3 + FeO) + MgO$.

We thus obtain finally 64 groups, for which the formulæ are given on Table III., pages 150, 151, and 152. The respective constants are recorded in the general table, page 90. The whole of the 1000 rocks are arranged in their groups from page 92 to page 149, the final results being given in the above-mentioned table, pages 150, 151, and 152. Any other igneous rock not included in this list of 1000 rocks may be arranged under its proper group by reference to the same constants, page 90. The following examples will show this:—

Rock I.—Felsite porphyry: Knock Mahon, Co. Waterford; recorded in the *Q. Journal of the Geological Society*, year 1900, p. 686, sp. grav. = 2·66.

Rock II.—Alnöite: Naversdile, Orkneys; *N. Jahrb.*, 1902, vol. ii. p. 67.

Rock.	SiO_2	TiO_2	H_2O	Al_2O_3	Fe_2O_3	FeO	MgO	CaO	Na_2O	K_2O
I.	72·3	...	1·8	9·0	6·3	1·1	·0	1·9	5·8	1·5
II.	35·5	2·0	6·0	11·7	5·9	6·3	13·6	15·8	1·9	2·2

From these data we obtain the following totals:—

		Rock I.	Rock II.
(1)	$Al_2O_3 + (Fe_2O_3 + FeO) + MgO + CaO$	18·3	53·3
(2)	$(Fe_2O_3 + FeO) + MgO + CaO + Na_2O$	15·1	43·5
(3)	$MgO + CaO + Na_2O + K_2O$	9·2	33·5
(4)	$CaO + Na_2O + K_2O + Al_2O_3$	18·2	31·6
(5)	$Na_2O + K_2O + Al_2O_3 + (Fe_2O_3 + FeO)$	23·7	28·0
(6)	$K_2O + Al_2O_3 + (Fe_2O_3 + FeO) + MgO$	17·9	39·7

On looking up the table of constants, page 150, we obtain the following formulas:—

Rock I.—*a b a a a a*
Rock II.—*b b b b b a*

We find thus that Rock I. belongs to group 17, and Rock II. to group 63.

Table II. (page 91), which contains the same numbers to be found in Table I. (page 90), is intended to illustrate more clearly the dichotomous evolution of the 64 groups, and will require no further explanation.

The following diagram, pages 154 to 156, is intended to illustrate the relation between the various groups. It is constructed as follows:— Vertical lines are drawn at distances equal according to the scale (page 154) to the percentage of the respective bases which are present in the average of 1000 rocks (see page 152). A line is drawn from a point on the left edge of the first band at an inclination of 45° downwards, and from the point (*d*) in which this line cuts the next vertical a distance is measured upwards equal to the percentage of Al_2O_3 present in the group. Proceeding thence to the intersection (*f*) of the inclined line and the third vertical, we measure upwards a distance equal to $Al_2O_3+Fe_2O_3$, then from the intersection (*g*) a distance equal to $Al_2O_3+Fe_2O_3+FeO$. We thus continue until all the bases are added, and by connecting the upper ends of all the vertical distances we obtain the quasi-curves of the diagram. We notice that a rock with a composition equal to the grand average, if there were such a one, would be represented by a straight horizontal line. In contemplating the 64 curves we are struck by the marked effect of the continued dichotomous division. We have successions of 8 or of 4 groups which are similar to each other. There is also a pronounced similarity between successive odd and successive even groups, and successive pairs of groups are therefore also more or less similar. The first 32 groups are symmetrically opposed to the second 32 groups on respective sides of the straight line which represents the average composition.

As regards the actual composition of individual rocks, an examination of the list will show that the rocks are brought together sufficiently close for all possible requirements. The chief divergence happens to occur in the silica. It amounts repeatedly to 10 per cent., but is to a very large extent compensated for by an increase in the percentage of TiO_2 and T_2O_5 and also H_2O. The latter is no doubt in a large measure replacing silica as a solvent. It will be noticed that within the limit of each group the rocks have been arranged in descending order of their silica percentage. This arrangement brings similar rocks still closer together, and the actually neighbouring rocks are so near in composition that in a great many cases they are practically identical, although from widely distant localities and differently named.

There is also a theoretical test for the proximity of rocks in their groups, as has already been shown in my previous paper. The mean deviation of rocks from a given average may be calculated as follows:—We take, for instance, 500 rocks and write down for each rock how much its silica percentage differs from the average silica whether positive or negative. If we then add up the 500 figures which represent these differences we have only to divide the sum by 500 in order to find the mean deviation of the silica in all the rocks. We do the same for the alumina and afterwards for all the remaining oxides, certain of them being in the present example taken together in pairs Fe_2O_3+FeO, $MgO+CaO$, Na_2O+K_2O. Adding up the deviations of all the constituents, we obtain the total equal to 26·2. After dividing the rocks into 16 groups, I obtained such approximation that the total deviation within the groups amounted to an average of 9·0. Doing the same in our present classification, we find the total deviation within the 64 groups reduced to 7·3. This is a far greater advance than it might at first sight appear to be. In the case of a single item of composition the deviation would diminish in direct proportion to the number of groups, whilst with many items considered together the deviation changes only in proportion to the square root if not the cube root of the number of groups. It was found in a former case possible to reduce the total deviation to about half by means of only 4 groups. In the case of 16 groups the classification was not extended to the individual bases MgO, CaO, Na_2O, and K_2O, but to the sums $MgO+CaO$ and Na_2O+K_2O. In the case of 64 groups these bases were individually treated, as an examination of the rock list will show.

As far as possible the specific gravities of the rocks have also been added and means were calculated. In doing so all the vitreous or partly vitreous rocks were, however, treated separately, the figures for the latter being placed within brackets (see page 151). The specific gravity of the vitreous rocks averaged 2·44 and that of the non-vitreous ones 2·80. Taking all the rocks together, 362 in number, the grand average amounted to 2·76, and we have here again a coincidence. The mean specific gravity of 736 British rocks was found by Harker to be also equal to 2·76 (see *Geological Magazine*, 1899).

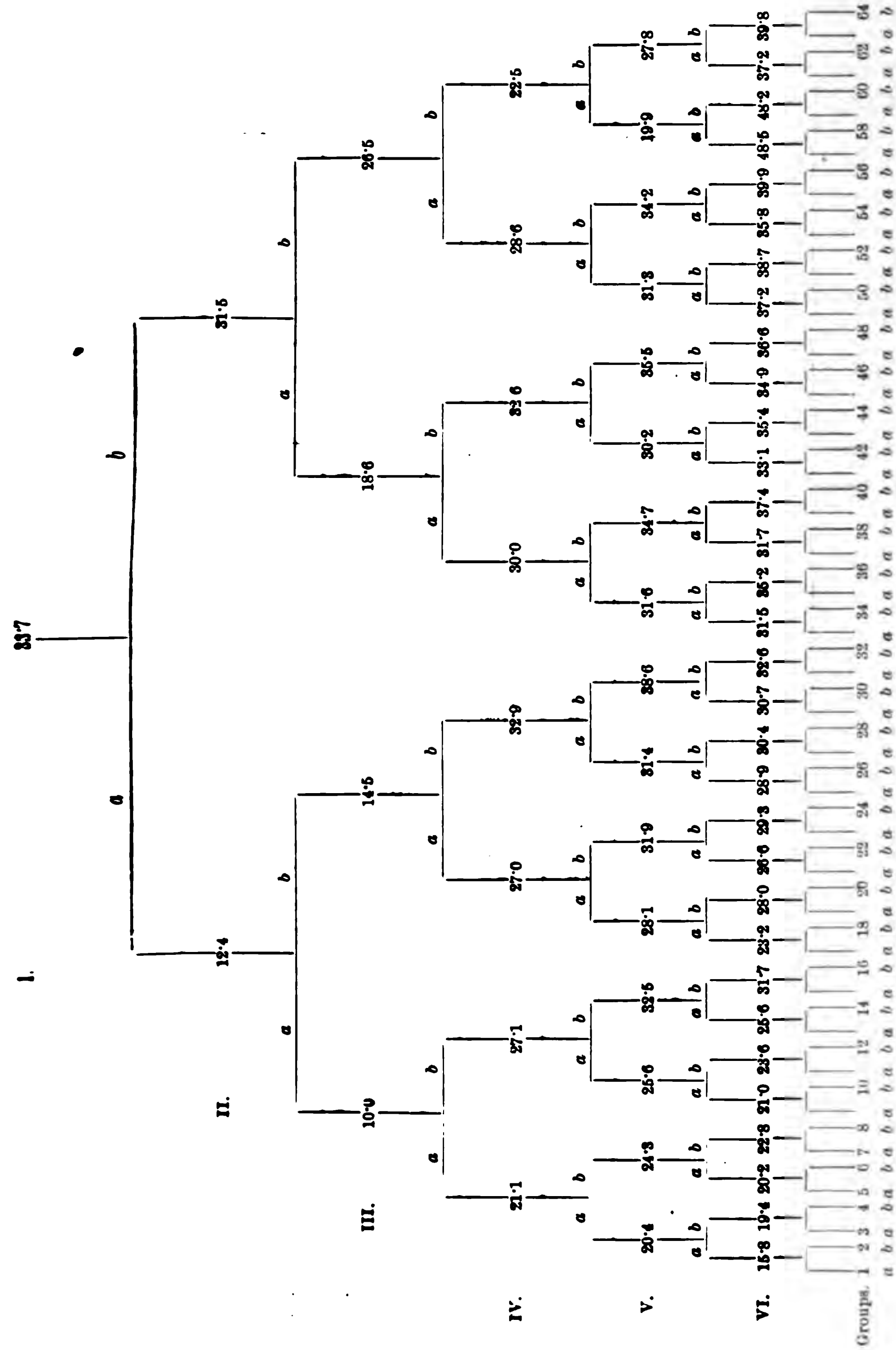
[TABLES.

TABLE I.

(1)	<33·7															
(2)	<12·4								>12·4							
(3)	<10·0				>10·0				<14·5				>14·5			
(4)	<21·1		>21·1		<27·1		>27·1		<27·0		>27·0		<32·9		>32·9	
(5)	< 20·4	> 20·4	< 24·3	> 24·3	< 25·6	> 25·6	< 32·5	> 32·5	< 28·1	> 28·1	< 31·3	> 31·3	< 31·4	> 31·4	< 38·6	> 38·6
(6)	< > 15·8	< > 19·4	< > 20·2	< > 22·8	< > 21·0	< > 23·6	< > 25·6	< > 31·7	< > 23·2	< > 28·0	< > 26·6	< > 29·3	< > 28·9	< > 30·4	< > 30·7	< > 32·6

(1)	>33·7															
(2)	<31·5								>31·5							
(3)	<18·6				>18·6				<26·5				>26·5			
(4)	<30·0		>30·0		<32·6		>32·6		<28·6		>28·6		<22·5		>22·5	
(5)	< 31·6	> 31·6	< 34·7	> 34·7	< 30·2	> 30·2	< 35·5	> 35·5	< 31·3	> 31·3	< 34·2	> 34·2	< 19·9	> 19·9	< 27·8	> 27·8
(6)	< > 31·5	< > 35·2	< > 31·7	< > 37·4	< > 33·1	< > 35·4	< > 34·9	< > 36·6	< > 37·2	< > 38·7	< > 35·8	< > 39·9	< > 48·5	< > 48·2	< > 37·2	< > 39·8
	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61	63

TABLE II.


I. 33·7
II. 12·4 31·5
III. 10·0 14·5 18·6 26·5
IV. 21·1 27·1 27·0 32·9 30·0 32·6 28·6 22·5
V. 20·4 24·3 25·6 32·5 28·1 31·9 31·4 38·6 31·6 34·7 30·2 35·5 31·3 34·2 19·9 27·8
VI. 15·8 19·4 20·2 22·8 21·0 23·6 25·6 31·7 23·2 28·0 26·0 29·3 28·9 30·4 30·7 32·6 31·5 35·2 31·7 37·4 33·1 35·4 34·0 36·6 37·2 38·7 35·8 39·9 48·5 48·2 37·2 39·8
Groups. 1 2 3 4 5 6 7 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 58 60 62 64
a b

LIST OF 1000 ROCK

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.
1	*a a a a a a*												
	85·7			1·1	7·7		·3	·4	·5	4·0	·2	99·9	2·57
	83·3			·8	8·4	·1	·3		·1	·7	5·8	99·5	2·63
	82·5		·2	1·3	8·4	2·5		·3	1·7	2·4	·8	100·0	2·65
	80·6			1·1	9·9	2·1		·8	2·2	·4	2·4	99·5	2·65
	80·5			1·4	8·3	3·4	1·0		1·2	2·1	1·9	99·8	2·63
	80·4			2·0	11·1		1·8	·6	·7	1·8	2·5	100·7	2·68
	79·9			7·4	5·6	3·4			2·5	·3	1·2	100·4	...
	78·7			·6	9·0	2·2	2·0	·4	·3	5·5	·6	99·3	...
	78·0			3·6	11·2	1·3			·9	4·9		99·9	...
	73·9			8·5	12·0	1·1		·3	1·1	1·6	2·3	100·7	(2·30)
10 rocks	80·4			2·8	9·2	1·5	·6	·3	1·1	2·4	1·8	100·0	2·64
	±1·9				±1·9	±1·2		±·7		±1·1		±6·8	(2·30)
2	*a a a a a b*												
	80·6			2·0	11·3	·3	1·4	1·0	·5	·7	3·0	100·7	2·55
	78·7			2·0	10·7	·8	1·0	1·5	·3	2·0	1·9	98·8	2·69
	78·3	·7	·1	·9	10·0	1·8	1·9	·9	1·7	2·7	1·4	100·4	...
	78·1			·8	13·6	1·1		·7	2·0	·2	2·7	99·2	...
	78·0				10·0	2·0	1·6		1·0		6·0	98·6	...
	77·6	·6	·1	1·5	12·7	·7		·2	·1	2·6	4·0	100·1	2·51
	77·3			·9	10·0	2·6	1·4	·5	2·3	2·1	2·4	99·4	2·67
	77·2			1·1	6·5	5·8		·6	1·8	3·0	3·7	99·8	2·64
	76·9			2·0	12·9	1·2		1·0	·7	·7	4·3	99·7	2·49
	76·8			·7	13·7	1·8	·3	·5	2·8	1·6	2·1	100·3	...
	76·1			2·7	14·8	1·5		1·0	·1	·4	3·3	99·7	2·70
	73·6			5·6	10·0	3·7		·2	1·9	2·4	3·3	100·7	...
12 rocks	77·4	·1		1·7	11·3	1·9	·7	·7	1·3	1·5	3·2	99·8	2·61
	±1·0				±1·8	±1·3		±·7		±1·1		(±5·9)	
3	*a a a a b a*												
	78·4			·6	10·8	1·3	·5		·4	2·7	5·5	100·2	2·61
	77·8			·6	13·2	·2	·7			5·1	2·1	99·7	2·63
	77·6	·2		·2	11·9	·6	·9		·3	3·8	5·0	100·5	...
	77·3			1·4	11·2	1·7		·2		3·1	4·6	99·5	...
	77·3	·3		·7	11·6	1·6	·8	·8	·6	5·8	1·0	100·5	...
	77·2			·6	(12·0 –	2·6)		·4		7·6	·2	100·6	not given
	76·9			·7	12·5	·5	1·1	·3	·8	3·2	3·5	99·4	...
	76·9			·8	11·6	·7	2·6	·6	·9	5·6	·7	100·4	...
	76·3	·3		·5	11·6	2·4	·3	·1	·6	5·5	2·8	100·4	...
	76·3	·4		·7	12·1	1·1	·9	·1	·7	2·9	4·5	99·6	...
	76·2			1·1	10·6	1·2	3·4	·9	·9	5·3	·6	100·4	...
	76·1	·1		·9	11·4	2·2		·1	·6	7·3	1·2	99·8	2·61

RANGED UNDER 64 GROUPS.

N. *Jahrb.*, 1901, suppl. 14, p. 38.	Porphyroid.	Harz.
N. *Jahrb.*, 1901, suppl. 14, p. 38.	Porphyroid.	Harz.
N. *Jahrb.*, suppl. 1893, p. 568.	Quartz keratophyre.	Westphalia.
Q. *Journ.*, 1900, p. 679.	Felsite.	Co. Waterford.
Q. *Journ.*, 1900, p. 679.	Felsite.	Co. Waterford.
Q. *Journ.*, 1895, p. 141.	Greisen.	Skiddaw.
Z. *Ges. Nw. Halle*, vi., p. 70, 1887.	Pitch stone.	Persia.
N. *Jahrb.*, 1901, suppl. 14, p. 38.	Porphyroid.	Harz.
Q. *Journ.*, 1893, p. 558.	Felsite.	Arran.
Zeitsch. d. d. geol. Ges., part i., 1867–8, xx., p. 539.	Pitch stone.	Meissen.
Q. *Journ.*, 1883, p. 297.	Felsite tuff.	St Davids.
N. *Jahrb.*, 1901, suppl. 14, p. 38.	Porphyroid.	Harz.
G. H. Williams, 15, *A.R.U.S.G.S.*, p. 670, 1895.	Granite gneiss.	Maryland.
Q. *Journ.*, 1899, p. 763.	Felsite, perlitic.	Co. Waterford.
Prestwich, i., p. 37.	Obsidian.	Mexico.
Q. *Journ.*, 1899, p. 467.	Rhyolite.	Omahu, N. Zealand.
C. Chelius, *Erl. G. Kte. Hesse, I. Bl. Rossdorf*, p. 35, 1886.	Granite.	Hessen.
Q. *Journ.*, 1900, p. 679.	Felsite, pale green.	Co. Waterford.
A. Geikie, p. 135, edit. 1882.	Quartz porphyry.	Meissen, Saxony.
L. Milch, *N.J.B.B.*, xii., p. 180, 1899.	Granite.	Silesia.
O. Mügge, *N.J.B.B.*, viii., p. 667, 1893.	Keratophyre tuff.	Westphalia.
Duparc, Pearce, and Ritter, *Mem. Soc. Phys. Gen.*, xxxiii., 2, p. 115, 1900.	Liparite.	Algeria.
N. *Jahrb.*, 1904, vol. ii., p. 407.	Quartz porphyry.	N. Hampshire.
Q. *Journ.*, 1895, p. 319.	Soda felsite.	Co. Wicklow.
Whitman Cross, Table XIV.	Granite.	Cape Anne, Mass.
G. O. Smith, *Geol. of Fox Is. In. Diss.*, 1896, p. 51.	Aporhyolite.	Maine.
P. J. Holmquist, *Afh. Sver. G. Und.*, No. 181, p. 83, 1899.	Granite.	Sweden.
Hatch, *Petrology*, p. 138.	Rhyolite.	Co. Wicklow.
K. Futterer, *Mt. Bad. G. La.*, ii., p. 41, 1893.	Granite.	Baden.
Q. *Journ.*, 1884, p. 528.	Dimetian.	Pembrokeshire.
F. Bascom, *B.U.S.G.S.*, 150, p. 348, 1898.	Aporhyolite.	Pennsylvania.
N. *Jahrb.*, 1900, vol. ii., p. 221.	Granite porphyry.	Sundswall.
Q. *Journ.*, 1884, p. 528.	Dimetian, two analyses.	St Davids.
N. *Jahrb.*, 1893, vol. ii., p. 100.	Quartz porphyry.	Klinzerberg.

LIST OF 1000 ROC

Group.	SiO₂.	TiO₂.	P₂O₅.	H₂O.	Al₂O₃.	Fe₂O₃.	FeO.	MgO.	CaO.	Na₂O.	K₂O.	Total.	Sp.(
3	76·0	·2		·9	12·3	·9	·9	·2	·2	4·0	4·4	100·0	...
	75·9	·5	·3	·8	12·3	1·1	1·4	·3	·9	3·2	3·4	100·1	...
	75·2			·7	13·8	·6	1·4	·1	·7	3·8	3·3	99·8	...
	74·6			·2	10 0	3·8	1·7	·3	2·4	3·3	3·4	99·9	...
	74·1			4·4	12·4	·3	·2	·4	·3	3·2	5·1	100·3	...
	73·7			·4	11·2	1·3	3·2	·5	2·8	3·7	1·9	99·2	...
	73·4			1·3	12·1	·9	3·4	1·1	2·9	3·9	1·7	100·7	2·7
	72·6			5·2	12·4	·7	1·1		·9	1·7	4·7	99·3	(2·3
	72·3			4·9	11·6	1·4	1·1	·5	1·3	4·1	4·0	101·3	(2·3
21 rocks	75·8	·1		1·3	11·8	1·3	1·2	·3	·9	4·2	3·0	100·1	2·6
	±1·0				±·6	±·9		±·7		±·8		±4·0	(2·3
4	*a a a a b b*												
	76·4			1·5	14·2		1·6		·6	1·8	4·2	100·3	2·5
	76·4		·1	1·0	11·7	·6	·6	·3		1·6	7·0	99·3	2·6
	76·3			2·5	12·6	3·1	·5		·4	1·3	3·5	100·2	...
	75·9			1·0	12·1	2·8	·8	·1	·1	4·0	4·3	101·0	...
	75·7			1·2	12·6	1·7		·7	1·6	1·8	4·5	99·8	...
	75·5			1·1	14·6	1·4	1·0	·3	1·3	1·0	3·5	99·7	...
	75·4				15·5	1·2		1·4			3·8	97·3	...
	75·3				11·4	5·4		·6	·8	1·4	6·1	101·0	2·6
	74·8			·3	13·5	5·0		·7	·9	1·5	3·3	100·0	2·6
	74·8			·6	11·6	3·5	·2	·2	·1	4·3	4·9	100·2	...
	74·7			·9	11·9		2·8	·3	·7	2·9	5·3	99·5	2·6
	73·0			3·3	12·6		2·5	·3	·7	1·7	5·9	100·0	...
	73·0		·1	2·0	13·7	2·0	1·1	2·0	1·5	3·9	1·5	100·8	2·6
	72·8	·2		2·8	12·2	·4	3·1	·2	·1	3·4	4·7	100·1	...
	72·6			3·0	16·2	2·7	·5	1·4	·2		3·4	100·1	...
	72·1			1·7	14·0	2·1	2·4	1·0	2·4	1·1	3·3	100·0	2·5
	71·4			1·4	11·8	5·6	1·4	·6	·2	4·2	4·3	100·9	...
	70·6			6·0	15·0	2·6		·6	1·2	2·4	1·6	100·0	(2·3
18 rocks	74·3			1·7	13·2	2·1	1·1	·6	·7	2·1	4·2	100	2·6
	±1·1				±1·3	±1·1		±·8		±1·7		(±6·0)	(2·3
5	*a a a b a a*												
	77·3			·6	14·6			·4		7·6	·2	100·3	...
	77·1		·1	·4	13·3	·8		·2	1·3	4·4	3·2	100·8	2·6
	76·6				13·5	·5		·1	·7	3·5	5·5	100·4	...
	76·4			·2	15·5			·4	2·1	3·0	2·3	99·9	...
	76·3	·1		·5	12·5	1·5	·3	·1	·2	3·9	4·7	99·6	...
	76·0			·6	11·8	2·0		·3	·5	3·4	5·6	100·1	...
	75·8			·4	12·4	·2	1·3	·3	·8	4·0	4·6	99·8	...
	75·4			·6	11·8	1·8	1·0		·3	3·7	5·4	100·0	2·6
	75·4		·1	1·4	13·6	·2	·4	·1	·4	4·6	4·4	100·7	...

-continued.

N. Jahrb., 1903, vol. i., p. 430.	Rhyolite.	Aroostock area.
Q. Journ., 1899, p. 467.	Rhyolite.	Mt. Sheridan, California.
J. P. Iddings, *M.U.S.G.S.*, xxxii., p. 426, 1899.	Rhyolite.	Yellowstone Park.
N. Jahrb., 1901, vol. ii., p. 244.	Granite rock.	Fox River.
G. W. Card, *Rec. G.S.N.S.W.*, iv., p. 116, 1895.	Trachyte.	N. S. Wales.
N. Jahrb., 1901, vol. ii., p. 244.	Rhyolite gneiss.	Fox River.
Q. Journ., 1883, p. 297.	Felsitic tuff.	St Davids.
Hatch, *Petrology*, p. 129.	Pitch stone.	Arran.
Q. Journ., 1893, p. 545.	Pitch-stone porphyry.	Arran.
Teall's *Br. Petrology*, 1888, p. 348.	Rhyolite.	Tardree, Antrim.
N. Jahrb., 1893, vol. ii., p. 100.	Quartz porphyry.	Alvensleben.
N. Jahrb., 1905, vol. ii., p. 57.	Quartz porphyry.	Bohemia.
N. Jahrb., 1905, vol. i., p. 247.	Liparite, mean of 1 and 2.	Somali.
Q. Journ., 1899, p. 765.	Felsite, nodular.	Co. Waterford.
N. Jahrb., 1901, vol. ii., p. 62.	Leptite.	Suoniemi, Finland.
Prestwich, vol. i., p. 41 (Durocher).	Felsite.	Brittany.
T. H. Holland, *Mem. G.S.I.*, xxviii., p. 142, 1900.	Charnockite.	Mt. Thomas.
(T. H. Holland).	Charnockite.	Madras.
Whitman Cross, Table XIII.	Comendite.	Sardinia.
Roth's *Beiträge z. Petrog.*, 1869, p. 51.	Eurite.	Moena, Tyrol.
Prestwich, i., p. 37.	Rhyolite.	Hungary.
N. Jahrb., 1904, vol. ii., p. 223.	Quartz-diorite porphyrite.	Montenegro.
N. Jahrb., 1903, vol. i., p. 430.	Trachyte (resembl. bostonite).	Aroostock.
Q. Journ., 1883, p. 297.	Felsitic tuff.	St Davids.
L. Ricciardi, *Atti. Ac. Gioen. Catan.*, xviii. (18), 1885.	Porphyry.	Piedmont.
N. Jahrb., 1905, vol. i., p. 247.	Liparite.	Somali Desert.
A. Geikie, p. 140.	Pitch stone, average.	
Hatch, *Petrology*, p. 138.	Soda rhyolite.	Wicklow.
N. Jahrb., 1906, vol. i.	Alkali granite.	Campa, Elba.
N. Jahrb., 1900, vol. i., p. 224.	Granitite.	Kullen.
Q. Journ., 1895, p. 319.	Pumice.	*Challenger* Expedition.
J. E. Spurr, *A.G.*, xxv., p. 231, 1900.	Alaskite.	Alaska.
H. O. Lang, *Nyt. Mag.*, xxx., p. 40, 1886.	Quartz porphyry.	Thüringerwald.
W. Lindgren, *B.U.S.G.S.*, 150, p. 151, 1898.	Rhyolite obsidian.	California.
N. Jahrb., 1904, vol. ii., p. 407.	Quartz porphyry.	N. Hampshire.
N. Jahrb., 1905, vol. ii., p. 234.	Liparite.	John Day Basin.

LIST OF 1000

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.
5	75·2			·8	16·3			·1	·9	3·9	3·8	100·8
	75·1			1·3	12·2	1·6		·1	·9	4·1	4·6	99·9
	74·8	·3		·8	13·8	·4	·3	·1	·2	4·3	4·8	99·8
	74·8	·2		1·6	12·6	1·2		·3	3·6	5·1	·2	99·6
	73·2			2·8	13·4	1·8		·1	1·5	3·2	4·7	100·6
	73·1	·6	·1	1·3	13·5	2·6	·1	·2	1·1	4·0	3·2	99·8
	73·1			·7	13·4	2·6 (Fe₂O₃ + FeO)		1·0	2·3	3·9	1·6	98·6
	72·7			2·9	12·7	1·9		·2	1·6	3·6	4·1	99·7
	72·1			4·8	13·3	1·4		·4	1·1	3·6	3·6	100·2
	72·1			4·2	13·7	·3	·9	·4	1·4	3·2	3·3	99·6
	72·0			2·7	14·9	·3	·2	·3	·8	4·1	4·2	99·5
	71·8			3·1	14·8	1·9 (Fe₂O₃ + FeO)			·7	4·9	2·4	99·6
	71·6			1·1	16·9	·3	·9	·6	2·2	4·7	1·3	99·6
	70·5			6·1	13·4	·4	·9	·5	1·0	4·0	3·5	100·5
23 rocks	74·2	·1		1·7	13·7	1·0	·4	·3	1·1	4·1	3·5	100·0
	± ·9				± 1·0	± ·7		± ·7		± ·9		(± 4·2
6	*a a a b a b*											
	77·6				13·5	1·2			·4	·2	7·1	100·0
	75·6			·5	12·8		1·9	·4	·1	3·0	5·6	99·9
	75·4			·7	13·2	·8	·6	·2	·4	1·1	7·0	99·4
	75·3			1·5	13·6	2·3		·2	1·0	3·0	4·1	101·0
	74·8			1·8	13·9	1·7		·5	1·5	1·5	4·7	100·0
	74·7			1·2	16·2		1·8	·5	·3	1·3	3·6	99·6
	74·1			·3	13·0	2·6		·3	1·0	3·8	4·6	99·9
	74·2	·2		2·0	14·5	1·3	·7	·2	·3	3·0	3·7	100·3
	74·2			1·0	13·2		3·2	·3	1·5	1·9	5·4	100·7
	74·0		·1	1·2	13·1	·7	1·3	·2	·9	3·5	5·0	100·0
	73·6			·6	13·8	·5	2·4	·7	·6	3·2	4·3	99·7
	73·6	·1		2·0	13·4	·6	·8	·3	1·4	3·5	4·5	100·3
	72·9	·5		1·2	13·9	1·9	·9	·5	·7	3·7	3·7	100·0
	72·8			2·4	13·8	1·7		·3	1·2	3·4	4·4	100·0
	72·8			1·1	13·8	3·3		·6	1·9	4·1	3·0	100·6
	72·4			1·1	15·2	1·8	·7	1·5	2·0	2·1	3·5	100·3
	71·6			1·9	12·8	1·6	1·8	1·1	3·1	2·7	3·0	99·7
17 rocks	74·2	·1		1·1	13·7	1·3	1·0	·5	1·1	2·7	4·6	100·1
	± 1·0				± ·6	± ·6		± ·8		± ·8		(± 3·
7	*a a a b b a*											
	74·7			·5	13·0	2·1	·8	·3	·7	3·8	8·8	100·
	74·5			·9	14·9	2·5	·2		·3	3·5	3·7	100·
	73·9	·4	·2		12·3	2·9	1·6		·3	4·7	4·6	100·
	73·7			·6	14·5	·4	1·5		1·1	4·2	4·4	100·

—continued.

Duparc and Maazec, *Mem. Soc. Phys. Gen.*, xxxiii., No. 1, p. 82, 1898.	Aplite.	Mont Blanc.
A. Lagorio, *T.M.P.M.*, viii., p. 454, 1887.	Nevadite.	Colorado.
Weed and Pirsson, *B.U.S.G.S.*, 139, p. 101, 1896.	Quartz tourm. porphyry.	Montana.
J. E. Spurr, *B.U.S.G.S.*, 168, p. 228, 1900.	Tonalite aplite.	Alaska.
Duparc, Pearce, and Ritter, *M. Soc. Ph. Genev.*, xxxiii., p. 77, 1900.	Liparite.	Algeria.
Q. Journ., 1899, p. 467.	Rhyolite.	Waihi, New Zealand.
N. Jahrb., 1901, vol. ii., p. 244.	Metarhyolite.	Fox River.
Duparc, Pearce, and Ritter, *M. Soc. Ph. Genev.*, xxxiii., p. 77, 1900.	Liparite.	Algeria.
Duparc, Pearce, and Ritter, *M. Soc. Ph. Genev.*, xxxiii., p. 77, 1900.	Liparite.	Algeria.
N. Jahrb., 1893, vol. i., p. 285.	Liparite.	Cabo de Gata.
N. Jahrb., 1906, vol. i., p. 222.	Pumice.	Manindjau, Sumatra.
Merrill.	Hornblende granite.	Utah.
Geol. Magazine, 1889, p. 546.	Soda felsite.	Co. Wicklow.
N. Jahrb., 1893, vol. i., p. 285.	Liparite.	Cabo de Gata.
N. Jahrb., vol. i., 1900, p. 27.	Aplite, reduced to 100.	Harzburg.
Q. Journ., 1900, p. 679.	Felsite, dark grey.	Co. Waterford.
Hatch, *Petrology*, p. 138.	Rhyolite.	Wrekin.
Q. Journ., 1893, p. 545.	Quartz felsite.	Arran.
L. Ricciardi, *Att. Ac. Gioen.*, xviii., p. 12, 1885.	Porphyry.	Piedmont.
Q. Journ., 1902, p. 170.	Granite.	Carn Brea, Cornwall.
Prestwich, vol. i., p. 37.	Obsidian.	Fossa Bianca, Lipari.
Q. Journ., 1899, p. 467.	Rhyolite.	Willowlake, California.
Von Sommaruga, *Jahrb. d. k. geol. Reichs.*, 1866, p. 464.	Rhyolite.	Hungary.
Q. Journ., 1906, p. 117.	Granite.	R. Madeira, S. America.
Geol. Magazine, 1889, p. 546.	Pot. soda felsite.	Co. Wicklow.
Q. Journ., 1899, p. 467.	Rhyolite.	Slate Creek, California.
N. Jahrb., 1900, vol. ii., p. 221.	Rapakiwi.	Rodoen Island.
A. Lagorio, *T.M.P.M.*, viii., p. 444, 1887.	Obsidian.	Hungary.
Holland, *Quart. Journ. Geol. Soc.*, p. 435, 1889.	Soda eurite.	Cader Idris.
L. Milch, *N.J.B.B.*, xii., p. 163, 1899.	Granitite.	Silesia.
N. Jahrb., 1904, vol. i., p. 65.	Granite.	Steiermark.
N. Jahrb., 1902, vol. ii., p. 231.	Granulite.	Borry, Moravia.
Q. Journ., 1902, p. 170.	Granite.	Botallak, Cornwall.
Whitman Cross, Table XIII.	Granite.	Blue Hills, Mass.
Q. Journ., 1880, p. 13.	Red granite.	Peterhead.

List of 1000

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.
7	73·5			·4	12·5	3·6			·3	5·6	4·0	100·0
	73·5			·4	14·4	·5	1·5	·3	1·3	4·0	4·3	100·2
	73·5	·3		·8	14·8		1·1	·4	·5	4·4	4·2	100·0
	73·5	·1		1·2	15·3	·8		·8		4·7	3·9	100·2
	73·0			1·3	15·2	1·9		1·0	·6	3·4	4·1	100·5
	72·8	·5		·6	12·8	2·6	1·9	·3	·6	3·2	5·2	100·3
	72·3			1·3	13·3	2·3	2·0	·4	·1	4·1	4·4	100·2
	72·1			1·1	14·8	2·2		·3	1·6	2·8	5·1	100·0
	71·8			1·2	15·9	4·2		·5	3·2	4·1	1·1	101·9
	71·2			1·6	13·8	3·2	·8	·3	1·3	6·3	1·6	100·1
	71·2			1·5	14·4	3·4		·6	1·5	3·1	4·1	99·8
	70·0	1·0		1·3	13·3		4·9	·7	1·8	3·3	3·5	99·8
	69·3			1·2	17·3	2·5		1·2	3·8	4·3	1·7	99·9
	69·0	·5		3·2	16·9	·9			1·0	4·6	3·9	100·0
	68·5			2·6	14·7	3·9		·3	3·9	2·3	3·4	100·6
19 rocks	72·3	·2		1·1	14·5	2·1	·8	·3	1·2	4·0	3·7	100·2
	±1·1				±1·0	±1·1		±·8		±·9		(±4·9)
8	*a a a b b b*											
	73·7				12·4	4·2			·9	2·2	6·6	100·0
	72·2			1·5	14·5	1·8	·9		·9	1·9	6·1	99·8
	72·1			·5	15·8	2·7		1·3	2·0	3·1	3·1	100·6
	71·4			1·7	15·4	·3	2·3	·2	·5	2·8	5·5	100·1
	70·7			1·2	16·2	1·5	·5		·6	·5	8·7	99·8
	70·6			1·6	17·6	1·7			2·0	·8	5·1	99·6
	69·6			·7	17·4	1·0	2·0	·2	1·4	3·5	4·1	99·9
	69·3				17·4	2·3		·3	2·3	3·6	2·8	98·0
	69·2		·2	·6	17·5	1·1	3·1	1·0	2·9	2·7	3·0	101·3
	68·6			3·4	16·4	·7	1·5	·1	1·8	2·8	4·7	100·0
	68·4	1·6	·3	1·6	13·8	2·6	2·7	·7	·7	3·6	4·5	100·5
	68·4			1·9	16·9	3·0			1·5	4·3	4·0	99·9
	66·5			·9	18·9	3·6	1·3	·4	·2	4·8	4·5	100·1
	65·3			1·2	20·2	5·6	1·2	·2	2·1	2·5	1·2	99·4
14 rocks	69·7	·1		1·2	16·5	2·1	1·3	·3	1·4	2·8	4·6	99·9
	±1·7				±1·5	±1·2		±·8		±1·3		±6·5
9	*a a b a a a*											
	76·3			·3	14·2				1·2	5·0	3·9	100·9
	76·0			·3	13·3		1·7	·3	·8	5·3	3·8	101·5
	75·8			·4	13·2		1·1	·4	·9	3·7	5·4	100·9
	75·6			·7	12·9					3·4	7·1	99·7

—*continued.*

G. T. Prior, *Min. Mag.* xii., p. 262, 1900.	Grorudite.	Abyssinia.
Q. Journ., 1902, p. 666.	Rhyolite obsidian.	Medicine Lake.
Q. Journ., 1902, p. 666.	Granular felsite.	Co. Wicklow.
N. Jahrb., 1905, vol. i., p. 437.	Quartz costonite.	Madagascar.
C. Barrois, *Guide*, Exc. viii., *Cong. G.*, vii., p. 21, 1900.	Aplite.	Brittany.
N. Jahrb., 1900, vol. ii., p. 221.	Felsite porphyry.	Storholmen.
N. Jahrb., 1905, vol. i., p. 247.	Liparite, mean of 6 and 7.	Somali Desert.
A. Geikie, p. 157, third edit.	Granite, mean of twelve localities.	
N. Jahrb., 1895, vol. ii., p. 82.	Porphyry.	Leinster.
N. Jahrb., 1901, suppl. 14, p. 39.	Quartz keratophyre.	Harz.
Q. Journ., 1856, p. 199.	Granite.	Newry, Co. Down.
Roth's *Beiträge z. Petrog.*, 1869, p. xlvi.	Granite.	Syene, Egypt.
U. S. Grant, 21, *A.R.G.Nh.S. Minn.*, p. 43, 1893.	Granite.	Minnesota.
Q. Journ., 1901, p. 36.	Rhyolitic rock, tuffaceous.	Westmoreland.
Duparc, Pearce, and Ritter, *M. Soc. Ph. Genev.* xxxiii., No. 2, p. 59, 1900.	Dacite.	Algeria.
N. Jahrb., 1900, vol. i., p. 27.	Granite, reduced to 100.	Bairischer Wald.
Q. Journ., 1877, p. 457.	Rhyolite (perlitic).	Wrekin.
N. Jahrb., 1901, vol. ii., p. 233.	Granite.	Quérigut, Pyrenees.
Q. Journ., 1875, p. 335.	Eurite (fine-grained Elvan).	Cornwall.
Q. Journ., 1902, p. 170.	Granite.	Chywoon, Cornwall.
K. Muraközy, *F.K.*, xxii., p. 54, 1892.	Rhyolite.	Hungary.
Q. Journ., 1880, p. 8.	Granite.	Lamorna, Cornwall.
Merrill, p. 132.	Granite.	N. Carolina.
N. Jahrb., 1905, vol. ii., p. 382.	Granitite.	Brix.
W. Cross, *B.U.S.G.S.*, 148, p. 179, 1897.	Rhyolite.	Colorado.
E. Haworth, *A.R.Mo.G.S.*, viii., p. 181, 1895.	Granite.	Minnesota.
C. W. Schmidt, *Z.D.G.G.*, xxxvii., p. 744, 1885.	Liparite.	Iceland.
N. Jahrb., 1905, vol. i., p. 247.	Liparite.	Somali Desert.
N. Jahrb., 1893, vol. i., p. 285.	Andesite.	Cabo de Gata.
N. Jahrb., 1906, vol. i., p. 375.	Granite.	Campo, Elba.
Q. Journ., 1902, p. 666.	Rhyolite.	Euganean Hills, Italy.
Duparc and Mrazec, *M. Soc. Phys. Gen.*, xxxiii., No. 1, p. 107, 1898.	Quartz porphyry.	Mont Blanc.
N. Jahrb., 1901, vol. ii., p. 59.	Pegmatite.	Silfpickarehald.

LIST OF 1000]

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.
9	75·5			·9	11·2	·4	·1	·3	·8	3·0	8·0	100·1
	75·4			·4	7·7	1·4	·1	1·3	1·6	8·1	4·5	100·6
	75·2			·6	11·1		1·8	1·1	1·6	4·0	4·5	100·0
	74·3	·4		·2	14·7	·8			·9	4·6	4·5	100·0
	73·8			2·9	10·8	1·8		1·3	1·2	4·3	3·9	100·0
	72·9	·5		1·2	12·9	·7	1·1	·8	·8	3·7	5·0	99·6
	72·7			1·1	13·8	1·0		·7	2·1	4·9	4·3	100·5
	71·5			1·7	13·5	1·2	·9	1·4	3·2	2·6	4·0	100·0
	71·3			·1	15·7	1·5	·3	·9	3·4	4·8	2·3	100·3
	70·5			1·2	15·9	1·0		1·4	1·6	4·9	2·7	99·2
14 rocks	74·1	·1		·9	12·9	·7	·5	·7	1·4	4·4	4·6	100·2
	± 1·4				± 1·6	± ·6		± 1·2		± 1·1		± 5·9
10	*a a b a a b*											
	73·3			·4	13·0	1·0	1·8		1·0	3·5	5·6	99·6
	73·0		·1	2·0	13·7	2·0	1·1	2·0	1·5	3·9	1·5	100·8
	72·5			·7	11·5	2·1		2·7	1·8	3·4	5·2	99·1
	72·2			1·1	13·6	1·3	1·5	1·5	2·2	2·7	3·6	99·7
	71·8			·6	16·0	1·8		1·8	1·7	3·5	4·2	101·4
	70·8	·4	·1	·6	15·1	1·0	1·4	·7	3·2	3·1	3·6	100·1
	70·4			1·4	13·9		3·4	2·5	3·4	1·5	3·6	100·1
	70·1	·7	·1	·8	15·1	1·7	1·2	1·2	2·6	3·6	2·8	100·0
	69·8			3·2	14·2	·1	2·8	·6	1·7	3·3	4·5	100·2
	69·4		·1	1·2	14·3		3·6	2·4	3·2	2·7	2·7	99·6
	69·4	·2		·3	12·8	1·1	2·6	·7	4·7	3·1	5·2	100·2
	68·9			1·6	14·7	·9	2·2	1·6	3·0	3·7	2·8	99·4
	68·5			2·3	14·7	·4	3·3	1·4	2·8	2·9	4·1	100·5
13 rocks	70·8	·1		1·2	14·0	1·0	2·0	1·5	2·5	3·1	3·8	100·0
	± 1·2				± ·9	± ·5		± 1·0		± 1·1		± 4·7
11	*a a b a b a*											
	72·9			·6	12·9	3·0		·4	·9	7·1	3·0	100·8
	72·7			·5	12·8	2·6	1·5	·1	·6	6·5	3·9	101·2
	72·6			·7	14·8		·9	·2	1·2	4·9	5·3	100·6
	72·3			·4	13·4	1·1	1·6		·7	4·3	5·6	99·4
	72·2	·6		·7	14·8	1·0	·9	·3	·7	4·2	5·2	100·6
	72·1			·3	14·9	1·3		1·3		4·1	5·9	99·8
	71·6			·7	13·6	2·4		·2	2·3	5·6	3·5	99·9
	71·4			1·5	14·8	1·7	·7	·6	·1	4·8	5·2	100·6
	71·3	·3		·8	13·9	1·3	1·2	·5	1·0	3·3	6·3	99·8
	71·1	·3	·1	·3	15·9	·6	1·6	·5	2·7	3·5	4·1	100·6
	71·1			1·0	14·6	1·7		·2	1·5	3·3	6·0	99·3
	71·0			·6	13·8	3·7		·6	1·1	5·2	4·0	100·0

—*continued.*

O. Nordenskjöld, *B.G. Inst. Un. Upsala*, i., p. 216, 1894.	Rhyolite (Hälleflinta).	Sweden.
G. F. Becker, M. xiii., *U.S.G.S.*, p. 159, 1888.	Rhyolite obsidian.	Borak Lake.
Q. Journ., 1895, p. 141.	Skiddaw granite.	Whitegill.
N. Jahrb., 1904, vol. ii., p. 403.	Aplite.	Atlanta, Georgia.
Watt's *Dictionary of Chem.*	Pumice.	Cotopaxi.
Weed and Pirsson, *B. U.S.G.S.*, 139, p. 96, 1896.	Aplitic granite.	Montana.
A. Lagorio, *T.M.P.M.* viii., p. 444, 1887.	Obsidian.	Ecuador.
N. Jahrb., 1899, suppl. 12, p. 162.	Granitite.	Querseifen.
P. Marshall, *Tr. N.Z. Inst.*, xxvi., p. 379, 1894.	Tridymite trachyte.	N. Zealand.
A. Lagorio, *Guide*, Exc. vii., *Cong. G. Int.*, xxxiii., p. 27, 1897.	Keratophyre.	Russia.
N. Jahrb., 1904, vol. ii., p. 407.	Quartz porphyry.	N. Hampshire.
N. Jahrb., 1904, vol. ii., p. 223.	Quartz diorite porphyrite.	Montenegro.
Q. Journ., 1893, p. 558.	Porphyritic felsite.	Arran.
N. Jahrb., 1899, suppl. 12, p. 232.	Granitite.	Riesengebirge.
C. Barrois, *Guide*, Exc. viii., *Cong. G. Int.*, vii., p. 21, 1900.	Granite.	Bretagne, France.
H. W. Turner, 17, *A.R. U.S.G.S.*, i., p. 702, 1896.	Biotite granite.	Amador county.
N. Jahrb., 1901, suppl. 14, p. 38.	Porphyroid.	Harz.
J. B. Harrison, priv. contrib.	Granitite gneiss.	Br. Guiana.
N. O. Holst, *Ahr. Sver. G. Und.*, No. 110, p. 37, 1890.	Rhyolite.	Sweden.
Merrill, p. 207.	Granite.	Columbia.
Q. Journ., 1906 (Dr Evans), p. 119.	Granite.	Madeira, S. America.
N. Jahrb., 1904, vol. i., p. 49.	Porphyry (dark).	Bohulibi.
N. O. Holst, *Abh. Sver. G. Und.*, No. 110, p. 37, 1890.	Rhyolite.	Sweden.
N. Jahrb., 1902, vol. ii., p. 237.	Quartz keratophyre.	Massachusetts.
A. Renard, *Challenger Reps. Pet. Oc. Islands*, p. 52, 1889.	Obsidian.	Ascension Island.
N. Jahrb., 1904, vol. ii., p. 403.	Granite.	Atlanta.
N. Jahrb., 1904, vol. ii., p. 406.	Quartz porphyry.	N. Hampshire.
N. Jahrb., 1900, vol. ii., p. 221.	Quartz porphyry.	Storflasen.
Q. Journ., 1895, p. 178.	Pot. soda felsite.	Carn Gelli, Fishguard.
Q. Journ., 1895, p. 129.	Granophyre.	Carrockfell.
H. S. Washington, *J.G.*, vii., p. 293, 1899.	Keratophyre.	Massachusetts.
N. Jahrb., vol. ii., p. 221.	Rapakiwi porphyry.	Rödön.
N. Jahrb., 1901, vol. i., p. 242.	Biotite granite.	Yosemite Valley.
Q. Journ., 1895, p. 178.	Liparite.	Island of Ponza.
A. Geikie, p. 141.	Obsidian.	Average.

LIST OF 1000]

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	Cao.	Na_2O.	K_2O.	Total.
11	71·0	·5		1·7	15·4	·7	·7	·2	·4	3·2	6·2	100·0
	70·9			1·5	16·4		·4	·6	·3	4·5	5·5	100·1
	70·7			·6	15·2	3·8		·4	3·3	4·7	2·3	101·0
	70·5			·5	16·0	·8	1·8	·8	2·6	3·8	3·6	100·3
	70·4	·2	·1	·5	15·5	1·0	1·3	·4	3·0	2·8	5·1	100·3
	70·2		·1	2·2	15·0	2·0	·2	·4	·3	5·0	5·0	100·4
	70·1	·5	·1	·3	15·2	1·8	1·1	·7	2·4	5·2	2·6	100·0
	69·7	·3		2·3	13·0	2·3	1·9	·2	1·2	3·1	5·8	99·8
	69·4			·8	15·2	1·7	·6	·9	2·0	5·1	4·5	100·3
	69·1			2·4	15·1	3·1		1·1	2·1	3·8	4·2	100·9
	69·0	·8			16·1	2·6	1·4	1·1	3·2	4·0	1·8	100·0
	69·0	·3		1·3	16·1	2·5	1·0	·4	1·3	5·4	3·3	100·5
	68·9			1·0 ?	15·5	2·5	·6	1·0	2·1	4·7	3·7	100·0
	68·2	·5	·1	·5	15·6	2·3	·9	1·0	2·8	5·2	2·5	99·7
	67·9	·2	·1	1·3	17·3	2·4	·3	·7	3·0	5·1	1·7	100·1
	65·0	·5	·1	4·3	13·7	·4	2·2	·8	4·4	3·7	4·8	100·1
	64·6	·3	·1	5·6	13·6	1·2	1·2	·7	5·1	3·5	4·1	100·0
29 rocks	70·2	·2		1·2	14·8	1·7	·9	·6	1·8	4·5	4·3	100·2
	±1·1				±·9	±·7		±1·3		±1·0		±5·0
12	*a a b a b b*											
	71·2			1·1	13·4	1·4	1·3	·5	·3	2·0	9·8	100·9
	71·0			1·5	14·2	·8	·7	1·1		·7	9·6	99·6
	70·8	·2		·9	15·8	1·2	1·3		·3	3·4	6·7	100·6
	70·4			·7	16·2	2·5		·9	2·1	3·5	4·3	100·3
	70·0	·2	·1	1·7	15·1	·4	1·1	·6	1·6	2·7	6·4	100·1
	69·9		·2	·6	15·7	4·7		·9	1·8	4·3	3·2	101·3
	69·6		1·6		13·7	2·0	2·4	·3	1·6	3·6	5·2	100·0
	69·5	·3	·1	1·0	15·5	1·7	1·3	·4	1·3	4·5	4·7	100·3
	68·6			·8?	16·2	2·3	·5	1·0	2·4	4·1	4·1	100·0
	67·9			3·7	15·7	3·0		1·5	1·4	1·5	5·6	100·3
	67·4			·5	15·6	3·1		·6	1·9	2·5	7·1	98·7
	67·4	·6	·1	·8	15·5	1·8	2·2	1·1	3·6	3·5	3·7	100·2
	67·3		·3	2·4	15·8	1·9	2·2	·7	2·4	3·8	3·6	100·1
	67·1	·5	·2	·7	15·0	1·6	2·3	1·7	3·5	2·8	4·5	99·9
	67·0			1·0	16·1	4·7	1·2	1·2	3·1	2·0	4·6	100·9
	66·7			3·1	16·7	2·1	·9	1·2	1·4	2·5	5·2	99·7
	66·6			2·8	15·1	·7	3·1	1·4	1·5	2·0	6·7	99·9
	66·3			1·2	15·3	5·4		2·6	·9	2·8	4·6	99·1
	65·7	·7	·3	·6	15·3	2·5	1·6	1·6	2·7	3·6	4·6	99·5
	62·2	·5	·2	5·9	14·7	3·8	·4	1·9	2·9	2·8	5·0	100·4
20 rocks	68·1	·2	·2	1·6	15·3	2·4	1·1	1·1	1·8	2·9	5·5	100·1
	±1·4				±·6	±·9		±1·0		±1·0		±4·9
13	*a a b b a a*											
	71·6				12·8	·1			1·5	3·8	9·7	99·7
	70·3			·7	16·3	·8		·2	3·7	5·9	4·2	99·1

—*continued.*

Q. Journ., 1901, p. 36.	Rhyolitic rock, tuffaceous.	Westmoreland.
Zeitsch. d. d. geol. Gesell., pt. i., 1867-68, xx., p. 328.	Quartz porphyry.	Toscana.
Hatch, *Petrology*, p. 113 (Sollas).	Granite.	Aughreni.
C. R. Keyes, 15, *A.R.U.S.G.S.*, p. 697, 1895.	Biotite granite.	Maryland.
H. W. Turner, *J.G.*, vii., p. 143, 1889.	Biotite granite.	Amador Co.
J. H. Sears, *B.M.C.Z.*, xvi., p. 170, 1890.	Keratophyre.	Massachusetts.
N. Jahrb., 1905, vol. i., p. 254.	Hyp. dacite.	Yellowstone Park.
N. Jahrb., 1900, vol. ii., p. 221.	Felsite porphyry.	Gorgviker.
L. Milch, *Z.D.G.G.*, *L.I.*, p. 69, 1899.	Quartz trachyte andesite.	Sumatra.
N. Jahrb., 1901, supp. 14, p, 38.	Porphyroid.	Harz.
Judd. Roy. Soc., Krakatoa report, pt. i., p. 32.	Hyperst. aug. andesite.	Krakatoa.
N. Jahrb., 1893, vol. ii., p. 100.	Quartz porphyry.	Mühlental.
Q. Journ., 1891, p. 278.	Granite.	Shapfell.
N. Jahrb., 1905, vol. i., p. 254.	Hyp. dacite.	Yellowstone Park.
J. S. Diller, *B. U.S.G.S.*, 148, p. 195, 1897.	Hornbl. andesite.	California.
N. Jahrb., 1892, vol. ii., p. 413.	Quartz porphyry.	Saar Nahe.
N. Jahrb., 1892, vol. ii., p. 413.	Quartz porphyry.	Münster a/St.

N. Jahrb., 1893, vol. i., p. 285.	Liparite.	Cabo de Gata.
Q. Journ., 1900, p. 679.	Felsite (brown, compact).	Co. Waterford.
H. Loretz, *Jb. Pr. G.L.A.*, ix. p. 295, 1889.	Quartz porphyry.	Baden.
Duparc, Pearce, and Ritter, *M. Soc. Phys. Gen.*, xxxiii., No. 2, p. 18, 1900.	Tourmaline granite.	Algeria.
W. H. Weed, *B. U.S.G.S.*, 168, p. 119, 1900.	Quartz porphyry.	Montana.
N. Jahrb., 1906, vol. i.	Normal granite.	Campo, Elba.
Q. Journ., 1906, p. 107.	Quartz felsite.	S. America.
Darton and Keith, *A.G.S.*, vi., p. 307, 1898.	Felsophyre.	Virginia.
Q. Journ., 1891, p. 276 (Cohen).	Granitite.	Shapfell.
Q. Journ., 1901, p. lxxxiii.	Quartz felsite (Waller's analysis).	Cheviots.
N. Jahrb., 1901, ii., p. 62.	Orthoclase porphyry.	Teiske, Finland.
N. Jahrb, 1901, vol. ii., p. 240.	Granodiorite.	Silverlake, Eldorado.
W. Cross, 14, *A.R. U.S.G.S.*, p. 227, 1894.	Quartz porphyrite.	Colorado.
W. H. Weed, *J.G.*, vii., p. 739, 1809.	Granite.	Montana.
N. Jahrb., 1904, vol. i., p. 65.	Granite.	Steiermark.
N. Jahrb., 1892, vol. i., p. 326.	Porphyry, schistose.	Michigan.
A. Schmidt, *cf. N.J.*, 1889, i., p. 95.	Porphyry.	Münstertal.
N. Jahrb., 1895, vol. ii., p. 100.	Kersantite.	Balme.
W. Cross, Telluride folio U.S., *U.S.G.S.*, p. 6, 1899.	Quartz monzonite.	Colorado.
N. Jahrb., 1892, vol. ii., p. 413.	Quartz porphyry.	Kreutznach.

H. Berghell, *Finl. G. Und. Bl.*, 23, p. 18, 1892.	Microcline granite.	Finland.
N. Jahrb., 1904, vol. i., p. 70.	Granulite adds too high.	Aiguilles Rouges.

LIST OF 1000 I

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.
13	68·7	·3		·6	17·1	·9	·4	·4	2·3	7·0	3·8	101·5
	68·6	·4		(1·9)	16·5	·9	1·1	·2	4·2	1·9	5·6	101·8
	68·2			1·4	16·9	1·6		1·1	2·2	5·3	3·0	99·8
	68·0			·7	17·5	·4		·5		5·8	7·1	99·9
	67·5			·5	17·8	2·5	·1	·4	1·7	5·0	4·4	99·9
	66·3		·2	1·3	17·6	2·2	·6	1·0	3·1	5·2	2·5	100·0
	65·9	·4		1·4	16·8	1·6	1·2	1·5	2·7	4·7	3·2	99·4
	65·6			·9	17·2	·2	3·2	1·4	3·4	3·8	3·1	98·8
	65·5			1·5	19·9	1·6		1·4	2·0	5·8	2·4	100·2
11 rocks	67·8	·1		1·0	16·9	1·2	·6	·7	2·4	4·9	4·5	100·0
	±1·5				±1·0	±·9		±1·1		±1·8		±6·3
14	*a a b b a b*											
	69·2			·8	15·6	1·1	1·3	·1	1·3	1·7	8·9	99·9
	66·7			·8	13·6	3·4		·7	1·0	5·8	9·0	101·0
	66·3				18·8	1·4		·5	1·2	3·0	8·8	100·0
	65·8			1·9	15·9	5·1			1·7	3·5	6·2	100·0
	65·7			1·1	17·9		4·3	·5	1·3	5·7	3·5	100·0
	65·4			1·8	17·1	1·7	1·1	·4	2·5	4·8	6·9	101·7
	65·3	·4		2·1	15·9	1·2	2·2	1·5	3·0	2·8	5·7	101·0
	64·0				20·4	2·8		1·3	4·5	3·3	3·5	99·6
	63·8	·7	·2	·4	17·4	·1	1·5	·9	1·7	6·7	6·0	99·3
	62·7			1·7	17·3	·5	2·3	1·4	3·2	4·5	6·3	99·9
	61·1			3·5	18·6	2·6		1·1	2·9	3·2	6·8	99·8
11 rocks	65·1	·1		1·3	17·1	1·7	1·3	·8	2·2	4·1	6·5	100·2
	±1·5				±1·4	±·8		±1·3		±1·6		±6·6
15	*a a b b b a*											
	64·5				19·8	2·2	·2	1·0	·6	4·1	7·6	100·0
	61·9	·4		1·4	18·2	·3	2·9	1·0	1·7	5·5	6·2	99·5
	60·4			·6	22·6	·4	2·3	·1	·3	8·5	4·8	100·0
	60·1	1·2	·1	1·6	20·0	2·4	1·3	·8	·9	6·3	6·0	100·7
	59·4		·5	(·5)	18·4	4·3	·4	·6	2·2	4·7	6·7	98·3
	58·3			5·9	20·9	4·2		1·1	2·2	4·1	3·9	100·5
	58·2			1·7	23·2	1·5			2·4	6·9	6·6	100·5
	57·6	·2		3·2	17·5	3·5	1·2	·2	1·4	5·8	9·2	99·9
	57·4	·4		1·7	23·1	1·9		·1	1·7	8·1	5·7	100·2
	55·7			4·3	20·6	3·1		·4	1·4	7·1	5·6	98·3
10 rocks	59·5	·2	·1	2·1	20·5	2·4	·8	·5	1·5	6·1	6·2	99·9
	±1·6				±1·7	±·9		±·6		±1·3		±6·1

continued.

N. *Jahrb.*, 1904, vol. ii., p. 400.	Red granite.	Osteroe, Norway.
N. *Jahrb.*, 1900, vol. ii., p. 221.	Quartz porphyry.	Storflasen.
). Nordenskjold, *B.G. Inst. Un. Ups.*, i., p. 177, 1894.	Quartz syenite porphyry.	Sweden.
). *Journ.*, 1895, p. 177.	Pot. soda felsite.	Goodwick, Pembrokesh.
W. Cross, 17, *A.R. U.S.G.S.*, ii., p. 324, 1896.	Mica dacite.	Colorado.
I. S. Diller, *B.U.S.G.S.*, 148, p. 191, 1897.	Dacite porphyry.	California.
Weed and Pirsson, *B.U.S.G.S.*, 139, p. 106, 1896.	Porphyry.	Montana.
N. *Jahrb.*, 1902, vol. ii., p. 390.	Hyaloplasm dacite.	Chrety, Russia.
N. *Jahrb.*, 1893, vol. ii., p. 90.	Granitite.	Saxony.
B. Frosterus, *T.M.P.M.*, xiii., p. 188, 1892.	Granite.	Finland.
N. *Jahrb.*, 1905, vol. ii., p. 219.	Syenite.	Monzoni.
N. *Jahrb.*, 1900, vol. i., p. 27.	Leucite granite porphyry (red to 100).	Tocos de Caldes.
R. Zuber, *Sb. Wien. G.R.A.*, xxxv., p. 750, 1885.	Quartz porphyry.	Galicia.
Roth, *Zeitsch. d. d. geol. Gesell.*, 1866, p. 623.	Trachyte.	Ischia.
N. *Jahrb.*, 1905, vol. ii., p. 219.	Syenite.	Monzoni.
J. F. Williams, *N.J.B.B.*, v., p. 411, 1887.	Trachyte.	Tuscany.
N. *Jahrb.*, 1900, vol. i., p. 224.	Banatite.	Kullen.
N. *Jahrb.*, 1901, vol. ii., p. 239.	Pyroxene syenite.	Kuusamo, Finland.
A. Verri, *B.S.G.Ital.*, viii., p. 403, 1889.	Trachyte.	Viterbo.
A. Geikie, p. 165, third edit.	Porphyry.	Pieve di Cadore, It.
N. *Jahrb.*, 1903.	Trachyte.	Celebes.
N. *Jahrb.*, 1905, vol. i., p. 437.	Pulaskite.	Madagascar.
Bull. U.S. Geol. Surv., No. 148, p. 65.	Elaeolite syenite.	Maine.
H. S. Washington, *J.G.*, ix., p. 610, 1901.	Foyaite.	Arkansas.
N. *Jahrb.*, 1892, vol. ii., p. 255.	Trachyte.	Monte Santo, Naples.
N. *Jahrb.*, 1892, vol. ii., p. 418.	Trachyte pumice.	Laacher See.
N. *Jahrb.*, 1892, vol. ii., p. 418.	Trachyte pumice.	Laacher See.
Weed and Pirsson, *A.J.S.*, ii., p. 192, 1896.	Tinguaite.	Montana.
N. *Jahrb.*, 1892, vol. ii., p. 418.	Trachyte.	Laacher See.
Merrill, p. 217.	Phonolite.	Bohemia.

List of 1000 Rock

Group.	SiO₂.	TiO₂.	P₂O₅.	H₂O.	Al₂O₃.	Fe₂O₃.	FeO.	MgO.	CaO.	Na₂O.	K₂O.	Total.	Sp.Gr.
16	*a a b b b b*												
	62·4			2·0	20·2		5·3	·9	1·1	3·9	5·8	101·6	2·50?
	57·9			2·4	19·1	2·5		1·1	1·2	6·6	9·2	100·0	...
	57·0			2·9	16·7	2·5	5·3	·7	2·0	1·4	11·7	100·2	2·68
	56·3	·5		1·8	23·6	·8	2·7	·3	·5	7·8	5·7	99·9	...
	55·1			1·1	23·3	3·3			1·5	6·8	8·9	99·8	...
	53·5			1·4	24·3	1·1	1·2	·1	·8	8·6	8·9	99·9	...
6 rocks	57·0	·1		1·9	21·2	1·7	2·4	·5	1·2	5·8	8·4	100·2	2·60
	± 2·5				± 2·5	± 1·6		± ·6		± 2·1		± 9·3	
17	*a b a a a a*												
	75·5			1·4	9·6	1·7	4·3	1·3	1·0	5·1	·6	100·4	...
	72·3			1·8	9·0	6·3	1·1	·0	1·9	5·8	1·5	99·8	2·66
	72·2	·4	·1	·9	13·8	1·5	2·0	1·1	3·5	4·4	·4	100·3	...
	71·6			1·0?	10·5	3·9	2·2	5·4	2·7	1·9	·8	100·0	...
	70·4	(1·8)		·3	7·9	7·0	3·0	·5	·3	4·1	4·5	100·6	...
	70·3			·8	6·3	9·2	1·4	·9	·8	7·7	2·5	100·0	...
	69·8			(2·9)	12·3	4·7		·7	1·7	6·7	2·0	100·8	...
	69·3			·6	13·4	4·8		1·6	5·1	3·3	1·8	99·9	...
	68·0			·3	16·1		4·4	·5	5·9	4·3	·5	100·0	...
	67·8	·5	·2	1·0	14·0		5·2	1·0	4·3	5·2	1·2	100·3	2·71
	64·9	·4	·2	3·0	14·3	5·7		2·0	5·5	3·5	·9	100·4	2·87
11 rocks	70·2	·3	·1	1·2	11·6	3·8	2·4	1·4	3·0	4·7	1·5	100·2	2·76
	± 2·1				± 2·6	± 1·7		± 2·2		± 1·9		± 10·5	
18	*a b a a a b*												
	69·1				16·3	3·7	1·4	1·1	5·1	2·9	1·1	100·7	2·46
	67·9	·9	·2	·7	12·2	4·2	3·0	1·2	2·0	3·8	4·5	100·5	...
	66·9			·2	15·2		6·5	2·4	3·7	3·3	·9	99·1	2·72
	66·8	·6	·2	·6	15·2	2·7	1·8	1·6	3·7	3·1	4·5	100·8	...
	66·7	·4	·1	·9	16·1	1·5	2·5	1·7	4·6	3·4	2·7	100·6	...
	66·3			1·1	14·3	5·5	·3	2·4	4·6	3·9	1·6	100·0	2·65?
	65·9	·8	·2	3·6	13·7	·5	5·7	2·3	3·0	2·8	1·6	100·3	...
	65·9	·4		·8	14·9	1·8	3·1	2·9	4·6	2·1	4·2	100·8	2·90
	65·2			1·1	15·6	2·1	3·7	2·4	6·7	2·5	1·5	100·8	...
	65·1		·3	2·6	16·2	3·3	·9	1·8	4·3	3·4	2·4	100·2	2·57
	63·9	·6	·1	1·9	15·8	1·9	2·8	2·1	4·8	3·3	3·1	100·3	...
	63·9	·7	·2	1·2	15·8	2·1	2·7	2·1	4·1	2·8	4·3	99·9	...
	63·5			2·9	12·4	6·4	2·2	1·3	4·2	4·9	1·8	99·6	2·52
	62·2			2·3	15·6	5·3	1·4	2·6	6·6	2·5	1·6	100·0	2·57
	62·0			4·6	15·2	2·1	2·0	3·2	5·5	3·2	1·7	99·3	...
	61·0	1·2	·3	5·4	13·9	1·6	3·7	1·6	4·0	2·8	4·2	99·8	2·63
16 rocks	65·1	·4	·1	1·8	14·9	2·8	2·7	2·0	4·5	3·2	2·6	100·2	2·64
	± 1·1				± 1·0	± 1·0		± 1·2		± 1·2		(± 5·5)	

—continued.

Zirkel, *Petrog.*, ed. 1, Bd. ii., p. 178.	Trachyte.	Hessen.
Zeitsch. d. d. geol. Gesell., pt. ii., xix., p. 477.	Pumice.	Krufter Ofen.
N. Jahrb., 1901, suppl. 14, p. 38.	Keratophyre.	Harz.
Whitman Cross, Table XIV.	Miascite.	Mt. Lobatchio, Siberia.
N. Jahrb., 1892, vol. ii., p. 146.	Phonolite (tinguaite).	Sierra de Tingua.
H. S. Washington, *J.G.*, ix., p. 667, 1901.	Foyaite.	Arkansas.
Q. Journ., 1884, p. 528.	Dimetian.	Pembrokeshire.
Q. Journ., 1900, p. 686.	Felsite porphyry.	Knock Mahon, Waterford.
H. W. Turner, 14, *A.R. U.S.G.S.*, ii., p. 484, 1894.	Quartz porphyrite.	California.
Q. Journ., 1891, p. 82.	Porphyroid.	Charnwood Forest.
N. Jahrb., 1905, vol. i., p. 437.	Aeg. riebeck granite.	Madagascar.
Whitman Cross, Table XIII.	Pantellerite.	Pantellaria.
Prestwich, i., p. 37.	Pumice.	Santorin.
N. Jahrb., 1902, vol. ii., p. 390.	Dacite (andesite).	Kasbek, Russia.
W. C. Day, 19, *A.R. U.S.G.S.*, vi. (2), p. 214, 1898.	Granite.	Delaware.
Q. Journ., 1879, p. 56.	Granite.	Ontario.
N. Jahrb., 1904, vol. ii., p. 223.	Diorite porphyrite.	Montenegro.
Q. Journ., 1884, p. 445.	Andesite.	Izu San, Japan.
N. Jahrb., 1904, vol. i., p. 393.	Granite.	Lövstakken.
Roth, *Zeitsch. d. d. geol. Gesell.*, 1864, p. 257.	Tonalite.	Tyrol.
N. Jahrb., 1901, vol. i., p. 242.	Quartz monzonite.	Yosemite Valley.
N. Jahrb., 1901, vol. ii., p. 240.	Granodiorite.	Nevada City.
Doelter, *Tscherm. Mitt.* (*Jahrb. d. g. Reichs.*), 1873, p. 92.	Dacite.	Transylvania.
G. F. Becker, 18, *A.R. U.S.G.S.*, iii., p. 45, 1898.	Diorite.	Alaska.
H. Traube, *N.J.*, 1890, i., p. 218.	Syenite.	Silesia.
N. Jahrb., 1901, vol. ii., p. 62.	Porphyritoid.	Finland.
H. Lenk, *Btr. G. Mex.*, ii., p. 132, 1899.	Trachyte.	Mexico.
N. Jahrb., 1901, vol. ii., p. 240.	Granodiorite.	California.
Whitman Cross, p. 223.	Granite.	Butte, Montana.
H. Ziegenspeck, *In. Diss. Jena*, p. 46, 1883.	Augite andesite.	Patagonia.
N. Jahrb., 1893, vol. i., p. 285.	Dacite.	Cabo de Gata.
N. Jahrb., 1900, vol. i., p. 393.	Amphib. andesite.	Dubowka.
N. Jahrb., 1892, vol. ii., p. 413.	Porphyry.	Oberkirchen.

List of 1000 Roc

Group.	SiO₂.	TiO₂.	P₂O₅.	H₂O.	Al₂O₃.	Fe₂O₃.	FeO.	MgO.	CaO.	Na₂O.	K₂O.	Total.	Sp. G.
19	*a b a a b a*												
	70·6	·1		·1	8·6	2·5	6·3	·1	·6	6·8	4·5	100·2	...
	68·8				*5·9*	5·8	5·3	·1	2·1	7·5	4·3	100·0	2·4
	68·8			·1	11·3	4·9	4·3	·1	·5	6·1	4·4	100·5	...
	68·2			1·6	15·9	5·2	·5	·9	·3	7·5	·6	100·7	...
	67·5			1·0	9·7	7·4	2·2	·8	1·5	7·2	2·9	100·1	2·6
	67·5			1·6	14·5	7·0		2·3	1·1	3·4	3·4	100·8	...
	66·5			·6	7·7	11·4	2·9	·8	4·3	1·8	4·9	100 8	...
	66·1			2·1	13·4	6·3	·5	·9	·6	5·4	5·0	100·4	...
	64·0	·8		·2	10·4	6·3	4·2	·3	1·5	7·6	4·6	100·0	...
	63·6	·7	·3	·3	14·6	1·5	5·9	1·6	4·6	4·9	2·0	100·0	...
	63·5		·1	·7	14·8	1·8	5·6	1·6	3·2	5·7	3·0	100·0	...
	63·1			2·0	16·0	4·3	1·5	2·1	4·4	3·9	2·7	100·0	...
	62·9			1·4	14·8	9·2		·4	3·6	4·0	2·9	99·2	...
	62·0			(3·2)	17·0	7·0		1·0	3·5	4·0	2·3	100·0	2·65
14 rocks	65·9	·1		1·1	12·5	5·5	3·0	·9	2·3	5·4	3·4	100·2	2·6
	±2·0				±3·0	±1·1		±1·7		±1·9		(±9·7)	
20	*a b a a b b*												
	66·0			1·1	13·3	8·4		2·2	1·6	3·4	4·7	100·7	...
	64·8			1·8	18·0	5·6	2·2	2·3	1·7	1·6	2·5	100·4	2·73
	64·5			·6	16·9	6·2		3·1	2·5	2·2	3·9	99·7	...
	64·4			·8	14·1	6·1	4·0	2·0	4·5	·6	3·7	100·2	...
	63·8			2·8	17·6	3·0	3·3	3·4	2·5	1·8	2·4	100·5	...
	63·0			2·8	13·4	3·5	5·3	·6	1·3	5·5	5·2	100·5	...
	62·3			2·2	14·1	8·2		3·4	1·3	4·4	3·5	99·4	...
	61·8	1·2	·2	2·7	14·8	1·8	5·3	2·7	·7	3·6 ?	4·5	99·6	2·66
	61·5			3·0	16·3	4·4	4·0	3·0	3·1	2·8	1·6	99·7	...
	61·4			2·3	16·7	7·5		3·7	2·1	4·8	2·9	101·4	...
	61·4	1·0	·3	3·4	14·5	2·0	5·8	2·0	1·3	3·9 ?	4·7	100·2	2·66
	60·9			1·5	12·4	11·7		2·0	1·7	4·0	5·3	99·5	...
	60·6			3·2	18·0	7·3		1·4	2·4	2·4	3·8	99·1	...
	59·1			1·5	14·5	12·3		2·0		4·0	6·6	100·0	2·68
	59·0			3·3	15·7	11·7		·8	3·2	5·4	·7	99·8	...
	58·6	1·4	·4	5·1	15·3	5·6	3·3	3·0	1·0	2·4	3·8	99·9	2·67
	58·2	2·1	·3	(3·3)	15·5	6·5	2·0	2·6	4·4	3·0	2·6	100·5	...
	57·9	·4	·4	1·7	16·5	6·6	2·4	4·6	3·7	3·6	1·6	99·4	2·93
18 rocks	61·6	·3	·1	2·4	15·4	6·0	2·6	2·5	2·2	3·3	3·6	100·0	2·72
	±1·6				±1·4	±1·3		±1·4		±1·8		±7·5	

continued.

N. Jahrb., 1904, vol. ii., p. 58.	Rhyolitic obsidian.	Br. E. Africa.
H. Fürstner, *Z.K.*, viii., p. 179, 1884.	Pantellerite.	Pantellaria.
N. Jahrb., 1905, vol. i., p. 247.	Liparite (mean 4, 5).	Somali Desert.
N. Jahrb., 1901, vol. i., p. 63.	Quartz keratophyre (mean).	Bukowina.
H. Fürstner, *Z.K.*, viii., p. 186, 1884.	Pantellerite.	Pantellaria.
N. Jahrb., 1895, vol. ii., p. 100.	Kersantite.	Croix de fer Dauphiny.
N. Jahrb., 1904, vol. i., p. 53.	Syenite porphyry.	Malenic, Bohemia.
N. Jahrb., 1901, vol. i., p. 63.	Ægirine trachyte.	Siebenbürgen.
N. Jahrb., 1904, vol. ii., p. 58.	Phonolite obsidian.	Br. E. Africa.
N. Jahrb., 1905, vol. ii., p. 75.	Augite andesite.	Marian Island.
N. Jahrb., 1905, vol. ii., p. 79.	Gneiss (mean of 2).	Schapbach, Black Forest.
Q. Journ., 1890, p. 349.	Propylite (hornbl. mica andesite).	Washoe.
Q. Journ., 1890, p. 349.	Hornbl. propylite (T. H. Holland).	Mull.
A. Geikie, p. 166, third edit.	Trachyte, average.	
N. Jahrb., 1895, vol. ii., p. 100.	Kersantite.	Croix de fer.
N. Jahrb., 1902, vol. ii., p. 231.	Hornfels granulite.	Borry Mts.
Q. Journ., 1900, p. 686.	Trachyte.	Newtown Head, Co. Waterford.
C. Chelius, *Erl. G. Kt. Hesse, V. Bl. Breusbach*, p. 24, 1897.	Granite.	Hessen.
A. Ossan, *Z.D.G.G.*, xl., p. 701, 1888.	Cordierite andesite.	Spain.
N. Jahrb., 1893, vol. ii., p. 347.	Quartz syenite.	Fourche Mts., Ark.
N. Jahrb., 1895, vol. ii., p. 99.	Orthophyre.	Château noir.
N. Jahrb., 1893, vol. ii., p. 99.	Augite porphyrite.	Süpplingen.
Q. Journ., 1883, p. 296.	Basic tuff.	St Davids.
J. M. C. Henderson, *Z.D.G.G.*, xlvii., p. 539, 1895.	Mica syenite.	Saxony.
N. Jahrb., 1893, vol. ii., p. 99.	Augite porphyrite.	Altonhausen.
N. Jahrb., 1895, vol. ii., p. 99.	Orthophyre (mean of 3).	Château noir.
Q. Journ., 1895, p. 192.	Minette.	Nair Point, Porthalla.
Q. Journ., 1895, p. 192.	Tachylitic rock.	Garn Fawr, Fishguard.
Q. Journ., 1895, p. 192.	Augite porphyrite (Weisselb.), *see* Rosenbusch, p. 501.	
F. Klockmann, *Jb. Pr. G.L.A.*, 1890, xi., p. 210, 1892.	Porphyrite.	Near Magdeburg.
N. Jahrb., 1900, vol. ii., p. 221.	Porphyrite.	Gaflasion.
F. V. Wolff, *Z.D.G.G.*, li., p. 502, 1899.	Augite porphyrite.	Chile.

LIST OF 1000 ROCK

Group.	SiO₂.	TiO₂.	P₂O₅.	H₂O.	Al₂O₃.	Fe₂O₃.	FeO.	MgO.	CaO.	Na₂O.	K₂O.	Total.	Sp. Gr
21	*a b a b a a*												
	67·8			·3	16·9	1·1	2·3	1·3	3·3	4·4	3·4	100·7	...
	67·4		·1	·1	15·9	1·4	1·1	1·4	3·5	6·4	2·6	99·9	...
	67·0			·8	14·9	4·3		1·6	3·3	6·1	3·9	101·8	...
	66·7			1·0	15·8	·7	·3	2·1	3·9	7·1	2·4	100·0	...
	66·4			1·0	17·1		3·8	1·0	4·0	4·5	2·1	99·9	...
	66·0	·1		·4	17·1	2·1	3·0	2·0	4·6	4·0	1·4	100·7	2·54
	65·5	·4	·2	·6	16·5	1·4	2·5	2·5	4·9	4·1	2·0	100·6	...
	65·2	·7	·2	1·3	16·9	1·6	1·9	1·3	4·1	3·6	3·0	99·8	...
	65·0				19·5	2·5	·3	·5	3·7	6·1	2·0	99·5	...
	65·0			1·0	18·8	2·4		2·1	4·4	4·4	2·2	100·3	2·59
	65·0	·8		·6	19·4	1·8	1·1	·5	3·1	6·3	1·7	100·3	...
	64·0				17·6		4·2	1·3	4·4	4·8	2·6	98·9	...
	64·0			·5	19·1	2·4	1·9	1·1	4·6	4·2	1·9	99·6	...
	63·5			2·0	15·3	3·2	1·7	2·5	4·3	4·8	2·8	100·1	...
	62·0	·2	·3	1·7	17·8		4·4	2·6	5·4	4·3	1·5	100·1	...
	61·4	·8	·2	1·7	16·6	2·1	3·1	2·7	6·2	3·8	1·3	100·1	...
	60·7			1·4	14·8	7·4			4·9	8·7	·4	98·3	...
	59·9	1·0		4·3	17·0	3·6	1·3	1·5	5·9	3·2	1·5	99·2	2·33
18 rocks	64·6	·2	·1	1·0	17·1	2·1	1·8	1·6	4·4	5·0	2·1	100·0	2·49
		±1·4			±1·1	±1·0		±1·1		±1·3		±5·9	
22	*a b a b a b*												
	65·8			1·2	18·4	2·0	1·5	1·5	3·7	4·0	4·1	102·2	...
	64·3	·3	·2	1·3	17·8	3·4	1·3	2·0	3·4	3·8	2·5	100·3	...
	63·7		·3	1·2	17·1	2·0	2·7	2·0	3·9	4·1	3·1	100·1	...
	63·2		·2	·5	17·0	·2	6·4	·9	4·2	4·4	2·9	100·2	...
	63·1			·6	18·9	3·5	2·0	2·0	6·2	3·1	1·3	100·7	...
	62·9			·6	19·0	1·0	3·2	1·7	4·3	3·9	3·4	100·1	...
	62·9	·2		1·0	18·3	1·8	4·0	1·6	5·6	2·9	1·5	99·8	2·46
	62·8	·4	·5	2·3	16·2	3·1	1·7	1·5	4·8	3·5	3·1	99·9	...
	62·8			2·7	18·7	1·1	4·4	2·4	3·6	3·6	1·3	100·6	...
	62·7			·2	17·2	3·8	2·7	1·8	5·5	3·5	3·0	100·3	2·79
	62·6	·4	·3	2·6	16·4	2·5	2·1	1·8	2·9	4·6	3·9	100·1	...
	62·3	1·1	·3	1·2	17·3	3·0	1·7	1·1	3·4	4·2	4·5	100·3	2·49
	61·6		·3	1·6	18·8	4·7		2·0	6·6	4·3	1·5	101·4	...
	61·3	·7		1·0	17·7	6·0	·3	2·5	5·6	4·3	1·4	100·6	2·44
	61·2			·7	18·3	6·2		3·8	5·1	3·1	2·4	100·7	2·61
	60·9	·3		·7	18·1	2·4	3·8	3·5	5·6	4·2	1·0	100·5	2·64
	60·6			2·2	17·5	2·8	2·5	2·8	3·8	3·3	4·5	100·0	2·65
	60·4	·6	·3	1·9	16·7	2·3	3·2	2·2	4·3	5·2	2·7	100·0	2·68
	60·3		·1	2·8	17·1	4·7	1·1	2·9	3·5	5·1	2·1	99·7	2·61
	60·0			1·6	16·9	2·4	4·8	3·6	5·1	3·9	1·3	99·6	2·85?
	59·9			·9	18·4	3·7	3·0	2·0	6·6	3·4	1·7	99·6	2·44
	59·8			4·1	18·6	4·8		2·9	4·7	2·8	2·7	100·5	(2·18)

continued.

N. *Jahrb.*, 1902, vol. ii., p. 72.	Hyp. amph. dacite.	Caucasus.
Whitman Cross, Table XII.	Aug. soda syenite.	Kekequabikl., Minn.
N. *Jahrb.*, 1904, vol. i., p. 70.	Amph. granulite.	Aiguilles Rouges.
L. Milch, *Z.D.G.G.*, li., p. 66, 1899.	Dacite.	Sumatra.
N. *Jahrb.*, 1901, vol. ii., p. 240.	Granodiorite.	El Capitan.
N. *Jahrb.*, 1900, vol. ii., p. 234.	Amphil. andesite.	Mexico.
N. *Jahrb.*, 1901, vol. ii., p. 240.	Granodiorite.	California.
W. Lindgren, 18, *A.R.U.S.G.S.*, iii., p. 640, 1898.	Biotite granite.	Idaho.
C. F. Kolderup, *Berg. Mus. Aarb.*, 1898, No. 7, p. 28.	Oligoclase rock.	Lofoden Island, Norway.
A. Lagorio, *T.M.P.M.*, viii., p. 458, 1887.	Andesite.	Mexico.
N. *Jahrb.*, 1904, vol. ii., p. 400.	White granite.	Lindaas, Norway.
N. *Jahrb.*, 1899, suppl. 12, p. 544.	Granite.	Granite Peak.
N. *Jahrb.*, 1902, vol. ii., p. 390.	Biotite andesite dacite.	Kasbek.
R. Küch, *G. Stud. Colomb.*, i., p. 172, 1892.	Pyr. hornbl. dacite.	Colombia.
Hague and Iddings, *A.J.S.*, xxvi., p. 230, 1883.	Hyp. andesite.	California.
N. *Jahrb.*, 1903, vol. i., p. 430.	Andesite.	Aroostock.
(Bernath), Prestwich, i., p. 37.	Trachyte.	Tokay, 21/23 $a\,b\,a\,b\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)a$.
H. S. Washington, *J.G.*, iii., p. 150, 1895.	Hornbl. andesite.	Greece.
	Granodiorite.	
N. *Jahrb.*, 1902, vol. ii., p. 72.	Hyp. amph. dacite.	Caucasus.
J. P. Iddings, 12, *A.R.U.S.G.S.*, i., p. 648, 1891.	Hornbl. andesite.	Yellowstone Park.
W. Cross, 14, *A.R.U.S.G.S.*, ii., p. 227, 1897.	Q. hornbl. mica porphyrite.	Colorado.
A. Osann, *Mit. Bad. G.L.A.*, ii., p. 385, 1893.	Malcite.	Hessen.
N. *Jahrb.*, 1900, vol. ii., p. 388.	Tonalite gneiss.	Wistra.
C. R. Keyes, 15, *A.R. U.S.G.S.*, p. 722, 1895.	Biotite granite.	Maryland.
H. S. Washington, *J.G.*, iii., p. 150, 1895.	Hornbl. hyp. dacite.	Greece.
W. Cross, 14, *A.R.U.S.G.S.*, ii., p. 227, 1894.	Diorite porphyrite.	Colorado.
N. *Jahrb.*, 1904, vol. i., p. 65.	Diorite.	Steiermark.
W. Cross, *B.U.S.G.S.*, 150, p. 242, 1898.	Diorite.	Colorado.
L. V. Pirsson, 20, *A.R.U.S.G.S.*, iii., p. 514, 1900.	Syenite porphyry.	Montana.
N. *Jahrb.*, 1900, vol. i., p. 71.	Biotite aug. latite (lava).	Sierra Nevada, California.
A. Michel Levy, *B. Serv. Cte. G. Fr.*, No. 57, p. 19, 1897.	Esterellite.	Esterel, France.
H. S. Washington, *J.G.*, iii., p. 150, 1895.	Hornbl. dacite.	Greece.
N. *Jahrb.*, 1900, vol. ii., p. 234.	Amphib. andesite.	Mexico.
N. *Jahrb.*, 1900, vol. ii., p. 234.	Amphib. andesite.	Mexico.
A. Geikie, third edit., p. 169.	Propylite.	Storm Cannon.
W. Cross, *B.U.S.G.S.*, 148, p. 181, 1897.	Diorite porphyry.	Colorado.
T. G. Bonney, *G.M.*, xxxvi., p. 4, 189.	Hornbl. andesite.	Argentine.
Teller and von John, *Jahrb. d. geol. Reichs.*, 1882, p. 589.	Quartz pyr. diorite.	Tyrol.
H. S. Washington, *J.G.*, iii., p. 150, 1895.	Andesite.	Greece.
N. *Jahrb.*, 1900, vol. ii., p. 234.	Pumice.	Mexico.

LIST OF 1000 R[…]

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp[…]
22	59·5	·9	·4	2·0	16·4	3·2	3·4	3·3	5·0	3·3	2·8	100·2	
	58·4			3·6	18·2		6·0	2·4	6·2	3·2	2·0	100·0	
24 rocks	61·8	·2	·1	1·6	17·7	2·7	2·8	2·2	4·7	3·9	2·5	100·3	2[…]
	±1·3				±·7	±·8		±1·4		±1·1		±5·3	(2[…]
23	*a b a b b a*												
	67·5	·1	·1	·5	18·6	1·1	·1	·2	·5	11·5	·1	100·3	.
	64·3			1·0	17·5	3·1	1·3	·3	·6	7·3	4·3	99·7	.
	63·7			·8	17·9	4·3	·5	·1	·8	7·2	5·2	100·5	.
	62·6			·8	18·2	·3	4·4	2·6	·7	6·5	4·0	100·1	.
	62·5				20·2	2·9		1·9	3·8	5·3	3·4	100·0	.
	62·5	1·1		·7	18·7	3·3	·5	·1	·5	11·8	·8	100·0	2·[…]
	62·4			·9	16·9	7·3		·8	3·5	4·4	2·9	99·1	2·[…]
	62·2		·1	1·6	18·6	2·2	1·1	·7	1·6	7·6	3·9	99·5	.
	61·7			·9	17·5	1·4	3·9	2·1	·2	8·5	3·4	99·5	.
	61·5				22·4	1·8			6·2	4·9	2·8	99·6	2·[…]
	61·3			1·7	18·2	6·0	1·8	·8	3·6	5·5	2·8	101·5	.
	60·1	·1		1·4	20·0	4·3	1·8	·2	·9	7·7	3·2	100·0	2·[…]
	59·8			1·0	17·2	7·6		1·3	6·0	4·0	3·1	100·0	.
	55·4	·9		4·9	18·3	1·1	5·9	3·5	3·2	7·1	·2	100·5	2·[…]
	54·5			5·2	20·0	2·3	3·3	·6	2·1	8·7	2·8	99·5	.
15 rocks	61·5	·1		1·4	18·7	3·3	1·6	1·0	2·3	7·2	2·9	100·1	2·[…]
	±1·5				±1·1	±1·6		±2·0		±1·9		±8·1	
24	*a b a b b b*												
	61·1	·3		·8	18·8	2·0	3·1	·4	1·3	6·6	6·0	100·0	2·[…]
	60·7		·2	·6	16·9	9·1		1·1	4·4	2·8	4·2	100·0	[illegible]
	60·6	·7	·2	·7	18·3	2·9	2·7	·5	1·0	6·7	5·7	99·9	..
	60·0				21·0		8·5	·7	3·2	4·3	2·0	99·7	2·[…]
	59·9			·6	17·2	10·0		·8	3·0	6·2	2·9	100·5	..
	59·7			1·9	18·9	4·9		·7	1·3	6·3	6·0	99·6	..
	59·4	·3	·6	·4	17·9	6·8	2·0	1·8	4·2	1·2	6·7	101·2	2·[…]
	58·7	1·0	·5	1·0	19·5	3·6	2·6	·8	3·0	5·7	4·5	100·9	..
	58·4			3·4	18·2	1·0	5·0	1·5	1·2	4·7	6·7	100·1	2·[…]
	58·1	1·9	·7		18·2	4·9	2·4	1·6	3·3	6·1	2·8	100·0	..
	56·2	·3	·1	1·2	23·2	·2	6·2	·4	2·4	3·8	5·3	99·3	..
	54·3	1·3		6·0	17·4	2·3	6·2	3·0	3·1	3·8	3·3	100·7	2·[…]
12 rocks	58·9	·5	·2	1·4	18·8	3·6	3·6	1·1	2·6	4·8	4·7	100·1	2·[…]
	±·9				±1·2	±1·5		±1·3		±1·8		±7·7	
25	*a b b a a a*												
	64·6			2·2	16·2		1·0	2·1	3·3	7·9	2·7	100·0	..
	63·0			2·0	14·2		6·7	1·1	5·4	5·7	3·5	101·6	..

-continued.

G. P. Merrill, *Pr. U.S. Nat. Mus.*, xvii, p. 651, 1895.	Hyp. andesite.	Montana.
Q. *Journ.*, 1890, p. 349.	Pyroxene andesite.	Washoe.
H. W. Turner, 17, *A.R.U.S.G.S.*, i., p. 727, 1896.	Soda syenite porphyry.	California.
N. *Jahrb.*, 1893, p. 497, vol. ii.	Acmite trachyte.	Montana.
G. T. Prior, *Min. Mag.*, xiv., p. 266, 1900.	Sölosbergite.	Abyssinia.
Hatch, *Petrology*, p. 163.	Trachyte.	Peppercraig.
N. *Jahrb.*, 1904, vol. i., p. 70.	Syenite porphyry.	French Alps.
N. *Jahrb.*, 1902, vol. ii., p. 395.	Mariupolite.	Sea of Azoff.
Bischof, *Lehrb. d. Geol.*, I. Aufl., vol. ii., p. 2181.	Andesite.	Siebengebirge.
N. *Jahrb.*, 1893, vol. ii., p. 497.	Acmite trachyte.	Montana.
Q. *Journ.*, 1895, p. 319.	Keratophyre.	Ruebeland, Harz.
Zeitsch. d. d. geol. Gesell., pt. iv., p. 375, 1868.	Quartz mica diorite.	Neurode.
N. *Jahrb.*, 1892, vol. i., p. 312.	Andesite.	Sulphur Island.
A. S. Eakle, *A.J.S.*, vi., p. 491, 1898.	Tinguaite.	Massachusetts.
Roley, p. 236.	Andesite.	Mean of several localities.
Q. *Journ.*, 1905, p. 595.	Trachytic rock.	St David's Head.
V. Hackman, *Fennia*, xi., p. 158, 1894.	Tinguaite.	Finland.
H. S. Washington, *J.G.*, vii., p. 118, 1899.	Sölosbergite.	Massachusetts.
Merrill, p. 215.	Gneiss.	Albemarle Co.
H. Rosenbusch, *Elemente*, p. 199, 1898.	Syenite porphyry.	Massachusetts.
Zirkel, *Zeitsch. d. d. geol. Gesell.*, 1859, p. 535.	Trachyte.	Eifel Gebirge.
N. *Jahrb.*, 1892, vol. i., p. 313.	Andesite pumice.	Sulphur Island.
N. *Jahrb.*, 1893, vol. ii., p. 347.	Elaeolite syenite.	Fourche Mts., Arkansas.
Whitman Cross, Table XIII.	Syenite.	Piedmont.
Whitman Cross, Table XII.	Laurvikite.	Laurvik, Norway.
N. *Jahrb.*, 1901, suppl. 14, p. 39.	Keratophyre.	Harz.
A. B. Lyons, *A.J.S.*, ii., p. 424, 1896.	Andesite.	Hawaii.
Q. *Journ.*, 1879, p. 44.	Foyaite, elaeolite syenite.	Portugal.
K. A. Lossen, *Jb. Pr. G.L.A.*, x., p. 316, 1892.	Porphyrite.	Harz.
Fuchs), Prestwich, i., p. 35.	Gneiss (reduced to 100).	Harz.
Q. *Journ.*, 1890, p. 349.	Propylite? (hornbl. andesite).	Tokay.

LIST OF 1000 ROCKS

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.
25	62·4			·3	18·1	·5	3·2	3·3	4·9	4·3	3·1	100·1	...
	62·1	·6	·2	·2	17·0	2·4	2·7	3·1	5·8	4·1	1·7	99·8	...
	61·7	·6	·3	·7	15·1	2·0	2·3	3·7	5·0	4·4	4·5	100·1	...
	61·6	·5	·2	1·3	17·0	1·7	2·9	3·7	6·3	3·9	1·3	100·4	...
	61·4	1·4		·9	14·4	2·8	4·6	2·7	4·3	4·0	3·8	100·2	...
	60·6			2·0	17·0	2·9	2·3	3·3	6·4	3·6	2·4	100·4	2·59
	60·5			1·8	16·1	3·0	3·5	3·2	5·8	3·9	1·7	99·5	2·77
	60·4	·2		2·0	17·0	1·5	3·4	3·8	5·4	3·4	2·0	99·1	2·58
	59·9			3·4	15·5	2·5	2·0	3·6	6·8	4·5	1·3	99·5	...
	59·6	·7	·3	1·3	16·0	1·1	5·4	5·0	5·5	3·7	1·1	99·5	...
	59·4	·4	·2	·5	18·5	1·8	3·9	3·1	6·4	4·3	1·3	99·8	...
	59·2	·6	·3	1·5	18·0	3·1	2·5	1·4	6·8	4·0	2·7	100·2	...
	57·9	·8	·2	2·9	17·2	5·0	1·5	2·8	6·7	3·2	2·3	100·6	2·68
	56·9	·8	·1	4·4	14·9	2·3	3·2	3·8	8·8	3·4	2·2	100·4	...
16 rocks	60·7	·4	·1	1·6	16·4	2·0	3·2	3·1	5·8	4·3	2·4	100·1	2·65
	±1·1				±1·1	±1·1		±1·2		±1·4		±5·9	
26	*a b b a a b*												
	61·9			1·1	13·2	3·6	2·3	4·6	3·5	2·7	6·1	99·0	...
	60·3		·1	1·7	15·8	5·4	·9	5·1	4·7	4·1	1·8	99·9	...
	60·3			·7	18·4	2·5	2·9	3·3	5·5	3·0	3·1	99·9	...
	59·9			1·6	12·1	10·6		5·2	4·5	3·8	1·0	98·6	...
	59·8	·7	·4	1·3	17·3	3·6	1·8	1·2	3·9	5·0	5·1	100·1	2·70
	59·5			·6	18·2	6·5		6·8	2·0	·6	5·3	99·5	...
	59·5	·9	·3	·8	17·2	2·1	4·6	2·7	6·6	3·5	2·3	100·2	...
	59·4	1·4	·6	1·0	16·7	2·5	3·5	1·8	4·2	3·7	5·0	100·0	2·61
	58·5	·7	·3	2·2	16·3	2·1	4·4	3·7	3·9	3·1	4·1	99·5	...
	58·0			5·5	12·5		7·1	5·3	3·8	3·3	4·8	100·2	2·67
	57·3	·7		2·7	15·7	4·5	3·5	4·3	5·4	4·0	1·9	100·0	...
	56·9	·8	·2	·5	18·8	2·1	4·3	4·4	7·5	3·9	·8	100·2	...
	56·8	1·1	·4	1·4	16·9	3·6	2·9	3·4	6·6	3·2	3·5	99·9	2·67
	56·4	2·1		1·3	12·9	2·4	3·5	7·8	4·1	1·3	7·8	99·6	...
	56·2	1·4	·2	5·4	17·2	2·8	1·0	4·9	1·8	4·3	4·8	100·2	2·56
	55·6	·8		2·9	10·7	2·0	5·8	4·6	8·4	1·5	7·5	99·7	2·67
	54·2	2·7	1·6	2·1	10·2	3·5	·7	6·6	5·0	1·2	11·9	100·2	2·70
	50·2	2·3	1·9	3·4	11·3	3·3	1·9	7·1	7·4	1·4	9·8	100·0	2·78
18 rocks	57·8	·8	·3	2·0	15·1	3·5	2·8	4·6	4·8	3·0	5·1	99·8	2·67
	±1·4				±2·5	±1·0		±1·9		±1·9		±8·7	
27	*a b b a b a*												
	63·0	·2		·2	14·3	2·8	5·3	1·3	2·7	4·9	6·4	100·9	2·73
	62·1			2·0	18·0		3·6	·7	3·6	6·5	4·1	100·6	...
	61·0			1·1	16·6	3·6	3·4	·1	3·3	5·9	5·2	100·1	...

—*continued.*

N. Jahrb., 1902, vol. ii., p. 390.	Andesite dacite.	Tscheri, Russia.
N. Jahrb., 1905, vol. i., p. 254.	Hyp. andesite.	Yellowstone Park.
N. Jahrb., 1903, vol. i., p. 432.	Aug. hornbl. syenite.	Montana.
J. S. Diller, *B.U.S.G.S.*, 148, p. 190, 1897.	Hornbl. andesite.	California.
N. Jahrb., 1902, vol. ii., p. 236.	Bronzite andesite.	Tibet.
Q. Journ., 1893, p. 141.	Anamesite (andesite).	Capraja, Italy.
N. Jahrb., 1905, vol. i., p. 108.	Quartz norite.	Penmaenmawr.
Q. Journ., 1893, p. 141.	Anamesite.	Capraja.
N. Jahrb., 1900, vol. i., p. 393.	Amph. andesite.	Mariupol.
N. Jahrb., 1905, vol. ii., p. 234.	Quartz basalt.	John Day Basin.
N. Jahrb., 1905, vol. i., p. 254.	Hyp. andesite.	Wizard, Yellowstone Park.
W. Cross, 14 *A.R. U.S.G.S.*, ii., p. 227, 1894.	Augite diorite.	Colorado.
Weiss and Grebe, *Erl. G. Kte. Pr. Bl. Lebach*, p. 34, 1889.	Bronzite porphyrite.	Prussia.
N. Jahrb., 1893, vol. ii., p. 339.	Quartz basalt.	Mytilene.
Whitman Cross, Table XII.	Pyroxene granitite.	Laveline, Vogesen.
N. H. Winchell, 21, *A.R.G.*, *Nh. S. Minn.*, p. 55, 1893.	Porphyrite.	Minnesota.
N. Jahrb., 1902, vol. ii., p. 390.	Andesite.	Blota, Kasbek.
Q. Journ., 1897, p. 45.	Volcanic breccia.	Canada (also 18 and 25).
W. Cross, *B.U.S.G.S.*, 168, p. 162, 1900.	Syenite.	Colorado.
Q. Journ., 1900, p. 689.	Igneous rock.	Bun Mahon.
N. Jahrb., 1901, vol. ii., p. 240.	Granodiorite.	Donner Pass.
N. Jahrb., 1900, vol. i., p. 71.	Augite latite (lava).	Sierra Nevada.
N. Jahrb., 1900, vol. i. p. 75.	Mica diorite.	Michigan.
Zeitsch. d. d. geol. Gesell., pt. i., 1867–68, xx., p. 331.	Augite porphyry.	Tuscany.
N. Jahrb., 1900, vol. ii., p. 221.	Syenite porphyry.	Svasken.
N. Jahrb., 1905, vol. i., p. 254.	Basalt.	Anna Creek, Yellowstone Park.
N. Jahrb., 1900, vol. i., p. 71.	Augite latite (lava).	Sierra Nevada.
N. Jahrb., 1901, vol. i., p. 225.	Mica trachyte.	Italy.
K. A. Lossen, *Jb. Pr. G.L.A.*, x., p. 290, 1892.	Mesokeratophyre.	Saar-Nahe.
N. Jahrb., 1903, vol. i., p. 414.	Minette.	Studené, Bohemia.
W. Cross, *A.J.S.*, iv., p. 130, 1897.	Orendite.	Wyoming.
Whitman Cross, Table XII.	Wyomingite.	Leucite hills, Wyoming.
N. Jahrb., 1902, vol. ii., p. 399.	Umtekite (alkali syenite).	Beverley, Mass.
Duparc and Mrazee, *Mem. Soc. Phys. Gen.*, xxxiii., No. 1, p. 48, 1898.	Protogine.	Mt. Blanc.
David, Smeeth, and Schofield, *V.R. Soc. N.S.W.*, xxix., p. 473, 1895.	Trachyte.	Antarctic Continent.

LIST OF 1000 ROCKS

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.
27	61·0	·5		3·1	18·5	2·1	·8	·9	1·9	7·3	4·8	101·3	2·56
	60·5			·2	16·5		7·9	1·4	5·8	5·0	2·3	99·7	2·75?
	59·7		·1	2·5	17·0	3·2	1·3	·8	2·3	8·4	4·2	99·6	...
	59·2			·9	18·8	4·3	1·7	1·9	6·6	4·7	2·0	100·1	...
	59·2			1·3	13·6	5·6	4·0	1·7	5·1	5·3	4·6	100·4	2·74
	58·8	·8	·3	1·2	17·5	2·4	2·5	1·0	2·6	6·8	5·9	100·0	...
	58·3	1·0	·4	2·5	19·4	1·4	3·1	1·3	5·1	4·4	3·8	100·5	...
	58·3	·2	·1	2·4	16·6	4·1	3·5	·4	1·7	7·3	5·5	100·0	...
	57·4	2·0	·6	1·0	19·4	1·7	2·9	1·2	2·7	6·5	4·3	99·7	...
12 rocks	59·9	·4	·1	1·5	17·2	2·6	3·3	1·1	3·6	6·1	4·4	100·2	2·70
		±1·2			±1·4	±1·8		±1·7		±1·7		±7·8	
28	*a b b a b b*												
	60·7				18·9	4·4		2·9	2·7	4·1	5·4	99·1	...
	59·8			1·3	16·8		7·0	2·6	4·4	2·4	6·5	100·8	2·73
	59·3	·2	·3	·8	18·6	4·5	·3	·7	2·6	4·5	6·9	98·7	...
	58·9	·4		1·9	17·7	3·9	2·9	·5	1·1	7·4	5·6	100·3	...
	58·5	·3	·2	1·4	16·6	5·7	2·6	·6	2·6	6·2	5·4	100·3	...
	58·2	·3	·6	·9	18·3	4·6	·3	·7	2·5	4·5	6·9	97·7	...
	58·0	·9		·6	16·9	3·3	4·5	2·0	3·6	5·1	5·2	100·1	...
	57·9	·7		·9	15·8	6·8	·2	1·7	3·0	6·0	7·3	100·2	2·52
	57·3	·7	·5	2·2	18·5	4·4	1·2	2·1	3·6	4·4	5·4	100·3	...
	53·8			1·3	15·2	7·1	4·2	1·1	1·7	10·5	5·1	100·0	...
	53·7	2·0		·3	16·8	6·6	2·6	1·7	2·5	9·1	4·1	99·4	...
	53·5	·9		1·8	16·4	8·7	1·9	1·1	1·5	10·0	4·6	100·3	...
	52·6	·8	·6	3·6	16·9	4·5	3·2	3·7	5·1	3·9	5·2	100·1	...
	52·0	1·0		(5·4)	18·1	2·2	5·4	2·8	4·6	3·8	4·7	100·0	...
14 rocks	56·7	·6	·2	1·6	17·2	4·8	2·6	1·7	3·0	5·8	5·6	100·2	2·63
		±1·7			±·9	±1·6		±1·7		±1·8		±7·7	
29	*a b b b a a*												
	61·0	·4	·1	1·4	19·0	1·8	1·4	·7	4·1	6·7	3·5	100·3	...
	60·4			·5	17·4	2·0	1·8	1·9	2·8	7·5	5·6	99·9	...
	60·1			3·4	20·5	1·0	·7	1·2	2·6	9·6	1·1	100·2	...
	59·6			·8	18·7	5·1		·8	1·8	7·0	5·6	99·4	...
	59·0	·8		·7	18·2	1·6	3·7	1·1	2·5	7·0	5·3	100·0	...
	58·8				18·6	5·0	1·8	1·0	3·8	7·9	3·1	100·0	...
	58·7		·1	2·6	19·3	3·4	·7	·8	1·4	8·6	4·5	100·0	...
	58·6	·2		2·9	19·6	2·2	·6	·4	1·2	8·4	5·3	99·7	2·52
	58·5			2·6	20·3	2·4	·6		·9	9·5	5·2	100·0	2·55?
	57·7	·7		3·1	20·4	2·3	1·5	·7	3·2	7·5	4·7	101·8	2·62
	57·7			3·2	20·6	3·5		·5	1·5	7·0	6·0	100·0	2·58
	56·9			2·5	21·0		3·4	·3	1·9	9·1	4·7	99·9	...
	56·3	1·0	·7	·7	19·8	1·9	2·0	1·2	2·6	8·9	5·3	100·4	...

—continued.

F. Rinne, *Jb. Pr. G.L.A.*, vii., p. 21, 1887.	Soda trachyte.	Rhöngebirge.
N. *Jahrb.*, 1869, p. 708.	Andesite.	Auvergne.
N. *Jahrb.*, 1893, vol. ii., p. 497.	Syenitic trachyte.	Montana.
N. *Jahrb.*, 1906, vol. i., 222.	Amph. pyr. andesite (mean of three analyses)	Sago, Sumatra.
Whitman Cross, Table XIII.	Andesite.	Siebengebirge.
N. *Jahrb.*, 1902, vol. ii., p. 399.	Umtekite.	Cabo Frio, Brazil.
W. Tarassenko, *cf. N.J.*, 1899, i., p. 463.	Labradorite rock.	Wolhynia.
N. *Jahrb.*, 1904, vol. ii., p. 57.	Phonolite.	Br. E. Africa.
N. *Jahrb.*, 1904, vol. ii., p. 410.	Pulaskite.	Mt. Royal.
N. *Jahrb.*, 1903, vol. i., p. 74.	Trachyte (mean of 13).	French Alps.
Hatch, *Petrology*, p. 147.	Syenite.	Plauenscher Grund.
N. *Jahrb.*, 1892, vol. ii., p. 255.	Trachyte.	Naples.
N. *Jahrb.*, 1904, vol. ii., p. 57.	Solosbergite.	Br. E. Africa.
N. *Jahrb.*, 1902, vol. ii., p. 399.	Nordnarkite.	Cabo Frio, Brazil.
N. *Jahrb.*, 1892, vol. ii., p. 255.	Trachyte.	Naples.
W. C. Brögger, *Eg. Kg.*, ii., p. 33, 1895.	Akerite.	Norway.
H. J. Johnston-Lavis, *Geol. Mag.*, Dec., iii., vi., p. 77, 1889.	Trachyte.	Naples.
J. P. Iddings, *J.G.*, iii., p. 947, 1895.	Quartz banakite.	Yellowstone Park.
N. *Jahrb.*, 1902, vol. ii., p. 382.	Lamprophylite, lujavrite.	Kola.
N. *Jahrb.*, 1902, vol. ii., p. 382.	Lujavrite.	Lujaor, Urt Kola.
N. *Jahrb.*, 1902, vol. ii., p. 382.	Lujavrite.	Lujaor, Urt Kola.
J. P. Iddings, *J.G.*, iii., p. 947, 1895.	Banakite.	Yellowstone Park.
N. *Jahrb.*, 1902, vol. ii., p. 67.	Bostonite.	Orkney Islands.
W. Cross, 14, *A.R. U.S.G.S.*, ii., p. 227, 1894.	Augite porphyry.	Utah.
N. *Jahrb.*, 1900, vol. ii., p. 236.	Trachyte andesite.	Sumatra.
E. O. Hovey, *A.J.S.*, iii., p. 291, 1897.	Keratophyre.	Connecticut.
N. *Jahrb.*, 1893, vol. ii., p. 347.	Resembles elaeol. syenite	Arkansas.
W. S. Bayley, *B.G.S.A.*, iii., p. 250, 1892.	Nepheline syenite.	New Hampshire.
N. *Jahrb.*, 1900, vol. i., p. 385.	Soda syenite.	Laupstadeid.
N. *Jahrb.*, 1893, vol. ii., p. 497.	Acmite trachyte.	Montana.
W. Cross, 16, *A.R. U.S.G.S.*, ii., p. 39, 1895.	Phonolite.	Colorado.
Bull. U.S. Geol. Surv., No. 148, p. 114.	Phonolite.	Dakota.
E. Moller, *N.J.*, 1888, i., p. 97.	Phonolite.	Rhöngebirge.
A. Geikie, p. 166, third edit.	Phonolite (average).	
J. D. Irving, *Amm. Y. Ac.*, xii., p. 272, 1899.	Phonolite.	Dakota
Whitman Cross, p. 201.	Laurdalite.	Norway.

List of 1000 Roc

Group.	SiO₂.	TiO₂.	P₂O₅.	H₂O.	Al₂O₃.	Fe₂O₃.	FeO.	MgO.	CaO.	Na₂O.	K₂O.	Total.	Sp.G
29	55·9		·2	3·7	20·4	2·2	1·6	·6	2·2	8·4	4·8	100·0	2·45
	53·4	1·3	·1	4·4	18·2	2·2	3·4	1·0	3·7	8·4	3·9	100·0	2·49
15 rocks	58·2	·2	·1	2·2	19·5	2·2	1·8	·8	2·4	8·1	4·6	100·1	2·53
	±1·1				±·9	±·9		±1·1		±1·0		±5·0	
30	*a b b b a b*												
	60·0		·1	·6	20·8	4·0	·8	·8	2·6	6·0	5·5	101·1	2·66
	59·5			·6	18·9		5·3	1·5	1·9	5·1	7·2	100·0	2·60
	52·9			1·4	20·0	4·7		1·1	2·4	5·5	5·8	100·0	2·53
	59·2			1·1	18 6		6·1	1·1	3·0	4·9	6·7	100·6	2·55
	57·5	·6	·2	1·2	15·4	4·9	·9	1·4	3·4	5·5	9·4	100·4	...
	57·2	·3	·1	2·1	18·5	3·7	1·2	·7	2·8	4·5	8·6	100·2	...
	56·8	·5		2·5	19·7	2·2	3·7	·4	2·2	4·3	7·1	99·4	...
	56·8	1·2		·2	18·4	2·2	3·0	2·0	4·7	4·9	5·9	99·4	...
	56·3	·7	·5	(·4)	20·4	2·8	3·5	1·5	3·8	6·0	4·1	100·0	...
	56·3	·6		·8	20·5	1·8	4·2	2·5	3·6	5·9	4·8	101·2	...
	54·6	1·4	·7	·7	19·1	2·4	3·3	2·0	3·2	7·7	4·8	99·8	...
	53·8			3·8	19·7	4·8		1·9	3·2	6·0	8·0	101·2	2·55
	53·8	·3	·5	4·4	18·5	6·2	·7	1·1	2·5	7·1	5·5	100·2	...
13 rocks	57·0	·4	·2	1·5	19·1	3·1	2·5	1·4	3·0	5·6	6·4	100·2	2·58
	±1·1				±1·1	±·6		±1·0		±1·1		±4·9	
31	*a b b b b a*												
	58·7			1·8	20·9	4·2		·2	·4	9·7	4·2	100·1	...
	58·6			·8	21·8	1·8	1·8	·6	·3	9·4	5·2	100·3	...
	57·8			1·5	18·7	5·8	·4		1·3	9·4	4·5	99·4	2·64
	56·5			2·1	22·3	2·7	1·0		1·5	11·1	2·8	99·9	2·54
	56·2	·3	·1	1·0	21·4	2·0	·7	·2	1·5	10·5	5·7	99·9	2·62
	55·6			3·2	19·7		5·5	·9	1·7	8·6	4·9	100·1	...
	55·0	·3	·1	2·6	21·7	2·0	2·2	·1	2·1	9·8	3·5	99·4	2·51
	54·9	·4		2·6	20·8	1·3	1·4	·4	1·0	9·0	7·3	99·7	...
	54·9	1·4		1·1	18·3	7·8		1·1	1·3	11·4	3·2	100·5	...
	55·2	·6		(3·2)	23·0	1·2			2·7	10·0	4·5	100·4	...
	54·1			5·4	21·7	3·5		·4	·4	8·9	4·8	99·3	...
	54·1	·9		·4	20·6	3·3	2·3	·8	1·9	9·9	5·2	100·5	...
	54·0	·6	·3	1·8	19·4	4·4	2·3	1·1	2·0	8·8	5·3	100·0	...
	54·0			1·9	20·3	4·7	·6	·2	2·7	8·6	6·8	99·8	...
	53·4			4·2	20·2	1·6	3·0	·3	3·3	7·9	6·2	100·2	...
	49·1	·6		6·0	19·5	2·3	3·9	·6	3·8	9·3	4·4	99·4	...
16 rocks	55·1	·3		2·5	20·6	2·8	1·8	·4	1·7	9·5	4·9	99·9	2·58
	±1·1				±1·1	±1·4		±·9		±·7		±5·2	

—continued.

G. F. Föhr, *In. Diss. Wurzburg*, 1883, p. 28.	Phonolite.	Staufen, Germany.
N. Jahrb., 1905, vol. i., p. 275.	Nepheline porphyry.	Bohemia.
N. Jahrb., 1893, vol. ii., p. 347.	Pulaskite.	Fourche Mts., Ark.
Zeitsch. d. d. geol. Ges., part iv., p. 298, 1868.	Leucite trachyte.	Viterbo.
N. Jahrb., 1893, vol. ii., p. 347.	Elaeolite syenite.	Fourche Mts., Ark.
Zeitsch. d. d. geol. Gesell., part iv., p. 291, 1868.	Trachyte.	L. Bolsena, Italy.
Weed and Pirrson, *Has.*, ii., p. 192, 1896.	Tinguaite.	Montana.
L. V. Pirsson, *B.U.S.G.S.*, 148, p. 152, 1897.	Trachyte.	Montana.
Hatch, *Petrology*.	Phonolite.	Traprain Law.
H. S. Washington, *A.J.S.*, viii., p. 290, 1899.	Trachyte.	Ischia.
N. Jahrb., 1904, vol. ii., p. 400.	Mica syenite.	Tunnaes, Norway.
N. Jahrb., 1905, vol. i., p. 437.	Micromonzonite.	Ambodimadiro.
Whitman Cross, Table XIV.	Laurdalite.	Norway.
Roth, *Beiträge zur Petrog.*, 1873, p. xxxviii.	Phonolite.	Aussig.
N. Jahrb., 1904, vol. ii., p. 58.	Trachydolerite (kenite).	Mt. Kenia, Afr.
N. Jahrb., 1893, vol. ii., p. 347.	Nephelite syenite.	Arkansas.
N. Jahrb., 1905, vol. i., p. 437.	Aegirine foyaite.	Ampangarinana.
N. Jahrb., 1904, vol. ii., p. 57.	Tinguaite.	Abyssinia.
Geol. Mag., 1871, p. 249.	Phonolite.	Cornwall.
W. Cross, *B.U.S.G.S.*, 168, p. 171, 1901.	Phonolite.	New Mexico.
J. D. Irving, *Ann. N.Y. Acad.*, xii., p. 272, 1899.	Phonolite.	S. Dakota.
F. G. Föhr, *In Diss. Wurzburg*, 1883, p. 28.	Phonolite.	Hohentwiel.
N. Jahrb., 1902, vol. ii., p. 399.	Nepheline aplite.	Cabo Frio, Brazil.
N. Jahrb., 1902, vol. ii., p. 382.	Lujavrite.	Kola, Russia (Finland).
W. Bhruns, *Vh. Nh. Ver. Bonn.*, xlviii., p. 317, 1891.	Nosean sanidinite.	Laacher See.
J. F. Williams, *A.R. Ark. G.S.*, 1890, ii., p. 370, 1891.	Tinguaite.	Arkansas.
N. Jahrb., 1902, vol. ii., p. 382.	Neph. syenite (chibinite).	Tschasnatschorr, Fin.
N. Jahrb., 1904, vol ii., p. 58.	Trachydolerite (kenite).	Mt. Kenia.
N. Jahrb., 1893, vol. ii., p. 347.	Leucite tinguaite.	Arkansas.
N. Jahrb., 1893, vol. ii., p. 347.	Elaeolite syenite (vein).	Magnet Cove, Ark.
N. Sahlbom, *N.J.*, 1897, ii., p. 97.	Nephelinite.	Sweden.

List of 1000 Rocks

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.
32	*a b b b b b*												
	56·5	·3	·1	1·8	20·1	1·3	4·5	·6	2·1	5·6	7·1	100·4	...
	56·1			1·2	20·1	3·8		·8	2·5	7·5	8·8	100·8	...
	56·0			·7	22·2	1·1	3·3	1·1	2·4	8·4	5·0	100·2	...
	55·4				19·8	9·5			3·9	6·0	5·3	99·9	...
	55·1			1·2	19·2		6·9	1·2	3·7	2·7	10·8	100·8	2·50
	54·9			·6	16·5	10·0		·7	4·0	7·0	6·3	100·0	...
	54·8			·3	23·5	5·0		·6	1·7	9·0	5·1	100·0	2·59
	53·8			1·5	19·7	6·2	3·6	·9	1·7	7·8	4·6	99·8	...
	53·8			1·7	23·2	1·3	3·2	·2	3·0	7·0	7·0	100·3	...
	53·7	·1		2·2	20·4	3·7	2·6	·5	2·7	7·9	6·1	99·9	...
	53·7	·4		·9	18·4	5·9	3·3	·9	2·1	9·5	4·9	100·0	...
	52·9	·5	·9	1·2	19·5	4·8	2·5	·3	2·6	7·1	7·9	100·2	...
	52·8			3·6	22·6	3·6		·2	1·8	8·1	7·1	99·7	...
	52·3	·6		·7	22·2	2·4	2·5	1·0	1·5	9·8	6·1	99·2	...
	51·9	·2	·1	2·4	20·3	3·6	1·2	·2	1·8	8·5	9·8	100·0	...
	51·0	·5		1·4	19·7	7·8		·4	4·4	7·7	6·8	100·0	...
	46·5	1·2	·2	3·7	19·0	4·7	2·3	2·5	4·4	8·5	6·8	99·9	2·58
17 rocks	53·6	·2	·1	1·5	20·4	3·8	2·7	·7	2·7	7·5	6·8	100·1	2·56
	±1·1				±1·4	±1·8		±·9		±1·2		±6·4	
33	*b a a a a a*												
	61·4			·2	19·4	·6	5·3	3·3	6·6	2·8	·4	99·8	2·79
	61·1			·7	15·4	8·0		3·6	7·1	2·7	1·4	99·9	2·66
	59·9	·2	·2	·4	17·1	7·5	·5	4·0	5·1	3·1	1·7	99·8	2·75
	59·8	·3	·5	·7	18·7	3·0	4·3	2·8	6·7	2·3	1·4	100·4	...
	59·6		·2	·4	16·1	6·3	4·8	3·1	6·3	3·1	·8	100·7	...
	59·6			1·9	20·4	5·9		3·7	6·8	1·5	1·3	100·9	...
	59·3				16·8	4·0	4·8	3·8	6·9	2·6	1·9	100·0	2·77
	58·6	·2	·1	3·6	16·8	3·0	4·5	2·1	7·7	3·2	1·2	101·0	2·81
	58·4			·3	16·7		7·2	4·8	6·8	4·7	1·2	100·1	...
	58·4			1·1	17·1	·8	4·6	5·2	7·6	4·2	1·0	100·0	2·94
	58·3				16·1	4·5	4·8	2·7	11·0	1·7	·9	100·0	2·84
	58·2			·8	14·9	8·3		3·0	8·0	3·3	3·5	100·0	...
	58·1	·7	·2	2·1	15·5	1·7	5·2	4·8	7·0	2·9	2·1	100·3	...
	58·1			1·5	13·2	10·1		4·5	7·0	2·6	1·6	98·6	2·77
	57·6		·8	·8	14·7	8·6		5·0	7·5	3·5	1·9	100·3	...
	57·6		·2	1·8	14·2	6·0	4·2	4·2	6·9	3·0	1·1	99·1	2·77
	57·5	1·1	·2	2·0	17·2	4·3	3·0	3·4	6·5	3·9	1·3	100·2	...
	57·4	1·0	·2	1·3	17·7	2·2	5·2	3·4	6·8	3·1	1·8	100·3	...
	56·9	1·1	·2	2·3	15·5	2·3	5·0	5·7	5·8	2·5	2·7	100·2	2·79
	56·2			3·0	16·5	10·6		2·2	7·0	2·9	·8	99·2	...
	55·7		·8	4·2	16·1	10·9		2·9	5·9	3·8	·5	100·8	...
	54·3	3·3	·1	·5	13·2	3·3	8·8	4·8	8·8	3·5	·2	100·8	...
	54·2	1·6	·4	3·8	16·3	5·1	3·5	3·0	6·3	4·0	2·0	100·2	2·66

—continued.

Whitman Cross, Table XIII.	Sodalite syenite.	Square Butte, Montana.
N. Jahrb., 1892, vol. ii., p. 146.	Foyaite.	Sierra de Tingua, Brazil.
W. C. Brögger, *Z.K.*, xvi., p. 38, 1890.	Nephelite porphyry.	Norway.
N. Jahrb., 1901, vol. ii., p. 75.	Nephelite syenite.	Siberia.
Zeitsch. d. d. geol. Ges., part iv., p. 290, 1868.	Leucitophyre.	Bolsena.
Q. Journ., 1895, p. 193.	Rhomben porphyry.	Rüs, Norway.
Geol. Survey of India.	Elaeolite syenite, reduced to 100·0.	Salem, India.
W. C. Brögger, *Z.K.*, xvi., p. 116, 1890.	Syenite pegmatite.	Norway.
N. Jahrb., 1893, vol. ii., p. 347.	Elaeolite tinguaite.	Magnet Cove, Ark.
Whitman Cross, Table XII.	Nephelite syenite.	Transvaal.
N. Jahrb., 1902, vol. ii., p. 382.	Eudialite lujavrite.	Lujaor Urt, Kola.
N. Jahrb., 1893, vol. ii., p. 347.	Leucite tinguaite.	Magnet Cove, Ark.
Whitman Cross, Table XII.	Nepheline syenite.	Cascada, Brazil.
N. Jahrb., 1902, vol. ii., p. 382.	Chibinite.	Umtok, Kola.
Whitman Cross, p. 203.	Tinguaite.	Bearpaw Mts., Mont.
N. Jahrb., 1893, vol. ii., p. 347.	Resembl. elaeol. syen. porphyry.	Arkansas.
N. Jahrb., 1903, vol. i., p. 427.	Leucitite.	Kamerun.
Geol. Survey of India, 1898, p. 9.	Hyp. hornbl. granite.	Perumbakam.
N. Jahrb., 1900, vol. ii., p. 234.	Hyp. andesite.	Mexico.
Chelius and Klemm., *Erl. G. Kte. Hesse*, iii., p. 8, 1894.	Hornbl. granite.	Hessen.
N. Jahrb., 1905, vol. ii., p. 382.	Tonalite gneiss.	Brix.
T. Wada, *Mt. D. Ges. Ostas.*, v., p. 74, 1889.	Augite andesite.	Japan.
Q. Journ., 1890, p. 349.	Augite andesite.	Hungary.
Q. Journ., 1884, p. 224.	Andesite.	Great Ayton, England.
N. Jahrb., 1904, vol. ii., p. 223.	Amphibole andesite.	Montenegro.
Roth's *Beiträge. zur Petrog.*, 1873, p. xlvi.	Andesite.	Tunguragua.
Q. Journ., 1905, p. 589.	Enstatite diorite.	Penmaenmawr.
Q. Journ., 1884, p. 235.	Felspathic dyke.	Teynmouth.
N. Jahrb., 1903.	Trachyte.	Mt. Cimina, Italy.
H. W. Turner, 17, *A.R. U.S.G.S.*, i., p. 731, 1896.	Diorite.	California.
Q. Journ., 1884, p. 224.	Andesite dyke.	Cockfield.
A. Bergeat, *Sb. Münch. Ak.*, xx., p. 219, 1899.	Ol. pyr. andesite.	Lipari Islands.
Q. Journ., 1884, p. 224.	Andesite (intrus. dyke).	Cockfield.
N. Jahrb., 1905, vol. ii., p. 234.	Hyp. andesite.	John Day Basin.
H. W. Turner, 17, *A.R. U.S.G.S.*, i., p. 702, 1896.	Quartz diorite gneiss.	California.
K. A. Lossen, *Jb. Pr. G.L.A.*, x., p. 309, 1892.	Enstatite porphyrite.	Harz.
A. Lagorio, *Guide*, Exc. vii., *Cong. G. Int.*, xxxiii., p. 27, 1897.	Quartz diorite.	Crimea.
E. Drasche, *Vh. Wien. G.R.A.*, 1884, p. 196.	Andesite.	Persia.
N. Jahrb., 1906, vol. i., p. 375.	Amphibolite.	Campo, Elba.
K. A. Lossen, *Jb. Pr. G.L.A.*, x., p. 316, 1892.	Basalt.	Harz.

List of 1000 Rock

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp.Gr
33	53·4			5·1	11·5	8·4	(4·9)	2·6	13·2	·7	·6	100·4	2·74
	51·3	1·4		(2·9)	15·5	3·2	7·1	4·4	9·2	3·8	1·2	100·0	2·92
25 rocks	57·6	·4	·2	1·7	16·1	4·9	3·7	3·7	7·4	3·0	1·4	100·1	2·79
	±1·1				±1·4	±1·6		±1·4		±·9		±6·4	
34	*b a a a a b*												
	59·7			2·1	16·2	1·9	8·6	2·7	4·8	1·0	3·1	100·3	...
	57·4			·8	16·9	2·5	5·2	5·5	7·3	3·3	1·9	100·7	...
	57·4			·7	15·7	2·1	5·2	8·8	4·9	3·1	1·5	99·9	2·83
	56·8			1·5	16·7	3·8	4·6	3·9	6·4	3·4	2·8	99·9	2·66
	56·7		·2	1·1	18·8	·2	7·0	5·6	7·3	2·3	·8	100·0	2·88
	56·4		1·1	2·3	16·8	3·3	7·1	3·5	5·6	1·2	3·1	100·4	2·63
	56·2		·3	1·0	16·1	4·9	4·4	4·6	7·0	3·0	2·4	99·9	2·74
	56·0	·7		·7	12·5	·5	15·9	6·1	4·2	2·0	·7	99·3	...
	55·8	1·1	·2	1·3	17·1	4·1	3·8	5·1	7·4	2·9	1·7	100·4	...
	55·4			1·2	16·8	9·3		5·2	7·6	3·6	1·4	100·5	...
	54·0			1·3	13·9	15·4		3·8	9·8	1·1	·7	100·0	2·90
	54·0	·9	·1	·7	16·7	1·4	7·7	5·7	8·8	3·0	·7	99·7	...
	53·8	1·7	·3	1·8	15·8	6·9	4·3	5·6	7·7	3·0	·7	101·5	2·91
	53·5			2·4	13·4		16·7	3·1	10·9			100·0	...
	52·8	2·1	·5	1·0	12·5	9·1	4·0	4·5	8·1	2·6	2·4	99·6	
	52·4	1·1	·1	1·8	13·6	2·7	10·1	5·5	10·0	2·3	·4	100·0	...
	52·2	1·4	·5	·3	14·6	10·8	3·2	5·0	8·7	1·8	·6	99·1	...
	52·2	2·1	·1	1·1	12·2	10·1	2·9	5·5	7·1	3·8	2·2	99·3	...
	51·3			1·5	17·6		12·0	5·7	9·7	·6	1·4	99·8	2·95
	51·2	2·4	·3	1·6	14·1	4·6	8·9	4·4	8·3	2·6	1·3	99·7	2·98
	50·7		2·8	2·0	14·1	8·3	4·6	6·6	7·7	2·7	1·2	100·7	...
	49·8	3·0		1·4	14·1	14·9	·1	5·6	9·4	1·7		100·0	...
	49·6	2·3	·3	3·6	15·1	1·7	8·8	5·4	7·3	4·2	·9	100·3	...
	49·0	2·8		3·3	15·3	2·6	8·2	4·9	8·2	2·5	2·6	99·4	...
24 rocks	53·9	·9	·3	1·5	15·3	5·0	6·4	5·1	7·7	2·4	1·5	100·0	2·83
	±1·6				±1·5	±2·4		±1·3		±1·2		±8·0	
35	*b a a a b a*												
	57·2				16·1		13·0	2·2	5·8	3·9	1·8	100·0	2·80
	57·0	·3	·1	2·3	14·6	9·2	4·9	1·6	4·6	4·9	·8	100·3	2·90
	56·8	1·4	·4	·4	14·3	5·8	9·5	1·6	5·3	3·4	1·7	100·6	3·02
	56·7			·7	16·9	4·1	6·3	4·6	7·2	4·6	·6	101·7	...
	56·2	2·2	1·1	·8	15·5	1·5	9·9	1·8	5·4	2·8	2·8	100·0	...
	55·8			2·4	16·0	12·5		2·2	7·1	2·2	1·9	100·1	2·70
	55·7			2·3	17·5	5·2	4·7	2·9	6·7	2·4	2·1	99·5	...
	53·8			3·3	18·4	8·3		5·6	3·2	7·0	1·2	100·8	...
	53·7			1·6	18·2	10·6		5·2	6·7	2·7	1·0	99·6	...
	52·5	1·3	·3	1·2	15·5	5·1	9·8	2·6	7·3	3·3	1·8	100·5	2·83
	52·4	·2		2·3	15·1	2·3	10·1	5·4	7·3	4·0	·9	100·0	2·97

—*continued.*

Q. Journ., 1904, p. 481.	Palagonite tuff.	Pontesford Hill.
Zeitsch. d. d. geol. Gesell., xix. part ii., p. 340.	Anamesite (mean of 2).	Dietesheim, Hessen.
C. Klein, *Sb. Berl. Akad.*, 1888, p. 113.	Leucite phonolite.	L. Bolsena, Italy.
J. P. Iddings, 12, *A.R.U.S.G.S.*, i., p. 627, 1891.	Pyroxene porphyrite.	Yellowstone Park.
G. F. Becker, *M.U.S.G.S.*, xiii., p. 159, 1888.	Basalt.	California.
Q. Journ., 1893, p. 141.	Anamesite.	Capraja.
W. Cross, *B.U.S.G.S.*, 148, p. 159, 1897.	Porphyry.	Colorado.
C. Klein, *Sb. Berl. Akad.*, p. 119, 1888.	Augite andesite.	Italy.
W. Cross, *B.U.S.G.S.*, i., p. 26, 1883.	Hyp. andesite.	Colorado.
A. Hamberg, *G.F.F.*, xxi., p. 523, 1899.	Hyp. andesite.	Spitzbergen.
J. P. Iddings, 12, *A.R.U.S.G.S.*, i., p. 648, 1891.	Pyroxene andesite.	Yellowstone Park.
C. v. John, *Jb. Wien. G.R.A.*, xxxviii., p. 350, 1888.	Augite diorite.	Hungary.
Roth's *Beiträge zur Petrog.*, 1873, p. xlvi.	Olivine basalt.	Java.
Q. Journ., 1905, p. 589.	Hypersthene gabbro.	Pennsylvania.
N. Jahrb., 1892, vol. i., p. 278.	Plagioclase basalt.	Bühl, near Cassel.
J. Siemiradzki, *N.J.*, 1886, ii., p. 176.	Corsite.	West Indies.
N. Jahrb., 1904.	Dolerite.	Breitfürst.
W. H. Hobbs, *B.U.S.G.S.*, 168, p. 35, 1900.	Olivine basalt.	Connecticut.
N. Jahrb., 1903–4, suppl. 18, p. 467.	Dolerite Schlüchtern.	Ofleiden.
N. Jahrb., 1903–4, suppl. 18, p. 467.	Dolerite Schlüchtern.	Ofleiden.
Q. Journ., 1884, p. 235.	Basaltic dyke.	Helt.
Q. Journ., 1884, p. 654.	Dolerite without olivine.	Whinsill, Durham.
N. Jahrb., 1903–4, suppl. 18, p. 467.	Dolerite.	Ofleiden.
Q. Journ., 1884, p. 654.	Diabase.	Eisfjord.
N. Jahrb., 1903, vol. i., p. 430.	Diabase.	Avoostock.
Hatch, *Petrology*, p. 179.	Dolerite.	Rowley.
A. Geikie, p. 167, third edit.	Augite andesite, average.	
N. Jahrb., 1904, vol. ii., p. 223.	Diorite porphyrite.	Montenegro.
N. Jahrb., 1901, vol. i., p. 245.	Trap.	Rocky Hill.
Q. Journ., 1905, p. 589.	Hypersthene norite.	Tyrol.
A. H. Brooks, *B.U.S.G.S.*, 168, p. 55, 1900.	Quartz gabbro.	Georgia.
Q. Journ., 1893, p. 545.	Augite andesite.	Arran.
Hatch, *Petrology*, p. 147.	Diorite.	Charnwood Forest.
N. Jahrb., 1902, vol. ii., p. 237.	Apoandesite.	Neposet Valley, Mass.
J. J. H. Teall, *Q.J.G.S.*, xl., p. 240, 1884.	Diabase.	Bellingham, England.
A. N. Winchell, *A.G.*, xxvi., p. 293, 1897.	Orthoclase gabbro.	Minnesota.
W. C. Day, 18, *A.R.U.S.G.S.*, v., p. 958, 1897.	Diabase.	Connecticut.

List of 1000 Rocks

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.
35	51·0	·7	·5	3·9	15·6	8·5	1·4	5·2	6·5	3·1	3·4	99·8	...
	50·7	1·9		2·0	14·8	3·5	9·3	5·9	8·2	2·8	1·4	100·5	2·94
	50·3		1·3	·9	13·6	10·8	6·5	2·3	7·1	4·2	1·8	98·8	2·88
	49·2	1·8	1·5	2·7	15·8	6·1	7·5	3·0	5·3	5·2	2·1	100·5	2·79
	49·1			4·3	17·2	·5	12·2	3·6	9·7	2·9		99·9	...
	46·8	2·3	1·0	5·2	14·9	7·8	5·2	2·9	6·4	5·0	2·4	99·9	...
17 rocks	53·2	·7	·3	2·1	15·9	6·0	6·5	3·5	6·5	3·8	1·6	100·1	2·88
	±1·8				±1·1	±1·7		±1·9		±1·0		±7·5	
36	*b a a a b b*												
	58·6			·6	22·9	2·5	7·0	4·0	1·7	3·2	·5	101·0	2·85
	58·0			·4	22·2	2·0	7·2	3·8	2·2	3·2	·7	99·7	2·85
	55·9				15·1		15·2	4·2	6·5	2·5	·9	100·0	...
	55·9	·8		·8	12·6	6·5	10·2	6·2	3·5	2·2	·7	99·4	...
	55·4	·8		1·6	20·0	2·7	5·1	6·6	3·6	3·0	1·9	100·7	...
	54·0				17·0	12·0		6·0	7·0	2·0	2·0	100·0	2·95
	52·3			·4	18·4	5·9	11·1	1·0	7·3	2·9	·5	99·7	...
	51·4		·1	·7	18·2	14·5		3·1	10·3	·8	·2	99·3	...
	51·3			5·0	20·4	3·0	4·1	7·2	4·5	1·8	2·9	100·3	2·84
	50·9	·5	·8	2·0	17·5	14·4		2·6	6·9	3·4	1·7	100·7	...
	50·0			2·0	16·0		14·0	5·0	8·0	4·0	1·0	100·0	2·98
	49·6	2·1		1·2	15·6	8·8	4·7	7·1	8·1	2·2	1·1	100·4	2·93
	49·3		·3	2·9	16·9	6·5	6·9	4·8	7·6	3·4	·7	99·3	...
	48·3			3·9	18·5	14·5		4·5	5·4	3·4	1·8	100·3	...
	46·9	3·2	1·1		16·7	11·4	5·6	3·6	6·1	3·9	·8	99·2	...
	42·3	2·9	·3	6·0	16·9	5·2	11·1	6·9	3·3	4·0	·8	99·8	...
1 rocks	51·9	·6	·2	1·7	17·8	6·9	6·4	4·8	5·8	2·9	1·1	99·9	2·90
	±2·3				±2·0	±2·6		±2·0		±·8		±9·7	
37	*b a a b a a*												
	60·5			·5	20·4	1·5	2·9	2·9	6·2	3·5	1·3	99·8	...
	59·5			1·8	20·2		6·7	1·3	6·8	2·8	1·3	100·4	...
	58·4			·2	19·3	5·8	·6	3·1	6·8	3·8	2·3	100·3	...
	58·3			·2	20·8	2·6	3·8	2·6	8·4	4·2	·7	101·6	2·82
	58·2	·2	·3	1·3	19·2	2·0	4·4	3·2	5·6	4·5	1·8	100·8	...
	57·3	·4		(·5)	24·9	1·1	·9	·3	8·0	5·4	1·2	100·0	...
	57·0		·2	·2	17·5	4·6	4·4	3·2	8·5	3·0	1·2	99·8	...
	56·5	·5	·1	·7	18·1	4·3	2·8	4·5	8·2	3·2	1·1	100·1	...
	56·1			5·1	17·2	4·8		2·3	11·2	2·0	1·4	100·2	...
	56·0				27·6	1·7		·7	11·9	3·8	·1	101·7	...
	55·9	·2		2·7	22·4	2·5	1·8	3·0	9·2	1·8	·4	90·8	(2·27)
	55·5	1·0	·5	·7	16·8	4·1	3·6	3·0	7·1	4·3	3·6	100·2	2·79
	55·1	·7	·4	1·1	20·3	1·5	4·4	1·8	7·2	4·3	2·8	100·0	...
	53·7	·9	·5	4·9	17·0	5·0	2·4	1·8	10·2	3·5	·8	100·6	2·74

—continued.

J. P. Iddings, *M.U.S.G.S.*, p. 340, 1899.	Hornblende basalt.	Yellowstone Park.
Q. Journ., 1884, p. 654.	Enstatite diabase.	Whinsill, Northumberland.
N. Jahrb., 1903–4, suppl. 18, p. 467.	Dolerite.	Schwarzenfelsen.
N. Jahrb., 1906, vol. ii., p. 71.	Mugearite.	Skye.
N. H. Winchell, 21, *A.R.G. Nh. S. Minn.*, p. 151, 1893.	Gabbro.	Minnesota.
N. Jahrb., 1903, vol. i., p. 430.	Teschenite.	Aroostock County.
N. Jahrb., 1900, vol. ii., p. 400.	Norite.	Le Pellet, France.
N. Jahrb., 1900, vol. ii., p. 400.	Norite.	Le Pellet, France.
Genth, *Ann. d. Chem. u. Pharm.*, 1848, p. 22.	Andesite.	Iceland.
N. Jahrb., 1901, vol. i., p. 72.	Basalt slag (two spec.).	König, Karl Land.
N. Jahrb., 1904, vol. i., p. 65.	Diorite.	Steiermark.
A. Geikie, p. 165, third edit.	Diorite, average.	
W. S. Bayley, *B. U.S.G.S.*, 150, p. 286, 1898.	Gabbro.	Minnesota.
S. P. Smith, *Eruption of Tarawera, Wellington*, 1887, p. 76.	Lapilli.	New Zealand.
Q. Journ., 1883, p. 296.	Basic tuff.	St. Davids.
N. Jahrb., 1901, vol. ii., p. 59.	Konga diabase.	Molle.
A. Geikie, p. 145.	Diabase, average, roughly.	
Zeitsch. d. d. geol. Ges., xix., part ii., p. 315.	Anamesite.	Hessen.
N. Jahrb., 1905, vol. ii., p. 232.	Basic diorite.	Tanjaro.
C. v. John, *G.R.A.*, xxxiv., p. 121, 1884.	Diabase.	Persia.
A. Wichmann, *Z.D.G.G.*, xxxvi., p. 494, 1884.	Mica porphyry.	Nain, Labrador.
H. E. Gregory, *B.U.S.G.S.*, 165, p. 179, 1900.	Diabase glass.	Maine.
N. Jahrb., 1901, vol. ii., p. 233.	Hornbl. granite.	Pyrenees.
N. Jahrb., 1893, vol. i., p. 72.	Pyroxene andesite.	Tokay.
N. Jahrb., 1906, vol. i., p. 222.	Pyroxene andesite (mean of three analyses).	Kaba, Sumatra.
Geol. Survey, India.	Granitite.	S. Arcot.
N. Jahrb., 1901, vol. i., p. 62.	Quartz diorite (tonalite).	Gilgen.
N. Jahrb., 1904, vol. ii., p. 399.	Andesite.	Bergen, Western Norway.
N. Jahrb., 1905, vol. ii., p. 75.	Augite andesite.	Marian Island.
N. Jahrb., 1893, vol. ii., p. 339.	Quartz basalt.	California.
Q. Journ., 1884, p. 214.	Andesite dyke (altered).	Cockfield.
Q. Journ., 1884, p. 432.	Pyroxene andesite.	Tokio.
H. S. Washington, *J.G.*, iii., p. 150, 1895.	Hornbl. andesite segregation.	Greece.
W. Cross, 16 *A.R.U.S.G.S.*, ii., p. 45, 1895.	Diorite.	Colorado.
N. Jahrb., 1903, vol. i., p. 432.	Pinto diorite.	Montana.
N. Jahrb., 1899, vol. i., p. 82.	Hypersthene andesite.	Mexico.

LIST OF 1000 ROCKS

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.
37	54·7	1·1		3·3	18·1	3·6	3·3	3·9	6·4	4·1	2·0	100·5	2·68
	54·2			1·0	15·2	2·8	9·5	2·9	8·6	5·8	·9	101·0	...
	53·9			4·6	18·0	4·9		4·6	7·6	3·9	1·1	98·6	...
	53·5	·4		1·5	22·2	3·6	3·0	2·0	9·4	4·3	·6	100·6	2·80
	52·8			·1	28·6	·2	·4	·3	12·2	4·8	·6	100·0	...
	52·4	·2	2·0	3·2	19·5	4·2	3·8	2·3	8·6	4·5	1·1	101·6	...
	48·3	1·0	·2	8·5	17·1	1·9	4·9	3·1	9·8	3·1	2·4	100·2	2·73
21 rocks	55·6	·3	·2	2·0	20·0	2·7	3·3	2·6	8·5	3·9	1·4	100·4	2·76
	±1·5				±2·6	±1·9		±1·4		±1·1		±8·5	
38	*b a a b a b*												
	58·3			1·0	22·3	4·9	1·2	2·0	7·3	2·8	1·4	100·9	...
	56·3			·6	20·0		8·0	4·1	7·2	2·7	1·7	100·6	...
	56·2			·7	21·2	1·6	4·2	4·0	6·7	3·0	1·7	99·5	...
	56·2	·7	·5	1·0	16·8	3·0	4·3	3·8	6·7	2·5	4·5	100·0	...
	55·8	·3		1·2	19·0	5·6	3·2	2·8	7·4	3·1	1·2	99·6	2·65
	55·2			·8	20·2		8·2	5·0	8·7	1·8	·3	100·1	...
	54·3			1·0	15·9	6·8	5·8	3·3	11·3	1·9	1·1	101·5	2·77
	53·6			1·3	20·8	1·0	7·6	1·6	9·7	3·3	1·6	100·5	...
	52·8	·9	·1	(·8)	18·8	3·3	4·9	5·2	9·6	3·2	·6	100·2	...
	52·0	·4	1·0	·6	17·2	8·2	2·0	5·4	8·2	3·8	·9	99·7	2·72
	49·9			3·0	18·9	1·5	7·8	5·6	8·5	2·5	1·6	99·3	2·88
	49·6	·5	·6	5	17·8	2·8	9·5	5·9	9·7	2·9		99·8	2·97
	49·5	·3		·7	20·4	1·3	9·5	5·3	10·0	2·7	·2	100·0	2·96
	47·3	·5		7·0	20·5	3·2	5·1	4·6	6·5	3·9	1·8	100·4	...
14 rocks	53·3	·3	·2	1·4	19·3	3·1	5·8	4·2	8·4	2·9	1·3	100·1	2·82
	±1·9				±1·5	±1·5		±1·9		±·9		±7·7	
39	*b a a b b a*												
	55·1			3·8	22·6	6·6		3·2	2·7	4·8	1·2	100·0	...
	54·8			1·4	25·5	1·6	1·6	2·0	6·1	5·7	1·9	100·5	...
	53·3	·3		1·4	16·6	8·3	3·2	1·1	7·3	5·3	3·5	100·4	2·67
	53·3			·1	17·3	9·0	3·4	·7	9·1	3·4	3·4	99·8	2·69
	53·0			2·6	20·1	9·4		2·6	6·1	4·5	1·3	99·7	...
	52·7	·6	·8	2·8	18·9	11·0		2·6	5·3	4·0	2·1	100·8	...
	52·6			3·3	17·3	11·8		2·6	6·5	4·2	2·4	100·7	2·72
	51·4	·8	·3	1·6	20·2		7·3	1·5	5·8	4·5	6·7	100·4	...
	51·3			·6	25·2	2·9	8·4	4·0	2·5	3·8	·8	99·5	2·88
	51·1			1·2	22·2		9·3	2·1	6·1	4·1	3·3	99·3	...
	50·6	·5	·8	2·0	18·5	13·8		2·7	6·2	3·7	1·9	100·7	...
	50·1	1·4	·4	4·5	18·9	3·5	3·8	2·1	6·7	4·1	4·6	100·2	...
	48·9	2·5	1·2	·7	19·4	4·3	5·1	2·0	8·0	5·4	1·9	99·4	...
	47·7	·8	1·2	5·2	19·3	3·9	4·9	3·9	6·2	3·6	4·6	101·2	...
	47·0	·6		6·4	22·0	3·6	4·7	4·1	6·1	4·0	2·1	100·6	...

continued.

K. A. Lossen, *Jb. Pr. G.L.A.*, x., p. 316, 1892.	Porphyrite.	Harz.
W. A. MacLeod, *Tr. N.Z. Inst.*, xxxi., p. 487, 1899.	Dolerite.	New Zealand.
N. Jahrb., 1892, vol. i., p. 313.	Hyp. andesite.	Peel Island.
Q. Journ., 1894, p. 323.	Quartz gabbro.	Carrock Fell.
N. Jahrb., 1904, vol. ii., p. 399.	Labradorite rock.	Holsenoe, Norway.
Chelius and Klemm, *Erl. G. Kte. Hesse*, iv. p. 37, 1896	Diabase.	Hessen, Darmstadt.
K. A. Lossen, *Jb. Pr. G.L.A.*, x., p. 280, 1892.	Melaphyre.	Harz.
N. Jahrb., 1903, suppl., p. 510.	Andesite.	Piatra Mori.
Fuchs, *N. Jahrb. für Min.*, 1862, p. 813.	Syenite.	Harz.
N. Jahrb., 1901, vol. ii., p. 233.	Granite.	Pyrenees.
N. Jahrb., 1900, vol. i., p. 71.	Augite latite (lava).	Sierra Nevada.
H. S. Washington, *J.G.*, iii., p. 150, 1895.	Hornbl. andesite.	Greece.
N. Jahrb., 1893, vol. i., p. 72.	Pyroxene andesite.	Tokay.
N. Jahrb., 1901, vol. i., p. 70.	Hyp. andesite.	Alboran Island.
Prestwich, vol. i., p. 41.	Gabbro.	Harz.
Q. Journ., 1902, p. 369.	Ashes.	St Vincent.
N. Jahrb., 1900, vol. ii., p. 232.	Plag basalt.	Puebla, Mexico.
F. Slavik, *cf. N.J.*, 1901, i., p. 63.	Mica diabase.	Bohemia.
W. S. Bayley, *J.G.*, iii., p. 10, 1895.	Gabbro.	Minnesota.
N. Jahrb., 1900, vol. ii., p. 400.	Gabbro.	La Morandière.
N. Jahrb., 1901, vol. ii., p. 388.	Nepheline rock—tephrite dolerite.	Puy de Dome.
N. Jahrb., 1903, vol. i., p. 69.	Diorite porphyry.	French Alps.
N. Jahrb., 1892, vol. i., p. 326.	Granite porphyry.	Horse Race.
E. v. Seyfried, *cf. N.J.*, 1898, ii., p. 61.	Tephrite.	Rhöngebirge.
A. Liversidge, *Journ. R. Soc. N.S.W.* xx., p. 237, 1887.	Glassy lava.	New Hebrides.
Q. Journ., 1888, p. 303.	Tachylite.	Ardtun.
N. Jahrb., 1901, vol. ii., p. 59.	Kullaite.	Kullen.
Q. Journ., 1880, p. 455.	Basalt glass.	Portree.
N. Jahrb., 1893, vol. ii., p. 347.	Felsitic porphyry.	Arkansas.
N. Jahrb., 1900, vol. ii., p. 400.	Norite.	Les Prinaux.
Fuchs, *N. Jahrb. für Min.*, 1862, p. 812.	Hornbl. diorite.	Harz.
N. Jahrb., 1901, vol. ii., p. 59.	Konga diabase.	Bökebolet.
F. Graeff, *cf. N.J.*, 1890, ii., p. 65.	Tephrite.	Kaiserstuhl.
N. Jahrb., 1904, vol. ii., p. 410.	Essexite.	Mt. Royal.
N. Jahrb., 1903, vol. i., p. 425.	Trachydolerite.	Celebes.
N. Jahrb., 1901, vol ii., p. 388.	Teschenite.	Puy de Dome.

LIST OF 1000 ROCK

Group.	SiO_2.	TiO_2.	P_2O_5	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.
39	46·3	5·4	·5		18·0	6·2	7·1	3·7	8·2	3·9	·9	100·3	...
	46·1	1·2		2·0	21·2	5·1	8·0	2·7	10·4	1·4	·2	98·3	...
17 rocks	50·9	·8	·3	2·3	20·2	5·6	4·3	2·6	6·4	4·1	2·5	100·1	2·94
	±1·7				±2·1	±2·3		±1·3		±1·5		±8·9	
40	*b a a b b b*												
	51·8			·8	26·4	11·1		5·1	1·5	3·1	·2	100·0	2·88
	50·4	·9		·9	22·2	9·9	3·6	1·5	8·4	·9	1·8	100·5	2·92
	50·1			1·0	27·6	3·7	9·1	3·9	1·5	3·0	·8	100·7	2·92
	50·0	·6		1·1	22·5	2·2	6·6	3·7	6·8	5·0	2·7	101·2	...
	49·7			·8	21·1	3·2	6·0	5·0	9·3	2·3	1·7	100·0	...
	49·4			·5	23·1	5·1	6·0	3·9	9·1	1·9	1·0	100·0	
	49·4			·6	28·0	12·3		3·5	2·7	3·5	·9	101·0	2·92
	49·2	·6	·7	1·3	19·1	13·6		3·1	7·2	3·6	1·9	100·3	...
	49·1			2·3	19·4	10·6	2·6	4·4	7·7	3·3	1·0	100 5	2·76
	48·8	1·0	·7	·6	18·0	·4	14·5	3·4	8·8	1·6	2·4	100·2	2·99
	47·9	·6		·8	19·9	4·9	9·8	4·6	8·6	2·8	·6	100·4	2·93
	47·3	·7		2·7	20·2	13·3		3·2	7·1	3·9	2·2	100·6	
	47·3		·7	·4	19·6	7·2	6·8	4·5	8·0	3·7	1·7	99·9	...
	47·3		·6	2·3	18·5	14 6		2·3	7·6	1·0	6·1	100·3	...
	46·4		·7	3·2	15·9	15·0		1·7	9·3	·5	6·9	99·6	...
15 rocks	48·9	·2	·2	1·3	21·5	7·2	5·6	3·6	6·9	2·7	2·1	100·3	2·93
	±1·0				±2·9	±1·5		±2·0		±1·3		±8·7	
41	*b a b a a a*												
	59·3			1·3	9·2	6·9	4·4	3·6	12·0	1·9	2·3	101·0	...
	57·6			·9	16·5	1·2	4·9	7·7	7·4	3·6	1·0	100·8	...
	57·3	·6	·2	·4	16·4	1·7	4·8	6·7	7·7	3·0	1·6	100·4	...
	56·7	·6	·2	·3	15·8	1·3	5·5	7·2	7·7	3·4	1·6	100·2	...
	56·5	·5	·2	3	17·5	1·3	5·1	5·9	8·1	3·5	1·5	100·6	...
	56·2			4	16·6	1·5	5·5	7·3	7·6	3·6	1·6	100·2	...
	55·9			·3	17·3	1·5	5·2	7·3	8·0	3·3	1·3	100·2	...
	54·6	·6	·1	2·6	12·1	1·8	5·2	11·9	7·7	2·4	1·0	100·0	...
	54·6	·5	·2	·3	16·0	1·0	6·2	8·7	8·9	3·1	1·2	100·6	...
	54·4	·7	·6	3·1	15·3	·7	5·2	6·7	8·3	4·2	1·0	100·2	...
	54·4	·8	·6	·6	14·3	3·3	4·2	6·1	8·2	3·4	4·2	100·2	...
	54·2	·9	·5	2·4	14·4	2·3	4·8	7·7	7·0	2·6	3·3	100·1	2·78
	53·6		·9	2·6	14·2	1·5	8·1	7·1	8·5	1·8	2·0	100·3	2·79
	53·0		·6	2·2	13·1	8·2	2·5	5·2	10·6	3·3	2·0	100·7	...
	53·0	1·2	·4	·8	16·7	3·8	3·5	7·0	8·7	3·6	1·3	99·9	...
	52·5			2·1	12·2	3·5	5·2	9·9	9·7	2·8	2·3	100·2	2·92 !
	52·3	·6	·6	(5·3)	14·9	3·5	3·7	5·8	6·3	2·9	3·8	99·9	...
	51·2	1·5		6·1	7·2	7·8	2·3	10·6	7·3	2·5	3·6	100·0	...
	50·7	1·1	·3	3·4	16·0	4·4	4·3	7·1	9·1	2·9	1·1	100·4	...

—*continued.*

A. B. Lyons, *A.J.S.*, ii., p. 424, 1896.	Basalt.	Hawaii.
N. *Jahrb.*, 1905, vol. ii., p. 248.	Andesine diabase.	Jenisej.
N. *Jahrb.*, 1900, vol. ii., p. 400.	Norite.	St Michel.
Geol. Mag., 1904 (H. Warth).	Dolerite.	Poonah.
N. *Jahrb*, 1900, vol. ii., p. 400.	Norite.	Pellet.
N. *Jahrb.*, 1905, vol. i., p. 437.	Micro-essexite.	Nosykomba.
N. *Jahrb.*, 1901, vol. ii., p. 233.	Mica diorite with quartz.	Valbonne Valley, Pyrenees.
Wartha.	Lava.	Teneriffe.
N. *Jahrb.*, 1900, vol. ii., p. 400.	Norite.	Pellet.
N. *Jahrb.*, 1901, vol. ii., p. 59.	Diabase porphyrite with olivine.	Tanga.
Hatch, *Petrology*, p. 191.	Olivine basalt.	Garlton Hills.
W. H. Hobbs, *B.M.C.Z.*, xvi., p. 9, 1888.	Diabase.	Medford, Massachusetts.
A. N. Winchell, *A.G.*, xxvi., p. 374, 1900.	Diabase.	Minnesota.
Merrill, p. 220.	Diabase.	Medford.
N. *Jahrb.*, 1904, vol. ii., p. 400.	Mangerite.	Manger, Norway.
N. *Jahrb.*, 1892, vol. i., p. 317.	Leucite basalt.	El Capitan, N. S. Wales.
N. *Jahrb.*, 1892, vol. i., p. 317.	Leucite basalt.	Pryrock, N. S. Wales.
N. *Jahrb.*, 1904, vol. i., p. 54.	Minette.	Strakoniz, Bohemia.
N. *Jahrb.*, 1893, vol. ii., p. 339.	Quartz basalt.	California.
J. S. Diller, *A.J.S.*, xxxiii., p. 49, 1887.	Quartz basalt.	California.
N. *Jahrb.*, 1893, vol. ii., p. 339.	Volcanic bomb.	California.
N. *Jahrb.*, 1893, vol. ii., p. 339.	Lapilli.	California.
N. *Jahrb.*, 1893, vol. ii., p. 339.	Lava.	California.
N. *Jahrb.*, 1893, vol. ii., p. 339.	Volcanic sand.	California.
Whitman Cross, Table XIII.	Quartz diorite.	California.
N. *Jahrb.*, 1893, vol. ii., p. 339.	Lava.	California.
Q. *Journ.*, 1905, p. 589.	Quartz norite.	St David's Head.
N. *Jahrb.*, 1903, vol. i., p. 432.	Monzonite.	Montana.
K. A. Lossen, *Jb. Pr. G.L.A.*, x., p. 309, 1892.	Olivine weisselbergite.	Prussia.
G. Mercalli, *Att. Soc. Ital.*, *Milano*, xxx., p. 371, 1887.	Andesite.	Tuscany.
G. Mercalli, *Gior. Min.*, iii., p. 102, 1892.	Basalt.	Vulcano.
N. *Jahrb.*, 1905, vol i., p. 254.	Basalt.	Redcove, Yellowstone Park.
Teall, *Brit. Petrog.*, p. 265.	Hornbl. angite diorite.	Sutherland.
H. Loretz, *Jb. Pr. G.L.A.*, viii., p. 112, 1888.	Kersantite.	Thüringerwald.
N. *Jahrb.*, 1903, vol. i., p. 414.	Minette.	Zampach, near Eule.
Hague and Jagger, *B.U.S.G.S.*, 168, p. 97, 1900.	Andesite.	Yellowstone Park.

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.
41	49·6	·7	6·0	·4	9·6	5·6	5·2	1·3	13·9	4·9	3·2	100·3	...
	48·4	·5	·4	3·8	13·3	4·4	3·4	8·4	10·5	3·4	3·0	100·0	...
21 rocks	54·1	·5	·6	1·9	14·2	3·2	4·7	7·1	8·7	3·1	2·1	100·3	2·79
	±1·4				±2·2	±1·4		±1·4		±·9		±7·3	
42	*b a b a a b*												
	57·3	·4		·2	14·7	1·2	4·4	7·8	6·9	1·3	6·4	100·6	...
	55·5	·2		·4	15·4	1·3	4·5	7·9	6·7	1·8	6·6	100·2	2·70
	55·0			·6	15·7	4·8	5·8	6·2	11·2	1·2	1·5	102·1	2·84
	54·5	1·1		·1	16·5	1·0	5·7	8·6	8·0	2·1	3·3	100·9	...
	54·4			5·5	12·9	7·1		12·8	5·1	2·1	·4	100·2	...
	54·1			1·5	15·0	4·1	5·1	7·3	7·7	2·0	3·5	100·3	...
	54·1	1·2		·6	16·4	1·7	5·3	8·4	8·1	2·2	3·3	101·3	...
	53·2			1·6	16·2	10·3		6·7	10·1	1·9	·4	100·4	...
	53·1			·7	15·6	2·3	8·2	5·8	11·7	1·9	1·8	101·2	(2·56)
	52·7			1·6	14·1	1·9	10·2	6·4	9·4	2·6	·9	99·8	2·97
	52·3	·6	·5	2·9	14·0	2·8	4·6	8·2	7·4	2·8	3·9	100·2	...
	52·2	·4	·1	1·4	14·7	4·1	7·7	9·4	8·4	1·5	·3	100·2	...
	52·2			2·2	15·4	4·3	5·1	8·9	8·6	2·1	·6	99·3	2·85
	51·9			1·0	15·5		12·9	4·0	13·8	·5	1·2	100·8	2·96
	51·7	1·2	·2	·3	15·2	2·1	8·5	8·2	8·7	2·3	1·8	100·2	...
	51·5			3·5	14·4	3·9	5·3	9·5	9·1	2·9	·2	100·3	...
	50·4	1·5	·5	2·3	12·3	5·7	3·2	8·7	7·1	1·0	7·5	100·4	2·88
	50·0	1·4	1·0	·8	12·3	2·9	5·8	9·2	10·2	2·2	5·0	100·0	...
	49·8	2·7	·7	4·0	15·3		7·7	6·6	7·2	2·7	4·4	101·1	...
	49·5	·8		2·3	16·8	2·0	6·6	9·3	11·2	1·6		100·0	...
	48·2	1·0	·1	2·7	18·3	1·3	6·1	10·8	9·4	1·3	·7	99·9	...
	46·0	·6	1·1	2·9	12·2	3·9	4·6	10·4	9·5	2·4	5·8	99·8	...
22 rocks	52·3	·6	·2	1·8	15·0	3·1	5·8	8·2	8·9	1·9	2·7	100·4	2·90
	±1·6				±1·2	±1·1		±1·5		±2·2		±7·6	(2·56)
43	*b a b a b a*												
	54·4	·3	·3	1·2	14·3	6·3	4·2	5·9	7·5	3·4	2·2	100·0	...
	52·8	·3	·4	·9	13·3	2·4	8·7	6·8	8·5	4·7	1·5	100·4	2·93
	52·6	1·1	·3	1·2	15·0	4·4	5·8	3·9	7·8	3·8	4·0	99·9	...
	52·0			2·0	14·0	13·0		3·0	8·0	3·0	5·0	100·0	(2·65)
	51·6	1·2		1·6	14·8	5·3	7·0	6·3	8·1	3·3	1·3	100·4	2·92
	51·2	1·0	·5	2·9	16·1	4·1	4·7	4·8	7·9	3·0	3·5	99·9	...
	51·0	1·1		1·3	15·2	8·7	3·4	4·7	11·4	2·4	1·1	100·4	2·92
	50·9	·6		·2	15·7	10·9		6·0	11·8	2·0	1·6	99·6	3·01
	50·3	1·0	·5	3·8	15·9	8·2	1·5	4·7	7·9	3·0	3·5	100·3	...
	50·2	1·1		2·4	16·0	5·1	5·8	5·6	10·9	3·4	·5	100·8	...
	49·5	2·9		·9	14·1	6·1	6·2	6·6	9·4	2·3	2·1	100·1	...
	49·5			4·8	18·0	5·1	3·0	6·2	7·8	2·5	2·3	99·1	...
	49·3	·4	·2	2·9	17·4	2·7	8·3	4·7	8·4	4·0	1·8	100·4	...

—*continued.*

N. *Jahrb.*, 1901, vol. ii., p. 239.	Pyr. apatite syenite (ijolite porphyry).	Finland.
N. *Jahrb.*, 1903, vol. i., p. 432.	Resembles augitite.	Montana.
N. *Jahrb.*, 1901, vol. i., p. 225.	Volcanic rock.	La Colonetta, Mt. Cimino.
N. *Jahrb.*, 1901, vol. i., p. 225.	Volcanic rock.	Fiescoli, Mt. Cimino.
N. *Jahrb.*, 1901, vol. i., p. 70.	Hyp. andesite.	Aboran Island.
N. *Jahrb.*, 1901, vol. i., p. 225.	Andesite.	Radicofani.
N. *Jahrb.*, 1902, vol. i., p. 313.	Andesite.	Peel Island, Japan.
Hatch, *Petrology*, p. 147.	Diorite.	Benan.
N. *Jahrb.*, 1901, vol. i., p. 225.	Andesite.	Radicofani.
N. *Jahrb.*, 1892, vol. i., p. 313.	Andesite.	Ototoshima, Japan.
N. *Jahrb.*, 1901, vol. i., p. 70.	Hyp. andesite.	Alboran Island.
Q. *Journ.*, 1884, p. 654.	Dolerite?	Mt. Holyoke, Mass.
N. *Jahrb.*, 1903, vol. i., p. 432.	Minette.	Montana.
J. B. Harrison, *Rep. G.N.W. Dist.*, ii., p. 11, 1598.	Diorite.	Br. Guiana.
N. *Jahrb.*, 1900, vol. ii., p. 232.	Hyp. plag. basalt.	Puebla, Mexico.
Q. *Journ.*, 1884, p. 235.	Basalt dyke.	Helt.
J. P. Iddings, *B. U.S.G.S.*, 148, p. 135, 1897.	Basalt.	Yellowstone Park.
N. *Jahrb*, 1892, vol. i., p. 325.	Gabbro.	Sturgeon Falls.
Whitman Cross, Table XII.	Syenite lamporphyre.	Tovo Buttes, Colo.
N. *Jahrb.*, 1903, vol. i., p. 432.	Shonkinite.	Montana.
J. S. Diller, *B. U.S.G.S.*, 148, p. 185, 1897.	Mica andesite.	New Mexico.
J. B. Harrison, *Rep. G.N.W. Dist.*, ii., p. 6, 1898.	Diabase.	Br. Guiana.
N. *Jahrb.*, 1900, vol. i., p. 75.	Norite.	Michigan.
L. V. Pirsson, *B. U.S.G.S.*, 148, p. 153, 1897.	Leucite basalt.	Montana.
J. B. Harrison, priv. contrib.	Diorite gneiss.	Br. Guiana.
N. *Jahrb.*, 1900, vol. ii., p. 232.	Plag. basalt.	Puebla, Mexico.
N. *Jahrb.*, 1906, vol. i., p. 69.	Monzoni.	Italy.
A. Geikie, ed. 1888, p. 149.	Vitreous basalt, average.	
Zeitsch. d. d. geol. Gesell., xix., part ii., p. 325.	Anamesite.	Louisa, Hessen.
Hague and Jagger, *B. U.S.G.S.*, 168, p. 98, 1900.	Augite andesite.	Yellowstone Park.
Zeitsch. d. d. geol. Ges., xix., part ii., p. 309.	Anamesite.	Eschersheim, Hessen.
Q. *Journ.*, 1897, p. 409.	Augite andesite.	Seven Pagodas, Madras.
Hague and Jagger, *B. U.S.G.S.*, 168, p. 97, 1900.	Gabbro porphyry.	Yellowstone Park.
N. *Jahrb.*, 1904, vol. i., p. 46.	Diabase.	Niedergrund.
N. *Jahrb.*, 1903–4, suppl. 18, p. 467.	Dolerite.	Grenzebach.
Q. *Journ.*, 1879, p. 166.	Mica trap.	Kentmore.
Mean of 2 = 285).	Dolerite.	Rowley Regis, mean.

LIST OF 1000 ROCKS

Group.	SiO₂.	TiO₂.	P₂O₅.	H₂O.	Al₂O₃.	Fe₂O₃.	FeO.	MgO.	CaO.	Na₂O.	K₂O.	Total.	Sp. Gr.
43	48·8	1·3	·2	4·1	16·6	5·6	5·0	6·9	8·9	2·5	·7	100·4	(2·78)
	48·5	2·2	·2	1·2	18·4	7·6	1·2	6·7	10·4	3·0	·6	100·0	...
	48·4	3·1		2·2	13·4	4·0	8·5	6·5	8·6	3·1	2·1	99·9	...
	47·7	2·6	·2	2·9	14·8	5·0	6·4	5·5	9·3	3·5	1·6	100·1	2·99
	47·4	3·1	·4	2·3 ?	16·7	4·6	6·7	4·1	10·8	3·0	·9	100·0	...
	43·7	2·8	·6	4·5	14·8	2·4	7·5	7·0	10·8	3·1	2·9	100·1	2·91
19 rocks	50·0	1·4	·2	2·3	15·5	5·2	5·6	5·6	9·2	3·1	2·1	100·1	2·95
	±1·0				±1·2	±1·1		±1·5		±1·1		±5·9	(2·71)
	b a b a b b												
44	51·2	·4	·1	·1	15·9	9·3	3·0	6·5	10·4	1·2	1·6	99·7	3·19
	51·0	1·8	·7	1·0	14·5	4·2	4·4	8·2	5·1	1·8	7·2	99·9	...
	50·2	1·9		·2	16·2	3·1	8·1	7·5	8·6	3·4	1·4	100·6	(2·79)
	50·0			1·4	16·5		12·3	5·5	9·5	2·3	1·5	99·0	...
	49·6			3·3	15·4		12·3	7·4	9·6	2·0	·9	100·4	...
	49·0	1·6		1·4	16·1	1·9	9·6	7·6	8·7	3·0	1·2	100·2	2·89
	49·0	2·2	·1	2·0	14·4	4·3	6·6	8·4	8·4	3·4	2·3	101·9	2·94
	48·8			3·6	18·1	3·5	7·2	4·9	8·4	3·7	1·9	100·1	2·79
	48·8	·9		1·0	18·7	7·2	3·4	6·0	9·5	3·2	1·1	99·9	2·91
	48·3	1·6		1·9	16·7	4·0	6·3	5·8	8·3	3·2	4·1	100·2	2·90 ?
	47·9	1·9	·3	·2	16·6	5·7	8·1	4·4	9·4	3·2	2·1	99·8	...
	47·7	1·8		1·4	19·0	·9	8·8	8·7	9·0	2·5	·5	100·3	2·89
	47·6	1·4		3·2	17·2	3·6	8·5	6·3	6·4	4·7	1·3	100·2	2·89
	46·8			1·8	17·9	5·3	5·6	7·3	8·2	3·5	2·2	98·6	...
	46·0			3·1	19·2	5·9	6·9	6·8	8·7	3·3	1·2	101·1	2·80 ?
	45·9			4·2	18·2	1·2	9·5	10·1	7·2	2·1	1·8	100·1	2·96
	45·8	1·7	·6	3·2	15·9	7·4	6·1	6·9	7·2	3·4	1·3	99·6	...
17 rocks	48·4	1·0	·1	1·9	16·9	4·0	7·5	7·0	8·4	2·9	2·0	100·1	2·93
	±1·1				±1·2	±1·1		±1·3		±1·2		±5·9	(2·79)
	b a b b a a												
45	55·7				20·4	6·4		3·8	8·3	5·7	1·9	102·4	...
	52·8	·5		1·2	17·8	1·2	4·8	4·8	12·9	3·0	·5	99·5	2·91
	51·3	·1		1·1	17·5	4·5	4·9	3·8	13·1	2·2	2·4	100·9	...
	49·8	·8	·1	2·0	20·0	6·3	·5	7·0	11·3	2·2	·6	100·6	...
	49·7		·8	2·4	14·4	4·2	3·8	6·3	10·1	3·2	5·5	100·2	...
	49·6		·3	3·6	19·2	2·1	5·0	4·9	10·0	5·6	1·0	101·4	2·78
	49·0	·6		5·3	17·7	2·1	6·5	2·1	8·4	6·8	2·1	100·6	...
	48·6	·1		3·1	20·2	1·3	3·0	7·6	14·0	2·2	·2	100·3	...
	47·8	2·3	1·3	·3	16·1	4·3	3·6	5·5	10·7	4·5	4·1	100·5	2·86
	47·8	·3	·5	·6	20·5	2·5	6·4	4·6	10·7	4·7	·5	100·1	...
	47·6	1·4	2·7	2·2	14·5	4·9	4·4	2·6	9·5	6·7	4·1	100·7	2·79
	47·3	·9	·7	1·4	20·8	1·9	4·5	6·4	13·0	2·8	·2	99·9	...
	46·6	3·0		·9	18·2	6·8		6·0	13·4	4·4	1·4	100·7	...

—continued.

F. L. Ransome, *B. U.S.G.S.*, 89, p. 58, 1898.	Basalt, not fresh.	California.
J. P. Iddings, *M. U.S.G.S.*, xxxii., ii., p. 438, 1899.	Basalt.	Yellowstone Park.
Hatch, *Petrology*, p. 179.	Dolerite.	Clee Hill.
N. Jahrb., 1892, vol. i., p. 278.	Plagioclase basalt.	Near Cassel.
N. Jahrb., 1899, suppl., p. 550.	Diorite.	Neila.
N. Jahrb., 1893, i., p. 322.	Monchiquite.	Brazil.
Q. Journ., 1897, p. 409.	Augite diorite.	Seven Pagodas, Madras.
A. Sauer, *Mt. Bad. G.L.A.*, ii., p. 258, 1892.	Durbachite.	Durbach, Bl. Forest.
N. Jahrb., 1900, ii., p. 238.	Glassy basalt.	Cockburn Is., Antarctica.
N. Jahrb., 1902, ii., p. 390.	Hornbl. mica gabbro.	Tchatch, Kasbek.
Q. Journ., 1879, p. 586.	Diabase.	Sp. Guiana, Venezuela.
N. Jahrb., 1900, ii., p. 238.	Basalt.	Cockburn Is., Antarctica.
N. Jahrb., 1892, i., p. 278.	Basalt, micaceous.	Staufenberg.
Q. Journ., 1905, p. 481.	Dolerite.	Hailstone Hill, Rowley.
N. Jahrb., 1905, ii., p. 62.	Amphibolite.	Umhausen.
Clarke, *Bull. U.S. Geol. Surv.*, No. 168, 1900, p. 140.	Olivine dolerite.	Valmont, Colorado.
W. C. Brögger, *Eg. Kg.*, iii., p. 83, 1899.	Essexite.	Norway.
A. N. Winchell, *A.G.*, xxvi., p. 181, 1900.	Olivine gabbro.	Minnesota.
A. W. Howitt, *Tr. R. Soc. Vict.*, xx., p. 53, 1884.	Diorite.	Noyana, Victoria.
N. Jahrb., 1903–4, suppl., p. 465.	Limburgite.	Steiermark.
Hatch, *Petrology*, p. 191.	Olivine basalt.	Garlton Hills.
Q. Journ., 1883, p. 303.	Diabase.	Rhosson, St Davids.
C. v. John, *Jb. G.R.A. Wien.*, xlix., p. 252, 1899.	Gabbro.	Tyrol.
Q. Journ., 1884, p. 447.	Augite andesite.	Miogi San, Japan.
Q. Journ., 1894, p. 653.	Gabbro, banded.	Isle of Skye.
N. Jahrb., 1904, vol. i., p. 68.	Monzonite.	Monzoni.
N. Jahrb., 1900, vol. i., p. 75.	Gabbro.	Michigan.
J. Steinecke, *Z. Nw. Halle*, vi., p. 12, 1887.	Leucitophyre.	Persia.
H. W. Fairbanks, *B. Dep. G. Un. Cal.*, ii., p. 30, 1896.	Augite teschenite.	California.
W. D. Matthew, *Tr. N.Y. Acad.*, iv., p. 213, 1895.	Diorite porphyrite.	New Brunswick.
Stokes, *Bull. U.S. Geol. Surv.*, No. 168, 1900, p. 225.	Diallage gabbro.	Beverly Creek.
Whitman Cross, Table XII.	Leucite tephrite.	Tetschen, Bohemia.
C. Chelius, *Notbl. Ver. Erdk.*, xviii., p. 24, 1897.	Olivine gabbro.	Hessen.
Whitman Cross, Table XII.	Tinguaite.	Two Buttes, Colo.
N. Jahrb., 1901, vol. i., p. 242.	Amphibole gabbro.	Beaver Creek, California.
N. Jahrb., 1905, vol. i., p. 437.	Nephel. gabbro.	Ampangarinana.

LIST OF 1000 ROCKS

Group.	SiO₂.	TiO₂.	P₂O₅.	H₂O.	Al₂O₃.	Fe₂O₃.	FeO.	MgO.	CaO.	Na₂O.	K₂O.	Total.	Sp. Gr.
45	46·2			7·7	17·0	5·2	·9	7·1	10·2	2·4	4·0	101·0	...
	45·7	3·3	·2	4·8	14·3	4·1	5·6	2·7	10·4	5·5	3·6	100·1	2·77
15 rocks	49·0	·9	·4	2·4	17·9	3·8	3·8	5·0	11·1	4·2	2·1	100·6	2·82
	±1·5				±1·9	±1·3		±2·5		±2·2		±9·4	
46	*b a b b a b*												
	50·3	1·8			18·7	6·1	4·3	4·7	9·4	4·1	1·4	100·6	...
	50·2	·3		(·5)	22·7	3·3	3·6	4·5	10·4	3·3	1·2	100·0	...
	50·2			1·1	19·4	8·2		7·1	10·2	3·8	1·2	101·3	not given
	49·7	·2		(·8)	20·8	1·0	5·5	6·5	10·8	3·5	1·4	100·2	...
	49·6	·4	·2	2·0	14·5	3·5	5·5	6·2	9·6	3·5	5·6	100·8	...
	49·3			·1	21·6	2·3	7·3	7·8	10·2	2·1	·3	101·0	2·97
	49·2		·3	·8	17·7	6·9	2·0	4·2	12·7	2·1	5·2	101·1	2·88
	48·4			1·1	20·3	4·0	6·4	8·1	8·8	1·8	2·0	100·9	...
	48·4				23·7	8·0		6·6	11·0	2·6	1·1	101·4	...
	47·9			5·0	19·1	4·3	2·0	6·4	6·2	2·5	5·5	98·9	...
	47·3	1·5	1·6	·1	18·3	2·2	6·9	6·8	7·9	6·0	1·0	99·6	2·86
	47·3	·3		·7	21·1	3·5	4·1	8·1	13·4	1·5	·3	100·2	2·95
	46·6			3·5	21·6	2·9	6·4	6·5	9·3	3·2	·9	100·9	...
	46·4	·3		·6	26·3	2·0	3·3	4·8	15·3	1·6	·2	100·8	2·88 ?
	45·9			1·0	21·2	2·2	7·1	7·8	10·5	3·2	1·1	100·0	2·98
	45·7			2·8	20·5	2·0	4·2	8·5	11·6	3·6	·8	99·8	...
	45·4	2·8		1·0	18·6	·8	6·8	7·5	13·2	2·3	1·3	99·5	...
	44·3			4·1	17·2	4·6	3·8	6·6	10·4	4·5	3·6	99·1	...
18 rocks	47·9	·4	·1	1·4	20·2	3·8	4·4	6·6	10·6	3·1	1·9	100·3	2·91
	±1·4				±2·0	±1·2		±1·8		±1·7		±8·1	
47	*b a b b b a*												
	51·4	·6	·7	1·1	18·5	2·9	5·2	1·3	7·3	6·7	4·4	100·1	2·75
	51·1	1·4		·9	21·1	·9	5·6	2·8	5·3	6·4	4·2	99·7	...
	49·9	1·2			24·4	6·1		3·9	9·6	5·3	·3	100·7	...
	49·3	·8	·6	3·5	16·1	7·9	3·4	2·7	8·0	5·2	3·4	100·9	2·63
	48·7	1·8			18·9	3·2	8·0	4·8	9·9	4·1	1·5	100·9	...
	48·7	2·7	1·1	1·0	17·9	3·1	6·6	3·1	7·4	6·0	2·6	100·0	...
	48·6			1·8	17·9	6·2	5·8	4·3	9·1	4·7	2·1	100·4	2·77
	48·5	1·7		2·1	21·3	1·0	5·5	4·1	7·4	4·9	3·2	99·7	...
	48·3	·3	1·7	3·0	16·6		6·5	1·2	7·8	9·4	6·5	101·3	...
	48·3			2·1	19·0	6·7	4·0	3·5	8·9	5·0	2·4	99·9	2·88
	47·4	·1			23·6	4·6	1·2	·7	4·4	15·1	2·0	99·1	...
	46·6			1·0	24·4	2·6	6·4	1·6	10·4	5·5	1·4	99·9	...
	46·5	1·0		4·7	16·2	6·2	6·1	4·0	7·4	5·9	3·1	100·9	2·72
	44·5	1·4		2·1	23·0	6·8		1·6	8·7	6·7	4·8	99·6	...
	43·7	·9	1·3	·9	19·8	3·3	3·5	3·9	10·3	9·8	2·9	100·3	...
	43·0	·6	·7		24·6	3·6	2·2	2·0	5·5	14·8	3·0	100·0	...
	42·8	1·7	1·7	1·0	19·9	4·4	2·7	1·9	11·8	9·3	1·7	98·8	...

—*continued.*

Q. Journ., 1879, p. 166.	Mica trap.	Near Staveley.
K. Gruss, *Mt. Bad. G.L.A.*, iv., p. 115, 1900.	Monchiquite.	Kaiserstuhl.
N. Jahrb., 1893, vol. i., p. 490.	Diabase.	Vogesen, Elsass.
N. Jahrb., 1904, vol. ii., p. 399.	Labradorite rock.	Lindaas, Norway.
Q. Journ., 1884, p. 456.	Diabase.	Kai Province, Japan.
N. Jahrb., 1904, vol. ii., p. 399.	Labradorite rock.	Rodoe, Norway.
L. V. Pirsson, *B.U.S.G.S.*, 148, p. 153, 1897.	Leucite syenite.	Montana.
N. Jahrb., 1900, vol. ii., p. 400.	Gabbro.	Quarry des Bois.
N. Jahrb., 1899, vol. i., p. 97.	Leucite basalt.	Morolo, Italy.
N. Jahrb., 1901, vol. ii., p. 233.	Mica diorite (mean of 2).	Valbonnev, Pyrenees.
N. Jahrb., 1900, vol. i., p. 224.	Gabbro.	Kullen.
Q. Journ., 1879, p. 174.	Mica trap.	Kendal district.
N. Jahrb., 1900, vol. ii., p. 234.	Plagiocl. basalt.	Mexico.
Q. Journ., 1902, p. 169.	Gabbro.	Skye.
Schilling, *Die Grünsteingen. Gesteine d. S. Harzes*, 1869, p. 26.	Diabase, fine-grained.	Harz.
Q. Journ., 1902, p. 169.	Gabbro.	Skye.
E. v. Seyfried, *cf. N.J.*, 1898, ii., p. 61.	Basalt.	Rhöngebirge (Kreuzberg).
N. Jahrb., 1893, vol. ii., p. 503.	Gabbro.	Rosswein, Saxony.
N. Jahrb., 1905, vol. i., p. 437.	Oliv. barkevikite gabbro.	Nosykomba.
J. E. Wolff, *B.U.S.G.S.*, 150, p. 201, 1898.	Theralite.	Montana.
N. Jahrb., 1904, vol. i., p. 275.	Hauyne tephrite.	Bohemia.
N. Jahrb., 1905, vol. i., p. 437.	Covite.	Nosykomba, Madagascar.
C. F. Kolderup, *Berg. Mus. Aarb.*, 1896, No. 5, p. 96.	Norite.	Norway.
N. Jahrb., 1904, vol. i., p. 58.	Sodal. aug. syenite.	Grosspriessen.
Q. Journ., 1906, p. 112 (Dr Evans).	Basalt.	Kilao.
N. Jahrb., 1904, vol. ii., p. 410.	Essexite.	Mount Royal.
R. Speight, *Tr. N.Z. Ins.*, xxvi., p. 409, 1894.	Dolerite.	New Zealand.
N. Jahrb., 1905, vol. i., p. 437.	Diabase essexite.	Jangoa, Madagascar.
Roth, *Zeitsch. d. d. geol. Gessel.*, 1864, p. 97.	Leucitite.	Rieden.
Q. Journ., 1904, p. 481.	Granulite dolerite.	Shropshire.
N. Jahrb., 1901, vol. ii., p. 239.	Ijolite (soda sussexite).	Kuusamo, Finland.
N. Jahrb., 1904, vol. i., p. 68.	Labradorite rock.	Monzoni.
N. Jahrb., 1893, vol. i., p. 322.	Monchiquite, glassy.	Brazil.
N. Jahrb., 1893, vol. ii., p. 347.	Elaeol. syen. porphyry.	Arkansas.
N. Jahrb., 1901, vol. ii., p. 239.	Ijolite.	Kuusamo, Finland.
N. Jahrb., 1901, vol. ii., p. 239.	Ijolite.	Kuusamo, Finland.
N. Jahrb., 1892, vol. i., p. 308.	Nephelinite (ijolite).	Finland.

LIST OF 1000 ROCKS

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp.Gr.
47	42·5			4·9	18·5	3·4	6·3	3·6	8·7	7·1	4·6	99·9	2·78
	42·1	1·0	2·4	1·8	18·7	1·7	4·8	3·5	10·8	11·0	1·9	99·7	...
19 rocks	46·9	·9	·5	1·7	20·0	3·7	4·6	2·9	8·4	7·5	3·0	100·1	2·76
	±2·6				±2·3	±2·0		±2·1		±2·9		±11·9	
48	*b a b b b b*												
	50·2			·4	20·1	2·5	5·7	3·7	7·8	3·0	7·8	100·8	...
	49·3				18·5	7·0	5·6	3·8	10·4	3·4	2·2	100·2	2·90?
	49·2			·6	19·1	1·8	10·3	5·0	8·8	3·9	1·2	99·9	...
	49·1			·8	16·0	7·1	4·5	5·0	8·3	4·5	4·8	100·2	...
	49·0		·3		19·8	2·6	5·3	2·8	8·1	3·2	9·1	100·2	...
	48·4	·1	·8	·4	19·9	2·5	5·3	5·2	8·0	5·5	4·0	100·0	...
	48·2			3·2	19·2	10·9		5·2	7·9	1·7	3·9	100·1	...
	48·1				20·0	7·0	5·0	4·1	10·1	2·2	4·5	101·0	...
	47·7	·4			18·4	2·5	5·7	4·8	9·4	2·8	7·6	99·3	...
	46·9				21·3	7·3	4·9	3·8	9·7	1·6	5·5	101·0	2·79
	46·0	·4		·4	17·1	4·2	5·4	5·3	10·7	2·2	9·0	100·7	...
	45·0		·4	·9	22·6		8·9	2·4	10·1	3·3	7·1	100·7	2·83
	44·4		·4	3·3	19·3	10·8	1·5	4·3	9·3	4·0	2·1	99·8	...
	42·2		·3	6·2	16·3	3·0	9·7	3·1	10·1	2·1	7·1	100·0	2·62
	41·0			1·6	24·2	9·5		5·1	11·0	5·7	1·8	99·9	...
15 rocks	47·0	·1	·1	1·2	19·5	5·2	5·2	4·2	9·3	3·3	5·2	100·2	2·78
	±1·6				±1·6	±1·7		±1·0		±2·1		±8·0	
49	*b b a a a a*												
	54·3			·6	10·1	7·1	5·8	6·5	8·9	4·1	2·2	99·6	2·90?
	53·1	1·8	·1	·7	8·9	3·3	9·6	14·4	6·8	·7	·5	99·7	3·09
	52·1	·7		·8	11·9	1·9	7·3	12·5	7·8	2·0	3·0	100·0	2·94
	51·8			1·4	13·4	10·1	2·9	7·4	10·9	2·5	·2	100·7	...
	51·8	1·4	·1	·6	12·8	3·6	8·7	7·6	10·7	2·1	·4	99·9	3·03
	51·7			1·0	11·4	12·6		7·6	10·8	3·5	·7	99·3	...
	51·2	1·5		1·1	8·0	8·9	8·1	7·1	9·9	2·8	1·3	99·9	2·88
	51·0			2·0	14·0	13·0		6·0	11·0	2·0	1·0	100·0	2·75
	50·6	·8		1·7	10·3	4·9	7·7	9·3	9·4	2·8	1·2	98·7	...
	49·7		1·5		9·4	15·7	1·2	9·4	10·3	2·8	·2	100·0	...
	49·7	1·7	1·1	1·1	12·9	7·4	4·8	6·6	10·3	3·0	1·3	100·0	2·92
	49·1	2·3		·3	13·4	6·5	5·9	9·6	8·9	3·4	1·0	100·4	...
	49·0			2·5	15·0	11·5		9·7	9·5	2·5	·3	100·0	...
	48·0	1·9	·5	2·2	13·6	2·9	8·4	8·7	8·4	3·4	2·0	100·0	...
	46·7			1·7	11·7	2·7	8·4	11·3	7·9	6·0	·8	97·2	...
	46·6	1·8	·1	3·2	15·2	3·5	7·8	8·7	10·1	2·4	·7	100·5	2·87
	46·6	3·1		2·1	9·6		14·7	10·1	8·6	2·6	1·8	99·2	2·90?
17 rocks	50·2	·9	·2	1·4	11·8	6·0	6·7	8·9	9·4	2·8	1·2	99·7	2·92
	±1·1				±1·7	±1·4		±1·0		±1·0		±6·2	

-continued.

Whitman Cross, Table XV.	Nephelinite (hauynophyre).	Mt. Vulture, Italy.
N. Jahrb., 1901, vol. ii., p. 239.	Ijolite.	Kuusamo, Finland.
H. S. Washington, *J.G.*, v., p. 370, 1897.	Leucite tephrite.	Orvieto, Italy.
A. Geikie, p. 149.	Basalt, average.	
Q. Journ., 1906 (Dr Evans), p. 112.	Basalt.	Mediterranean.
N. Jahrb., 1892, vol. ii., p. 418.	Trachyte.	Laacher See.
N. Jahrb., 1900, vol. ii., p. 388.	Leucite basanite.	Vesuvius.
H. S. Washington, *J.G.*, viii., p. 613, 1900.	Kulaite.	Kula, Asia Minor.
N. Jahrb., 1902, vol. i., p. 411.	Andesite.	Algiers.
N. Jahrb., vol. ii., 1899.	Lava.	Vesuvius eruption, 1858.
H. S. Washington, not published.	Leucite basanite.	Vesuvius.
Fuchs, *N. Jahrb. für Min.*, 1869, p. 79.	Leucite basalt.	Vesuvius.
Whitman Cross, p. 202.	Leucitite.	Capo di Bose, Italy.
F. Sabatini, *Mem. Cta. G. Ital.*, x., p. 163, 1900.	Leucitite.	Rocca di Papa, Italy.
N. Jahrb., 1905, vol. ii., p. 57.	Mica diabase.	Bohemia.
L. Ricciardi, *Gazz. Chim. Ital.*, xvii., p. 9, 1887.	Hauynophyre.	Mt. Vulture, Italy.
Hatch, *Petrology*, p. 203.	Augitite.	Madeira.
Roth, *Beiträge zur Petrog.*, 1869, p. cxxx.	Olivine dolerite.	Hessen.
Geol. Surv. India, 1897, p. 28.	Augite norite.	Eregur, Madras.
Q. Journ., 1900, p. 538.	Kentallerite.	Argyllshire.
E. Cohen, *N.J.B.B.*, v., p. 233, 1887.	Olivine diabase.	Orange River Colony.
N. Jahrb., 1892, vol. ii., p. 427.	Diabase.	New Haven.
Q. Journ., 1890, p. 327 (Cohen).	Lava, average.	Hawaii.
N. Jahrb., 1903, vol. i., p. 414.	Biot. hornbl. granite.	Zampach, near Eule.
A. Geikie, edit. 1888, p. 149.	Anamesite, average.	
Q. Journ., 1895, p. 318.	Basic pumice.	*Challenger* Expedition, 297.
P. Giacomelli, *Soc. Alp. Trident.*, xix., p. 406, 1894–5.	Basalt tuff.	Tyrol.
N. Jahrb., 1905, vol. ii., p. 213.	Mean of five basaltic rocks.	Röhn.
N. Jahrb., 1903–4, suppl. 18, p. 467.	Dolorite.	Londorf.
A. Geikie, p. 150.	Gabbro, average.	
Geol. Gesellschaft, vol. lvi., p. 20.	Basalt.	Bromburg, Hanover.
(Houghton), Prestwich, i., p. 37.	Lava.	Aden.
N. Jahrb., 1906, vol. ii., p. 71.	Olivine basalt lava.	Dvynoch, Skye.
Roth, *Beiträge zur Petrog.*, 1869, p. cx.	Olivine basalt.	Hessen.

LIST OF 1000 ROCK

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp.Gr
50	*b b a a a b*												
	54·1				7·9		12·9	16·6	6·2	·4	1·2	99·4	3·30
	50·9			·1	13·2	1·1	9·7	13·1	10·2	1·2	·3	99·7	3·10
	50·6	·8	·3	·9	7·9	1·4	15·0	18·6	3·4	1·0	·2	100·1	...
	50·0				11·7	2·4	10·6	12·8	11·2	1·6	·2	100·6	...
	49·6		·2	5·0	13·5	4·7	10·7	9·2	6·2	1·0	·4	100·6	2·72
	48·1			4·2	13·3	3·7	9·5	9·5	8·5	2·0	1·6	100·4	2·92
	45·9			·6	15·1	1·9	11·5	14·8	8·9	1·9	·2	100·8	...
	45·3	2·5	·4	·2	13·4	7·3	6·6	11·5	10·3	2·2	·2	99·9	...
	42·8			4·0	8·7		18·9	10·1	12·3	2·3	·6	100·0	2·83
	42·8			6·2	10·9	3·4	10·1	16·3	9·1	·9	·1	99·9	2·88
	42·4	·5	·6	4·4	13·4	6·4	6·5	11·0	11·1	2·8	·5	99·6	2·96
	41·3			7·1	2·4	9·4	15·0	21·4	3·3			99·9	...
	40·6	4·2		(5·0)	12·5	5·5	9·5	9·0	10·8	2·5	1·2	100·8	...
	40·2	2·9		3·4	12·8	4·0	10·4	11·9	10·4	2·7	·8	99·5	...
14 rocks	46·0	·8	·1	2·9	11·2	3·7	11·2	13·3	8·7	1·6	·5	100·1	2·96
	±2·2				±2·5	±2·3		±1·8		±·9		±9·7	
51	*b b a a b a*												
	53·6				13·9		15·7	3·4	8·1	5·3	·7	100·7	3·02
	51·8				15·7	8·5	7·3	5·0	9·6	2·2	1·0	100·9	...
	51·1				10·9	21·9		2·4	10·3	3·7	1·6	102·0	(2·56)
	51·0	2·4			12·8		17·1	5·2	6·1	4·2	1·2	100·0	...
	50·6			1·8	14·1	16·0		5·1	9·2	2·2	1·0	100·0	...
	50·0			2·0	14·0	14·2		6·0	10·0	3·5	·3	100·0	...
	49·8	1·3	·6	2·6	12·7	3·4	11·4	4·4	8·7	5·2	·6	100·7	2·91
	49·3			1·6	14·5	16·1		5·7	9·1	3·0	1·2	100·6	...
	49·1	·8		·8	13·8	6·8	12·6	3·2	8·7	2·5	1·3	99·5	...
	49·1	·6	·5		12·0	6·2	7·9	7·6	10·6	3·9	2·0	100·4	(2·33)
	48·4	1·3	·3	1·0	15·5	4·8	7·8	8·2	8·8	3·0	1·0	100·2	2·97
	48·3	1·8		1·4	13·9	9·6	4·2	6·8	9·2	3·5	1·0	99·7	3·13
	47·5			2·1	12·5	8·1	7·4	8·4	10·1	3·9		99·8	3·02
	46·9	·9		1·2	11·8	11·7	4·6	6·6	10·7	3·7	·4	98·0	...
	44·8	5·3	·3	(1·2)	12·5	4·6	13·2	5·4	10·2	2·5	1·0	100·7	3·09
	43·6			4·4	12·3	3·5	12·2	9·1	11·4	2·7	·8	100·0	?
16 rocks	49·1	·9	·1	1·3	13·3	6·8	9·2	5·8	9·4	3·4	·9	100·2	3·02
	±1·3				±1·1	±1·5		±2·2		±·8		±6·9	(2·45)
52	*b b a a b b*												
	48·2				16·1	3·6	12·8	7·6	10·0	?	?	98·3	...
	45·5			4·3	16·6	4·1	8·8	9·4	8·2	2·2	·7	99·8	2·99
	44·8			6·2	13·5	11·8	4·5	11·6	4·8	2·3		99·5	(2·56)

—*continued.*

G. P. Merrill, *Pr. U.S. Nat. Mus.*, xvii., p. 662, 1895.	Websterite.	Montana.
N. Jahrb., 1892, vol. ii., p. 427.	Ol. hyp. diabase.	Culpeper Co., Virginia.
B. K. Emerson, *B.U.S.G.S.*, 148, p. 77, 1897.	Wehrlite.	Massachusetts.
Q. Journ., 1895, p. 318.	Basic glass.	*Challenger* Expedition, 297.
N. Jahrb., 1893, p. 287.	Diabase.	Ravenna.
Q. Journ., 1883, p. 296.	Basic tuff.	St Davids.
Q. Journ., 1883, p. 257.	Picrite.	Ottenschlag.
L. v. John, *Jb. G.R.A. Wien*, xlvi., p. 291, 1896.	Basalt.	Azores.
Rosenbusch, *loc. cit.*, p. 54.	Limburgite.	Limburg.
Q. Journ., 1883, p. 256.	Hornbl. picrite.	Anglesey.
Geol. Gesellschaft, vol. lvi., p. 18.	Basalt.	Siebengebirge.
N. Jahrb., 1903–4, suppl., p. 292.	Picrite (partly serpentinised).	Nanzenbach.
W. C. Brögger, *Q.J.G.S.L.*, p. 26, 1894.	Camptonite.	Norway.
Hatch, *Petrology*, p. 203.	Augitite.	Garlton Hills.
T. H. Holland.	Norite.	Madras.
A. H. Phillips, *A.J.S.*, xlvii., p. 473, 1894.	Basalt, stalagmite.	Hawaii.
Q. Journ., 1884, p. 450.	Basalt.	Funabara, Japan.
N. Jahrb., 1906, suppl., p. 314.	Diabase, intrusive.	Hartenrod.
Rutley, p. 255.	Dolerite, average.	
A. Geikie, p. 149.	Dolerite, average.	
Q. Journ., 1904, p. 481 (Teall).	Dolerite.	Rowley.
N. Jahrb., 1901, vol. ii., p. 62.	Uralite porphyry.	Tammerfors, Finland.
N. Jahrb., 1901, vol. i., p. 72.	Basalt.	Cape Weissenfels, Sweden.
N. Jahrb., 1900, vol. ii., p. 232.	Basalt lava (vitrophyric).	Puebla, Mexico.
W. Cross, *B.U.S.G.S.*, 168, p. 171, 1900.	Plagioclase basalt.	New Mexico.
N. Jahrb., 1905, vol. ii., p. 62.	Amphibolite.	Umhausen.
P. Giacomelli, *Soc. Alp. Trident.*, xix., p. 410, 1894.	Basalt.	Tyrol.
N. Jahrb., 1903–4, suppl., 18 p. 466.	Basalt.	Schlüchtern.
J. F. Kemp, 18, *A.R.U.S.G.S.*, iii., p. 407, 1899.	Gabbro.	New York.
Merrill, p. 223.	Basalt.	Bohemia.
N. Jahrb., 1904, vol. i., p. 69.	Eclogite, light green.	Aiguilles Rouges.
Q. Journ., 1883, p. 303.	Diabase.	Clegyr Foig, St Davids.
N. Jahrb., 1906, vol. ii., p. 310.	Diabase glass.	Homertshausen.

LIST OF 1000 ROCKS

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.
52	43·1	1·7		4·5	15·7	12·9	2·9	7·1	10·9	1·0	·4	100·3	...
	42·9				10·0	14·9	17·6	8·6	1·3	4·7		100·0	2·90
	40·2	4·7		·5	9·5	9·7	13·0	8·0	13·1	·8	·2	99·7	3·36
	37·9	5·3		1·4	13·2	8·8	8·4	9·5	10·8	2·4	2·1	99·8	...
	35·0	5·2		1·3	10·8	1·4	21·3	19·3	·4	·2	5·4	100·3	3·28
8 rocks	42·2	2·1		2·3	13·2	8·4	11·2	10·1	7·4	1·7	1·1	99·7	3·13
	±3·0				±2·3	±4·8		±2·2		±1·6		±13·9	(2·56)
53	*b b a b a a*												
	51·5	1·1	·2		14·0	2·7	8·9	7·6	10·5	4·7		101·1	...
	51·4	·9		1·9	13·7	4·4	6·1	6·4	11·6	3·9	·6	100·8	...
	50·3	1·6	·2	·3	15·2	2·8	11·3	5·8	9·6	2·9	1·0	101·1	2·97
	49·6			1·3	11·9	2·6	9·2	8·0	12·8	1·1	3·6	100·1	2·88
	49·2	1·1		2·2	14·4	4·5	4·6	8·2	10·2	4·5	·4	99·4	...
	49·2	·2	·4	·9	15·0	1·4	9·0	8·0	13·6	1·1	1·5	100·4	...
	49·0			3·0	14·0	12·0		6·0	10·0	4·0	2·0	100·0	2·98
	48·4		·4	3·0	13·4	9·2	4·6	4·3	12·8	3·2	1·0	100·3	...
	48·0			3·0	16·8	4·3	4·2	9·1	13·3	1·2	·3	100·2	...
	47·0		·1	4·9	17·1	1·9	7·0	8·3	12·2	2·5	·5	101·4	3·20
	46·9	2·0			13·4	9·8	2·7	4·3	14·7	4·6	2·0	100·4	...
	45·6	1·3	·9	3·9	13·0	5·0	4·8	8·4	11·3	4·5	1·0	99·9	...
	44·1	1·4		6·8	14·0	9·1	4·4	4·9	11·6	3·3	1·1	100·6	...
	43·9	1·8	·1	3·4	17·6	4·0	3·9	8·2	13·1	2·8	1·3	100·1	...
	43·8	3·6		3·1	12·8	9·0	5·1	2·4	13·6	3·6	2·9	99·8	2·86
	43·7	4·0		1·0	11·5	6·3	8·0	7·9	14·0	2·3	1·5	100·2	...
	42·0	3·7	·6	1·1	13·6	7·6	7·1	6·4	14·2	1·9	1·0	99·2	...
	40·6	3·1	1·6	6·6	13·0	4·7	6·7	5·2	13·6	3·0	2·2	100·3	2·86
	40·2	4·7	1·1	3·4	12·1	7·0	6·9	6·6	13·3	3·6	1·6	100·7	2·92
19 rocks	46·5	1·6	·3	2·6	14·0	5·4	6·3	6·6	12·4	3·0	1·5	100·3	2·95
	±1·4				±1·2	±1·8		±1·8		±1·0		±7·2	
54	*b b a b a b*												
	48·6			1·7	15·9	2·5	6·3	11·5	11·7	2·0		100·2	...
	47·5	2·8		·9	16·7	6·7	6·9	6·4	8·7	2·8	1·1	100·5	...
	47·3			·9	15·2	1·2	10·7	9·9	11·3	3·0		99·6	...
	47·1	1·2		·9	16·0	4·0	9·9	4·9	13·3	1·1	2·0	100·4	...
	46·7				17·7	1·7	11·3	10·4	11·6	1·8	·2	101·5	...
	46·2		1·6		13·4	8·2	7·0	7·3	12·3	3·0	·6	99·9	3·01
	45·1			2·2	18·1	12·9		7·3	11·2	2·1	1·0	100·0	...
	44·2	1·5		2·5	14·7	6·8	4·8	9·5	10·4	3·0	1·8	99·2	2·90 ?
	42·7			2·5	18·4	5·3	7·0	12·9	10·1	1·7	·5	101·0	...
	42·2	1·9	·9	1·0	17·4	5·9	6·6	11·0	12·6	1·1	·9	101·5	...
	42·0	1·9	·1	3·2	13·9	5·8	6·2	10·4	11·4	3·6	·9	101·3	3·03

—continued.

N. *Jahrb.*, 1902, vol. ii., p. 37.	Basalt.	Seigertshausen.
Q. *Journ.*, 1897, p. 484.	Basalt.	Franz Joseph's Land.
Q. *Journ.*, 1894, p. 653.	Gabbro, dark banded.	Isle of Skye.
Whitman Cross, p. 171.	Hornblendite.	Gran, Norway.
M. Koch, *Z.D.G.G.*, xli., p. 165, 1889.	Biotite peridotite.	Harz.
N. *Jahrb.*, 1901, vol. i., p. 245.	Glassy trap (edge of dyke).	Rocky Hill.
N. *Jahrb.*, 1904, vol. i., p. 46.	Diabase.	Kohlhau.
N. *Jahrb.*, 1901, vol. i., p. 245.	Trap.	Rocky Hill.
L. Ricciardi, *Att. Soc. Ital. Mil.*, xxviii., p. 130, 1885.	Basalt.	Italy.
N. *Jahrb.*, 1904, vol. i., p. 46.	Diabase.	Tharandt.
C. Klein, *Sb. Berl. Acad.*, 1888, p. 111.	Leucite basanite.	L. Bolsena, Italy.
A. Geikie, p. 149.	Basalt, rough average.	
N. *Jahrb.*, 1905, vol. ii., p. 57.	Spilite (diabase).	Bohemia.
N. *Jahrb.*, 1892, vol. i., p. 325.	Gabbro diorite.	Quinesec.
F. L. Ransome, *B. Dep. G. Un. Cal.*, i., p. 231, 1894.	Fourchite.	California.
W. C. Day, 19, *A.R.U.S.G.S.*, vi., p. 222, 1898.	Diabase.	Pennsylvania.
Whitman Cross, Table XII.	Analcite basalt.	The Basin, Color.
N. *Jahrb.*, 1905–6, supp., p. 311.	Diabase, surface.	Niederschelde.
N. *Jahrb.*, 1904, ii., p. 91.	Hauyne monchiquite (heptorite.)	Siebengebirge.
K. Gruss, *Mt. Bad. G.L.A.*, iv., p. 126, 1901.	Leucite basanite.	Kaiserstuhl.
W. C. Brögger, *Q.J.G.S.*, l., p. 19, 1894.	Ol. gabbro diabase.	Gran, Norway.
N. *Jahrb.*, 1893, vol. ii., p. 347.	Fourchite.	Fourche Mts., Arkansas.
N. *Jahrb.*, 1905, vol. i., p. 271.	Nephelinite.	Schanzberg, Bohemia.
N. *Jahrb.*, 1904, vol. i., p. 275.	Hauynophyre.	Katzenkoppe, Bohemia.
N. *Jahrb.*, 1900, vol. ii., p. 397.	Gabbro.	Bologna, Apennines.
Whitman Cross, Table XII.	Basalt (also groups 44 and 55).	Grants, New Mexico.
N. *Jahrb.*, 1902, vol. ii., p. 390.	Hornbl. gabbro.	Kasbek, Russia.
N. *Jahrb.*, 1902, vol. ii., p. 32.	Felspar basalt.	Langenberg, Niederrheim.
Q. *Journ.*, 1895, p. 318.	Basic glass.	*Challenger* Expedition, 297.
P. Giacomelli, *Soc. Alp. Trident.*, xix., p. 407, 1894.	Basalt.	Tyrol.
David, Smeeth, and Shofield, *J.R. Soc. N.S.W.*, xxix., p. 477, 1895.	Basalt.	Antarctic Continent.
Mitscherlich, *Zeitsch. d. d. geol. Gesell.*, 1863, p. 372.	Olivine basalt.	Rolandoeck.
N. *Jahrb.*, 1903, vol. i., p. 417.	Ariègite.	Sherz, France.
N. *Jahrb.*, 1903–4, supp., 18, p. 465.	Limburgite.	Stellberg.
N. *Jahrb.*, 1892, vol. i., p. 278.	Nepheline basalt.	Wilhelmshohe.

LIST OF 1000 ROCKS

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.
54	40·1	3·0	1·3	·9	15·5	6·3	7·3	8·4	12·4	3·4	1·7	100·2	...
	39·8	·8	1·4	1·8	18·2	6·9	4·4	8·4	13·8	3·4	·8	99·7	...
13 rocks	44·6	1·0	·4	1·4	16·3	5·7	6·8	9·1	11·6	2·5	·9	100·4	3·00
	±1·6				±1·4	±1·1		±1·9		±·9		±6·9	
55	*b b a b b a*												
	48·8	1·2			15·2	5·7	10·4	4·6	10·4	2·1	·9	99·2	3·01
	48·3			·4	15·7	16·6		3·7	8·4	3·6	1·6	98·3	...
	47·0			·5	18·1	4·0	8·7	5·4	10·3	4·1	1·4	99·6	...
	46·6	·6		1·4	15·3	8·4	8·5	5·3	9·3	3·0	1·4	99·8	...
	46·3			3·1	16·9	14·7		6·0	3·7	7·9	·9	99·4	...
	46·3		·6	3·0	13·4	4·4	12·8	4·4	11·9	2·1	1·9	100·9	(2·40)
	46·2		·2	1·0	18·3	6·5	7·1	7·0	10·0	3·1	·8	100·3	...
	45·5			·5	15·8	15·3		4·0	9·3	5·0	3·0	100·0	2·84
	45·4			4·9	16·8		15·7	3·1	10·2	2·8	1·4	100·2	...
	44·4	1·6	·3	1·7	13·3	9·1	6·3	5·7	10·6	5·6	1·8	100·3	3·01
	44·1	2·3		·3	17·6	3·4	10·0	7·2	11·6	2·9	·9	100·2	3·54
	43·9	4·1	·7	1·6	16·2	4·0	10·1	5·1	9·6	2·9	1·5	99·7	...
	43·5	2·1		1·2	18·1	7·5	7·6	3·5	13·4	2·0	1·3	100·2	...
	43·5	1·8	·9	·9	22·0	3·5	7·8	3·4	14·1	3·0	·9	101·7	...
	43·1	·5	1·5	·7	15·2	5·3	8·6	7·5	11·9	4·0	2·5	100·8	3·21
	43·0			1·3	14·0	15·3		9·1	12·1	3·9	1·3	100·0	...
	42·9			3·1	21·1	3·7	8·3	5·2	11·6	4·2	·7	100·6	...
	42·4			1·1	21·0	4·7	8·6	3·2	15·9	2·8	1·2	100·9	...
	39·0	3·0	1·3	1·9	11·8	9·0	9·5	7·4	11·3	3·9	1·6	99·7	...
19 rocks	44·7	·9	·3	1·5	16·6	7·0	7·8	5·3	10·8	3·6	1·4	100·1	3·12
	±1·6				±2·1	±1·5		±2·1		±1·2		±8·5	(2·40)
56	*b b a b b b*												
	47·7				17·6	4·2	12·7	6·0	10·0	1·4	·2	99·5	...
	46·8		·2	·9	17·6	16·8		5·1	9·5	2·6	·5	100·0	...
	46·8			·3	18·0	6·2	8·8	8·4	10·2	2·2	·1	101·0	3·10
	46·1			1·6	17·9	·8	13·7	6·9	8·0	4·7	·7	100·2	...
	45·7			2·0	17·1	20·7		2·3	9·6	2·6		100·0	2·97 ?
	44·0			1·9	16·9	8·6	8·7	5·9	9·9	4·0	1·0	100·9	...
	42·3		·6	3·6	12·6	15·5	5·2	5·2	8·4	5·2	2·7	101·3	2·97
	42·0			4·8	17·6	6·2	8·6	8·0	8·5	2·1	2·8	100·6	3·01
	41·1	·2		·8	21·0	4·8	9·2	8·0	14·7	·6	·6	101·0	...
	39·8	·1		·9	19·7	7·7	8·9	7·3	13·5	1·6	·5	100·1	3·18
	39·6			1·3	17·0	6·6	9·3	6·7	10·6	6·0	3·1	100·1	...
	39·0			·5	21·6	9·0	6·8	4·5	12·6	3·8	2·6	100·4	...

-continued.

N. *Jahrb.*, 1905, vol. i., p. 437.	Ijolite.	Ambalike, Madagascar.
N. *Jahrb.*, 1903–4, supp. 18, p. 464.	Basalt.	Darmstatt.
O. Silvestri, *B.C.G. It.*, xix., p. 185, 1888.	Basalt.	Hawaii.
Q. *Journ.*, 1890, p. 327 (Sylvester).	Lava.	Hawaii, average.
N. *Jahrb.*, 1904, vol. i., p. 69.	Eclogite.	Aiguilles Rouges.
W. D. Matthew, *Tr. N.Y. Ac. Sci.*, xiv., p. 214, 1895.	Quartz diabase.	New Brunswick.
Loewinson-Lessing, *cf. N.J.*, 1899, ii., p. 234.	Augite porphyrite.	Caucasus.
L. Ricciardi, *B.S.G. It.*, v., p. 58, 1886.	Basalt.	Abyssinia.
N. *Jahrb.*, 1905, vol. ii., p. 235.	Gabbro.	Pajaro.
R. Scaritzer, *Jb. Wien. G.R.A.*, xxxiv., p. 718, 1884.	Basalt.	Arctic Ocean.
K. A. Lossen, *Jb. Pr. G.L.A.*, vi., p. 213, 1886.	Labradorite porphyrite.	Harz.
N. *Jahrb.*, 1905, vol. ii., p. 55.	Theralite.	Duppau.
N. *Jahrb.*, 1905, vol. ii., p. 62.	Eclogite.	Oetz Valley.
Not published.	Hornbl. gabbro.	New Hampshire.
N. *Jahrb.*, 1904, vol. ii., p. 91.	Monchiquite.	Arkansas.
N. *Jahrb.*, 1902, vol. ii., p. 32.	Limburgite.	Gensungen.
N. *Jahrb.*, 1905, vol. i., p. 271.	Felspar basalt.	Korkatch, Bohemia.
Bntley, p. 255.	Basalt.	Average.
N. *Jahrb.*, 1901, vol. ii., p. 388.	Olivine nephelinite.	Puy de Dome.
N. *Jahrb.*, 1904, vol. i., p. 69.	Augite diorite.	Malinverno, Monzoni.
N. *Jahrb.*, 1905, vol. ii., p. 75.	Nepheline basalt.	Caroline Island.
N. *Jahrb.*, 1904, vol. i., p. 70.	Amphibolite.	Aiguilles Rouges.
Merrill, p. 225.	Diorite.	Albemarle Co.
C. H. Smythe, *Jr. A.J.S.*, xlviii., p. 61, 1894.	Gabbro.	L. Wilmurt, N. York.
N. *Jahrb.*, 1904, vol. i., p. 69.	Quartz amphibolite.	Aiguilles Rouges.
Prestwich, i., p. 37 (reduced to 100·0).	Basalt.	Staffa.
N. *Jahrb.*, 1901, vol. ii. p. 388.	Olivine tephrite.	Puy de Dome.
Whitman Cross, Table XII.	Nepheline basalt.	Odenwald.
N. *Jahrb.*, 1902, ii., p. 67.	Camptonite.	Rennibuster, Orkney Isl.
N. *Jahrb.*, 1904, vol. i., p. 68.	Gabbro.	Monzoni.
N. *Jahrb.*, 1890, vol. i., p. 258.	Hornbl. gabbro.	Upper Italy.
F. Eigel, *T.M.P.M.*, xi., p. 98, 1890.	Teschenite.	Cape Verde.
N. *Jahrb.*, 1903–4, suppl. 18, p. 464.	Nepheline basalt.	Rhön.

LIST OF 1000 ROCK

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr
56	38·6			1·3	20·4	7·6	5·9	12·9	9·4	2·3	1·4	99·8	...
	38·1			5·5	17·9	14·1		8·9	11·7	1·0	2·0	99·3	...
	36·4	2·1	1·1	·8	16·1	12·9	6·9	5·0	15·5	2·4	1·1	100·3	...
15 rocks	42·2	·2	·1	1·8	17·9	8·3	8·1	6·7	10·8	2·8	1·3	100·3	3·08
	±3·0				±1·4	±2·0		±2·6		±1·9		±10·9	
57	*b b b a a a*												
	51·9			3·0	2·5	3·5	9·4	26·0	3·6			100·1	...
	50·8			·7	3·4	1·7	8·3	22·8	12·3			100·0	3·32
	49·4			2·6	3·4	1·3	5·9	35·0	2·1			99·7	...
	45·0	1·0	·1	3·8	6·3	3·4	8·3	21·0	8·8	·9	·7	99·2	...
	42·8			3·3	7·5	3·3	4·8	30·1	6·5	·5	·1	98·9	2·95
	41·5	·2	·2	4·7	3·6	6·3	8·9	25·3	6·6	·3	·1	97·5	...
	38·6			6·4	3·7	7·6	7·8	27·7	7·7		·2	99·7	...
7 rocks	45·7	·2		3·5	4·3	3·9	7·6	26·8	6·8	·2	·2	99·3	3·25
	±3·0				±1·4	±2·6		±2·6		±·4		±10·0	
58	*b b b a a b*												
	45·7			1·2	6·6	9·1		34·8	2·1			99·5	3·10
	43·8			1·1	1·1	9·2	·6	44·3	1·7			101·4	3·29
	43·2			·8	2·4	4·5	4·6	38·8	2·7	2·3	·6	99·9	...
	42·8			·6	·0		9·4	47·3				100·1	3·30
	42·0			1·7	4·4	2·8	4·4	41·1	3·3	1·2	·3	101·2	...
	41·5			·3	6·9	2·2	6·7	35·9	5·8	1·4	·3	101·0	...
	41·4			4·4	·8	2·5	6·4	43·7	·6			99·8	...
	40·0			2·1	3·6		8·6	41·3	4·2			99·6	3·25
	39·9			·2	(·3)	3·8	15·3	40·5	1·0			101·0	...
	39·2			·4	(·4)	·4	16·4	43·8	1·2			101·8	...
10 rocks	42·0			1·3	2·6	3·0	7·7	41·2	2·3	·5	·1	100·5	3·20
	±1·8				±2·2	±2·9		±2·4		±·8		±10·1	
59	*b b b a b a*												
	46·8			·7	9·8		16·3	18·1	9·6			101·3	3·33
	46·4			3·9	10·8	5·9	5·6	22·2	3·7	·3	1·2	100·2	...
	43·7			2·8	11·2	3·9	6·1	25·6	7·1	·5	·3	101·2	...
	42·9			6·1	10·9	3·5	10·1	16·3	9·1	·9	·2	100·0	2·88
	41·4			5·6	6·6	13·9	6·3	18·4	7·2	·2	·9	100·6	2·82
	40·8			4·1	10·4	3·5	6·4	23·3	8·5	1·7	·7	99·1	2·96
	40·4			5·0	9·9	4·8	8·3	21·6	4·7	3·6	·8	99·1	3·11
	40·1	·4	·2	4·0	7·8	7·4	8·7	23·7	6·5	1·2	·5	100·6	2·99
	39·9	2·7	·5	1·9	8·7	4·4	8·3	20·2	10·8	1·9	1·0	100·3	3·20
	39·6	1·0		5·7	7·3	4·4	10·5	24·8	4·8	1·0	·3	99·4	2·89

—continued.

N. *Jahrb.*, 1903, vol. i., p. 417.	Ariegite.	Lherz, Pyrenees.
N. *Jahrb.*, 1893, vol. ii., p. 347.	Ouachite.	Pot. sulphur springs, Arkansas.
N. *Jahrb.*, 1902, ii., p. 29.	Nepheline basalt.	Werrberg, near Homberg.
G. H. Williams, *B.U.S.G.S.*, 148, p. 83, 1897.	Pyroxenite.	Maryland.
G. H. Williams, *A.G.*, vi., p. 41, 1890.	Pyroxenite.	Maryland.
N. *Jahrb.*, 1900, ii., p. 397.	Norite.	Bologna, Apennines.
N. *Jahrb.*, 1900, vol. i., p. 75.	Peridotite.	Michigan.
Tschermak, *Porphyrgesteine Oest.*, 1869, p. 287.	Anorth. oliv. gabbro.	Transylvania.
Q. *Journ.*, 1883, p. 257.	Picrite.	Holler.
Hatch, *Petrology*, p. 211.	Oliv. diall. rock.	Westphalia.
Cossa, *Richerche chim e. micro.*	Pyrox. peridotite (lherzolite).	Baldissero.
Cf. J. J. H. Teall, *Brit. Petr.*, London, 1888, p. 103.	Saxonite.	St Paul's Rocks.
N. *Jahrb.*, 1904, vol. i., p. 208.	Olivine segregation.	Rhenish Prussia.
Hatch, *Petrology*, p. 211.	Dunite.	New Zealand.
N. *Jahrb.*, 1895, vol. ii., p. 267.	Lherzolite.	Prades.
N. *Jahrb.*, 1895, vol. ii., p. 267.	Hornbl. lherzolite.	Causson.
Diller and Clarke, *B.U.S.G.S.*, 60, p. 23, 1890.	Saxonite.	Oregon.
G. H. F. Ulrich, *Q.J.G.S.*, xlvi., p. 629, 1890.	Peridotite.	New Zealand.
N. *Jahrb.*, 1902, vol. ii., p. 387.	Dunite.	Ural.
N. *Jahrb.*, 1902, vol. ii., p. 387.	Dunite.	Ural.
Geol. Surv. India (T. H. Holland).	Pyroxenite.	Pallaverasse.
N. *Jahrb.*, 1901, vol. ii., p. 233.	Mica hornblendite.	Pyrenees.
N. *Jahrb.*, 1901, vol. ii., p. 233.	Bronzite hornbl. peridotite.	Pyrenees.
Hatch, *Petrology*, p. 205.	Picrite.	Anglesey.
Q. *Journ.*, 1883, p. 257.	Picrite.	Schriessheim.
Hatch, *Petrology*, p. 205.	Picrite.	Gümbelberg.
N. *Jahrb.*, 1903–4, suppl., p. 292.	Picrite.	Burg.
K. Busz, *N.J.*, 1895, i., p. 74.	Palæopicrite.	Devonshire.
Whitman Cross, Table XII.	Nephel. basalt.	Black Mts., Texas.
N. *Jahrb.*, 1903–4, suppl., p. 292.	Picrite.	Madenbach.

LIST OF 1000 ROCKS

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.
59	38·9			6·3	10·3	4·9	7·0	23·6	6·0	1·3	·8	99·1	...
	38·8	·9		9·9	6·8	8·8	2·0	26·3	3·9	·8	2·6	100·8	...
12 rocks	41·6	·4	·1	4·7	9·2	5·5	8·0	22·0	6·8	1·1	·8	100·1	3·16
	±1·6				±1·5	±2·3		±2·2		±1·0		±8·6	
60	*b b b a b b*												
	40·0	·6		4·3	8·3	2·0	12·0	27·6	4·2	·7	·3	99·8	2·99
	39·1			5·7	4·9	4·8	11·9	29·2	4·2			99·6	2·93
	37·1	·4	·1	5·1	5·0	8·9	8·0	26·9	6·1	·4	·5	98·6	...
	31·8			2·5	1·4	15·6	14·3	33·1	·9			99·6	...
4 rocks	37·0	·2		4·3	4·9	7·8	11·8	29·3	3·8	·3	·2	99·4	3·00
	±3·8				±1·8	±5·4		±·5		±·5		±12·0	
61	*b b b b a a*												
	51·3				13·6	·5	8·5	12·7	12·4	1·4	·3	100·8	...
	50·4			1·0	8·7	1·1	6·5	17·6	11·6	3·0	·5	100·3	...
	50·1			2·6	6·9	2·1	5·3	16·2	16·7	·7		100·6	3·09
	50·0			1·3	15·4		6·7	10·0	14·9	1·8	·3	100·4	...
	49·1			·2	8·5	6·4		20·8	12·9	1·7	·6	100·1	...
	48·4	·7	·5	2·8	11·6	4·1	3·6	12·6	8·1	4·1	3·2	99·9	...
	47·0	1·5	·2	(·6)	10·0	1·0	10·5	11·5	14·5	3·2	·3	100·3	...
	46·7	·8	1·5	1·3	10·0	3·5	8·5	9·7	13·2	1·8	3·8	100·9	...
	46·6			·4	9·5	3·4	9·1	13·5	15·8	1·7		100·0	...
	46·2			3·4	13·9	5·3	1·8	11·6	15·7	1·1	·3	99·2	...
	45·5	·2	·4	(·2)	19·3	·5	4·2	10·1	16·7	2·3	·6	100·0	...
	43·4			6·7	9·4	8·9		10·4	15·4	1·5	3·2	98·8	...
	42·3	1·5		2·5	12·7	10·6		12·7	13·0	2·7	·9	98·9	...
	36·5	(5·2)		3·4	8·2	8·3	7·3	8·2	18·9	2·1	1·1	99·2	...
14 rocks	46·7	·7	·2	1·9	11·3	3·4	5·7	12·7	14·3	2·1	1·1	100·0	3·09
	±1·9				±2·7	±2·5		±2·5		±1·3		±10·9	
62	*b b b b a b*												
	47·3	·8	·6	2·1	11·2	2·9	5·9	16·0	7·3	1·9	3·8	99·9	...
	47·3			·3	16·9	1·6	2·7	21·0	8·6	1·2	·4	99·9	...
	47·1			·8	17·0	1·6	3·6	19·9	9·2	·5	·3	100·0	...
	44·4			·6	17·6	1·4	3·9	15·1	16·0	·8	·1	100·0	...
	43·8	2·3	·5	1·0	10·9	3·5	10·1	12·8	13·8	2·2	·3	101·2	...
	42·4	1·8	1·0	1·8	12·3	3·9	7·3	13·1	12·7	2·7	1·0	100·0	...
	42·3			1·2	15·4	2·7	6·0	19·3	12·0	1·0	·2	100·1	...
	41·4	·3		1·1	9·8	3·3	5·2	13·4	16·6	1·6	7·4	100·1	2·76
	40·8			·3	5·2	11·9	10·4	14·2	18·4			101·2	...
	38·7			6·9	10·2	6·3	6·1	18·6	10·4	1·5	1·6	100·3	...
	38·3	1·6	·2	5·4	9·8	4·3	6·4	17·4	10·3	2·9	2·1	99·2	2·99

—*continued.*

Q. Journ., 1883, p. 257.	Picrite.	Söhle.
N. Jahrb., 1893, vol. ii., p. 347.	Picrite porphyrite.	Arkansas.
N. Jahrb., 1903–4, suppl., p. 292.	Picrite.	Wommelshausen.
N. Jahrb., 1903–4, suppl., p. 292.	Picrite.	Schwarzen Steinen.
Q. Journ., 1883, p. 257.	Picrite.	Schwarzenstein.
N. Jahrb., 1902, vol. ii., p. 387.	Dunite.	Ural.
N. Jahrb., 1892, vol. ii., p. 427.	Hyp. diabase.	Culpeper County.
G. F. Becker, *M.U.S.G.S.*, xiii., p. 101, 1888.	Pseudo-diorite.	California.
N. Jahrb., 1904, vol. i., p. 49.	Gabbro.	Studené.
Lasaulx, *Elemente d. Petrog.*, p. 312.	Gabbro.	Buchan.
N. Jahrb., 1895, vol. ii., p. 267.	Lherzolite.	Tuc d'Ess.
N. Jahrb., 1903, vol. i., p. 432.	Analcime basalt.	Montana.
N. Jahrb., 1904, vol. ii., p. 399	Eclogite.	Holsenoe, Norway.
Whitman Cross, Table XII.	Shonkinite.	Square Butte, Mont.
N. Jahrb., 1902, vol. ii., p. 387.	Olivine gabbro (mean of 2).	Ural.
R. Lepsius, *C. v. Attika. Berl.*, 1893, p. 98.	Gabbro.	Greece.
N. Jahrb., 1904, vol. ii., p. 399.	Olivine gabbro.	Skeie, Norway.
L. Brugnatelli, *B. Com. G. Ital.*, xiv., p. 318, 1883.	Melilite pyroxene rock.	Italy.
N. Jahrb., 1903–4, suppl. 18, p. 465.	Limburgite.	Elsass.
N. Jahrb., 1893, vol. ii., p. 347.	Elaeolite syenite.	Arkansas.
Hague and Jagger, *B.U.S.G.S.*, 168, p. 97, 1901.	Leucite absarokite	Yellowstone Park.
N. Jahrb., 1903, vol. i., p. 417.	Ariègite.	Lherz, France.
N. Jahrb., 1903, vol. i., p. 417.	Ariègite.	Lherz, France.
N. Jahrb., 1903, vol. i., p. 417.	Ariègite.	Lherz, France.
C. v. John, *Jb. G.R.A. Wien*, xlvi., p. 284, 1896.	Dolerite.	Cape Verde Islands.
Whitman Cross, p. 171.	Nephelite basanite.	Colfax Co., N. Mexico.
N. Jahrb., 1903, vol. i., p. 417.	Ariègite.	Lherz, France.
H. Rosenbusch, *Sb. Berl. Acad.*, 1899, p. 113.	Euctolite.	Umbria.
N. Jahrb., 1902, vol. ii., p. 388.	Koswite pyroxenite (mean of 2).	Ural.
Q. Journ., 1883, p. 257.	Picrite.	Schönau.
N. Jahrb., 1902, vol. ii., p. 401.	Limburgite.	Cabo Frio, Brazil.

List of 1000 Rocks

Group.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.	MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.
62	38·0	(4·4)	·3	3·1	9·3	6·0	5·9	17·1	10·4	3·5	2·0	100·2	3·07
	36·5	8·4			9·9	6·7	6·0	18·1	10·3	3·1	1·6	100·6	2·99
13 rocks	42·2	1·5	·2	1·9	12·0	4·3	6·1	16·6	12·0	1·8	1·6	100·2	2·95
	±1·9				±3·0	±3·4		±1·9		±2·1		±12·3	
63	*b b b b b a*												
	46·3	·3		1·1	14·4	4·4	5·8	12·0	11·7	2·4	1·5	99·9	3·45
	46·1	1·1	1·5	1·0	13·1	10·6		12·6	10·0	2·6	2·0	100·5	...
	44·2	1·6		1·1	10·2	9·7	6·3	12·0	11·3	2·7	1·1	100·2	...
	42·9	·1	·4	1·8	10·9	4·3	8·9	14·0	13·2	3·2	·6	100·4	...
	42·5			6·5	12·0	2·7	8·3	12·0	11·8	2·7	2·2	100·5	2·91
	42·1	1·9	·3	3·1	12·2	2·7	7·9	11·5	11·3	5·1	1·1	99·2	2·97
	42·0	·5		1·0	10·2	9·0	10·6	9·2	16·4	1·3	·6	100·8	...
	41·8	1·5		3·7	8·1	12·7	3·7	10·1	13·6	3·6	1·2	100·0	...
	41·8	2·1		2·9	12·4	6·3	4·8	13·6	10·9	3·4	1·7	99·9	3·01
	41·5	·8	2·0	1·5	14·0	8·3	5·3	7·7	13·6	4·0	1·6	100·3	3·05
	41·3	·7	·9	1·6	8·7	10·6	7·5	11·5	13·1	3·4	1·1	100·4	...
	38·9	1·6	·4	5·2	15·4	5·1	4·7	5·6	16·5	5·3	1·8	100·6	...
	38·0	3·5	·3	1·1	11·8	7·8	6·1	11·9	14·5	3·9	2·0	100·9	...
	36·5	3·1		1·4	10·2	8·3	9·3	8·2	18·8	2·1	1·1	99·1	...
14 rocks	41·9	1·3	·4	2·4	11·7	7·3	6·4	10·9	13·3	3·3	1·4	100·2	3·25
	±1·8				±1·7	±2·8		±1·5		±1·0		±8·8	
64	*b b b b b b*												
	44·8		·5	2·2	15·4	3·4	6·7	12·8	9·8	3·0	1·7	100·3	2·98
	42·3	2·2	·3	2·8	12·1	5·0	6·3	15·2	9·8	2·7	1·9	101·6	3·07
	41·3	1·2		1·5	14·4	5·4	9·7	10·0	11·5	4·6	1·0	100·6	...
	41·2		1·2	3·8	12·4	6·3	8·0	9·7	11·1	2·4	3·7	99·8	...
	40·9	·7	·9	1·4	10·5	3·4	10·4	14·6	12·6	3·2	1·1	100·0	3·14
	40·7	1·3		3·2	8·4	14·3	6·3	12·0	11·4	2·3	·8	100·7	...
	39·9		·8	·5	10·0	12·9	4·1	14·8	13·3	2·5	1·8	100·6	3·19
	39·0			3·4	19·8	3·0	4·5	16·4	12·1	·9	·4	99·4	...
	38·0	2·0	1·6	1·6	14·4	7·9	7·0	10·2	10·6	5·2	1·9	100·1	...
	37·8			·6	19·9	3·5	12·7	11·0	14·7	1·7		101·8	...
	36·4	·4	1·0	6·3	12·9	8·3	4·6	11·4	14·5	1·0	3·0	99·8	...
	35·8	·5	·7	1·0	13·4	16·8	4·0	8·8	15·0	3·8	·7	100·5	...
	35·6	8·0			11·3	9·3	6·7	14·7	9·0	3·9	1·8	100·1	3·05
13 rocks	39·5	1·3	·5	2·2	13·5	7·7	7·0	12·4	12·0	2·9	1·5	100·4	3·06
	±2·2				±2·6	±2·7		±2·2		±1·2		±10·9	

-*continued.*

F. Rinne, *Sb. Berl. Acad.*, 1891, p. 988.	Basalt.	Westphalia.
U. Grubenmann, *In Diss. Zürich*, 1886, p. 20.	Melilite basalt.	Hegau, Baden.
N. *Jahrb.*, 1905, vol. ii., p. 62.	Eclogite.	Oetz Valley.
M. Bauer, *N.J.*, 1891, ii., p. 159.	Basalt.	Nassau.
N. *Jahrb.*, 1903–4, suppl. 18, p. 466.	Basalt.	Fichtelgebirge.
J. B. Harrison, *Rocks of Grenada*, London, 1896, p. 10.	Olivine basalt.	West Indies.
N. *Jahrb.*, 1902, ii., p. 67.	Monchiquite.	Grainbank, Orkney Isl.
N. *Jahrb.*, 1903–4, suppl. 18, p. 465.	Limburgite.	Near Cassel.
N. *Jahrb.*, 1904, vol. i., p. 68.	Pyroxenite.	Mal Inverno, Monzoni.
N. *Jahrb.*, 1903–4, suppl. 18, p. 466.	Basalt.	Fichtelgebirge.
N. *Jahrb.*, 1888, vol. i., p. 116.	Nepheline basalt.	Rhön.
N. *Jahrb.*, 1905, vol. ii., p. 213.	Nepheline basalt.	Rhön.
N. *Jahrb.*, 1903–4, suppl. 18, p. 465.	Basanitoid basalt.	Fichtelgebirge.
N. *Jahrb.*, 1893, vol. ii., p. 347.	Elaeolite syenite.	Arkansas.
N. *Jahrb.*, 1903–4, suppl. 18, p. 464.	Nepheline basalt.	Fichtelgebirge.
J. F. Williams, *A.R. Ark. G.S.*, 1890, ii., p. 227, 1891.	Nepheline syenite.	Arkansas.
K. Oebbeke, *Jb. Pr. G.A.L.A.*, ix., p. 402, 1889.	Nepheline basanite.	Hessen.
N. *Jahrb.*, 1892, vol. i., p. 278.	Limburgite.	Essigberg, near Cassel.
N. *Jahrb.*, 1904, vol. i., p. 54	Basalt.	Schluckenau, Bohemia.
N. *Jahrb.*, 1903–4, suppl. 18, p. 465.	Basalt.	Near Giessen.
N. *Jahrb.*, 1905, vol. i., p. 271.	Nepheline basalt.	Schanzberg, Bohemia.
N. *Jahrb.*, 1903–4, suppl. 18, p. 465.	Limburgite (also group 52).	Fichtelgebirge.
N. *Jahrb.*, 1903–4, suppl. 18, p. 464.	Basalt.	Löbauer, Berg.
N. *Jahrb.*, 1903, vol. i., p 417.	Ariègite.	Lherz, France.
N. *Jahrb.*, 1902, vol. ii., p. 32.	Basanitoid basalt.	Langenberg.
Loewinson-Lessing, *G. Sk. Jushno. Dorpat.*, 1900, p. 166.	Magnetite gabbro.	Ural.
N. *Jahrb.*, 1903, ii., p. 347.	Ouachitite.	Pot. sulph. springs, Arkansas.
N. *Jahrb.*, 1903–4, suppl. 18, p. 464.	Basalt.	Rhön.
U. Grubenmann, *In. Diss. Zürich*, 1886, p. 35.	Melilite basalt.	Hegau, Baden.

TABLE III.

64 Groups.

No.	Formula.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.
1	*a a a a a a*	80·4			2·8	9·2	1·5	·6
2	*a a a a a b*	77·4	·1		1·7	11·3	1·9	·7
3	*a a a a b a*	75·8	·1		1·3	11·8	1·3	1·2
4	*a a a a b b*	74·3			1·7	13·2	2·1	1·1
5	*a a a b a a*	74·2	·1		1·7	13·7	1·0	·4
6	*a a a b a b*	74·2	·1		1·1	13·7	1·3	2·0
7	*a a a b b a*	72·3	·2		1·1	14·5	2·1	·8
8	*a a a b b b*	69·7	·1		1·2	16·5	2·1	1·3
9	*a a b a a a*	74·1	·1		·9	12·9	·7	·5
10	*a a b a a b*	70·8	·1		1·2	14·0	1·0	2·0
11	*a a b a b a*	70·2	·2		1·2	14·8	1·7	·9
12	*a a b a b b*	68·1	·2	·2	1·6	15·3	2·4	1·1
13	*a a b b a a*	67·8	·1		1·0	16·9	1·2	·6
14	*a a b b a b*	65·1	·1		1·3	17·1	1·7	1·3
15	*a a b b b a*	59·5	·2	·1	2·1	20·5	2·4	·8
16	*a a b b b b*	57·0	·1		1·9	21·2	1·7	2·4
17	*a b a a a a*	70·2	·3	·1	1·2	11·6	3·8	2·4
18	*a b a a a b*	65·1	·4	·1	1·8	14·9	2·8	2·7
19	*a b a a b a*	65·9	·1		1·1	12·5	5·5	3·0
20	*a b a a b b*	61·6	·3	·1	2·4	15·4	6·0	2·6
21	*a b a b a a*	64·6	·2	·1	1·0	17·1	2·1	1·8
22	*a b a b a b*	61·8	·2	·1	1·6	17·7	2·7	2·8
23	*a b a b b a*	61·5	·1		1·4	18·7	3·3	1·6
24	*a b a b b b*	58·9	·5	·2	1·4	18·8	3·6	3·6
25	*a b b a a a*	60·7	·4	·1	1·6	16·4	2·0	3·2
26	*a b b a a b*	57·8	·8	·3	2·0	15·1	3·5	2·8
27	*a b b a b a*	59·9	·4	·1	1·5	17·2	2·6	3·3
28	*a b b a b b*	56·7	·6	·2	1·6	17·2	4·8	2·6
29	*a b b b a a*	58·2	·2	·1	2·2	19·5	2·2	1·8
30	*a b b b a b*	57·0	·4	·2	1·5	19·1	3·1	2·5
31	*a b b b b a*	55·1	·3		2·5	20·6	2·8	1·8
32	*a b b b b b*	53·6	·2	·1	1·5	20·4	4·1	2·4
33	*b a a a a a*	57·6	·4	·2	1·7	16·1	4·9	3·7
34	*b a a a a b*	53·9	·9	·3	1·5	15·3	5·0	6·4
35	*b a a a b a*	53·2	·7	·3	2·1	15·9	6·0	6·5
36	*b a a a b b*	51·9	·6	·2	1·7	17·8	6·9	6·4
37	*b a a b a a*	55·6	·3	·2	2·0	20·0	2·7	3·3
38	*b a a b a b*	53·3	·3	·2	1·4	19·3	3·1	5·8
39	*b a a b b a*	50·9	·8	·3	2·3	20·2	5·6	4·3
40	*b a a b b b*	48·9	·2	·2	1·3	21·5	7·2	5·6
41	*b a b a a a*	54·1	·5	·6	1·9	14·2	3·2	4·7
42	*b a b a a b*	52·3	·6	·2	1·8	15·0	3·1	5·8
43	*b a b a b a*	50·0	1·4	·2	2·3	15·5	5·2	5·6
44	*b a b a b b*	48·4	1·0	·1	1·9	16·9	4·0	7·5
45	*b a b b a a*	49·0	·9	·4	2·4	17·9	3·8	3·8
46	*b a b b a b*	47·9	·4	·1	1·4	20·2	3·8	4·4
47	*b a b b b a*	46·9	·9	·5	1·7	20·0	3·7	4·6
48	*b a b b b b*	47·0	·1	·1	1·2	19·5	5·2	5·2
49	*b b a a a a*	50·2	·9	·2	1·4	11·8	6·0	6·7
50	*b b a a a b*	46·0	·8	·1	2·9	11·2	3·7	11·2
51	*b b a a b a*	49·1	·9	·1	1·3	13·3	6·8	9·2

64 Groups.

MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.	No. of Rocks.	Deviation.	
							SiO_2, etc.	Total.
·3	1·1	2·4	1·8	100·0	2·64 (2·30)	10	1·9	6·8
·7	1·3	1·5	3·2	99·8	2·61	12	1·0	5·9
·3	·9	4·2	3·0	100·1	2·65 (2·36)	21	1·0	4·0
·6	·7	2·1	4·2	100·0	2·62 (2·34)	18	1·1	6·0
·3	1·1	4·1	3·5	100·0	2·60 (2·35)	23	·9	4·2
·5	1·1	2·7	4·6	100·1	2·61 (2·42)	17	1·0	3·8
·3	1·2	4·0	3·7	100·2	2·64	19	1·1	4·9
·3	1·4	2·8	4·6	99·9	2·66 (2·38)	14	1·7	6·5
·7	1·4	4·4	4·6	100·2	2·64 (2·39)	14	1·4	5·9
1·5	2·5	3·1	3·8	100·0	2·67	13	1·2	4·7
·6	1·8	4·5	4·3	100·2	2·63 (2·40)	29	1·1	5·0
1·1	1·8	2·9	5·5	100·1	2·66	20	1·4	4·9
·7	2·4	4·9	4·5	100·0	2·61	11	1·5	6·3
·8	2·2	4·6	6·5	100·2	2·60	11	1·5	6·6
·5	1·5	6·1	6·2	99·9	2·75	10	1·6	6·1
·5	1·2	5·8	8·4	100·2	2·60	6	2·8	9·6
1·4	3·1	4·7	1·5	100·2	2·76	11	2·1	10·5
2·0	4·5	3·2	2·6	100·2	2·64	16	1·1	5·5
·9	2·3	5·4	3·4	100·2	2·60	14	2·0	9·7
2·5	2·2	3·3	3·6	100·0	2·72	18	1·6	7·5
1·6	4·4	5·0	2·1	100·0	2·49	15	1·4	5·9
2·2	4·7	3·9	2·5	100·3	2·61 (2·18)	24	1·3	5·3
1·0	2·3	7·2	2·9	100·1	2·72	15	1·5	8·1
1·1	2·6	4·8	4·7	100·1	2·64	12	·9	7·7
3·1	5·8	4·3	2·4	100·1	2·65	16	1·1	5·9
4·6	4·8	3·0	5·1	99·8	2·67	18	1·4	8·7
1·1	3·6	6·1	4·4	100·2	2·70	12	1·2	7·8
1·7	3·0	5·8	5·6	100·2	2·63	14	1·7	7·7
·8	2·4	8·1	4·6	100·1	2·53	15	1·1	5·0
1·4	3·0	5·6	6·4	100·2	2·57	13	1·1	4·9
·4	1·7	9·5	4·9	99·9	2·56	16	1·1	5·2
·7	2·7	7·5	6·8	100·1	2·56	17	1·1	6·4
3·7	7·4	3·0	1·4	100·1	2·79	25	1·1	6·4
5·1	7·7	2·4	1·5	100·0	2·83	24	1·6	8·0
3·5	6·5	3·8	1·6	100·1	2·88	16	1·8	7·5
4·8	5·8	2·9	1·1	99·9	2·90	17	2·3	9·7
2·6	8·5	3·9	1·4	100·4	2·76	21	1·5	8·5
4·2	8·4	2·9	1·3	100·1	2·82	14	1·9	7·7
2·6	6·4	4·1	2·5	100·1	2·74	17	1·7	8·9
3·6	6·9	2·7	2·1	100·3	2·93	15	1·0	8·7
7·1	8·7	3·1	2·1	100·3	2·79	21	1·4	7·3
8·2	8·9	1·9	2·7	100·4	2·90 (2·56)	22	1·6	7·6
5·6	9·2	3·1	2·1	100·1	2·95 (2·65)	19	1·0	5·9
7·0	8·4	2·9	2·0	100·1	2·93 (2·79)	17	1·1	5·9
5·0	11·1	4·2	2·1	100·6	2·82	15	1·5	9·4
6·6	10·6	3·1	1·9	100·3	2·91	18	1·4	8·1
2·9	8·4	7·5	3·0	100·1	2·76	19	2·6	11·9
4·2	9·3	3·3	5·2	100·2	2·78	15	1·6	8·0
8·9	9·4	2·8	1·2	99·7	2·92	17	1·1	6·2
13·3	8·7	1·6	·5	100·1	2·96	14	2·2	9·7
5·8	9·4	3·4	·9	100·2	3·02 (2·45)	16	1·3	6·9

64 GROUPS

No.	Formula.	SiO_2.	TiO_2.	P_2O_5.	H_2O.	Al_2O_3.	Fe_2O_3.	FeO.
52	*b b a a b b*	42·2	2·1		2·3	13·2	8·4	11·2
53	*b b a b a a*	46·5	1·6	·3	2·6	14·0	5·4	6·3
54	*b b a b a b*	44·6	1·0	·4	1·4	16·3	5·7	6·8
55	*b b a b b a*	44·7	·9	·3	1·5	16·6	7·0	7·8
56	*b b a b b b*	42·2	·2	·1	1·8	17·9	8·3	8·1
57	*b b b a a a*	45·7	·2		3·5	4·3	3·9	7·6
58	*b b b a a b*	42·0			1·3	2·6	3·0	7·7
59	*b b b a b a*	41·6	·4	·1	4·7	9·2	5·5	8·0
60	*b b b a b b*	37·0	·2		4·3	4·9	7·8	11·8
61	*b b b b a a*	46·7	·7	·2	1·9	11·3	3·4	5·7
62	*b b b b a b*	42·2	1·5	·2	1·9	12·0	4·3	6·1
63	*b b b b b a*	41·9	1·3	·4	2·4	11·7	7·3	6·4
64	*b b b b b b*	39·5	1·3	·5	2·2	13·5	7·7	7·0
		57·4	·5	·2	1·8	15·6	3·7	4·0

—continued.

MgO.	CaO.	Na_2O.	K_2O.	Total.	Sp. Gr.	No. of Rocks.	Deviation.	
							SiO_2, etc.	Total.
10·1	7·4	1·7	1·1	99·7	3·13 (2·56)	8	3·0	13·9
6·6	12·4	3·0	1·5	100·3	2·95	19	1·4	7·2
9·1	11·6	2·5	·9	100·4	3·00	13	1·6	6·9
5·3	10·8	3·6	1·4	100·1	3·12 (2·40)	19	1·6	8·5
6·7	10·6	2·8	1·3	100·3	3·08	15	3·0	10·9
26·8	6·8	·2	·2	99·3	3·25	7	3·0	10·0
41·2	2·3	·5	·1	100·5	3·20	10	1·8	10·1
22·0	6·8	1·1	·8	100·1	3·16	12	1·6	8·6
29·3	3·8	·2	·2	99·4	3·00?	45	3·8	12·0
12·7	14·3	2·1	1·1	100·0	3·09	14	1·9	10·9
16·6	12·0	1·8	1·6	100·2	2·95	13	1·9	12·3
10·9	13·3	3·3	1·4	100·2	3·25	14	1·8	8·8
12·4	12·0	2·9	1·5	100·4	3·06	13	2·2	10·9
4·6	5·8	3·7	2·9	100·1	2·80 (2·44)	1000		7·3

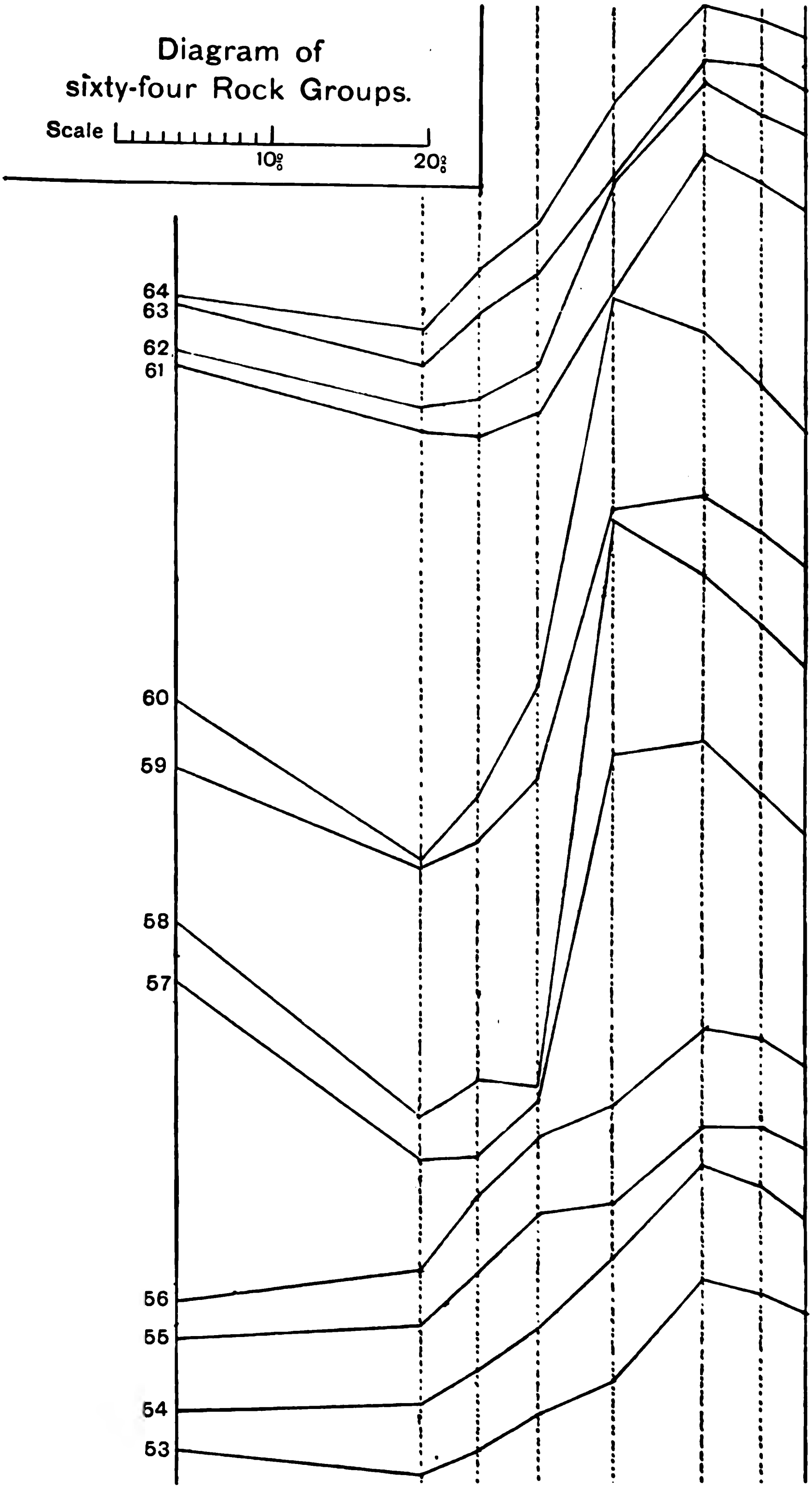
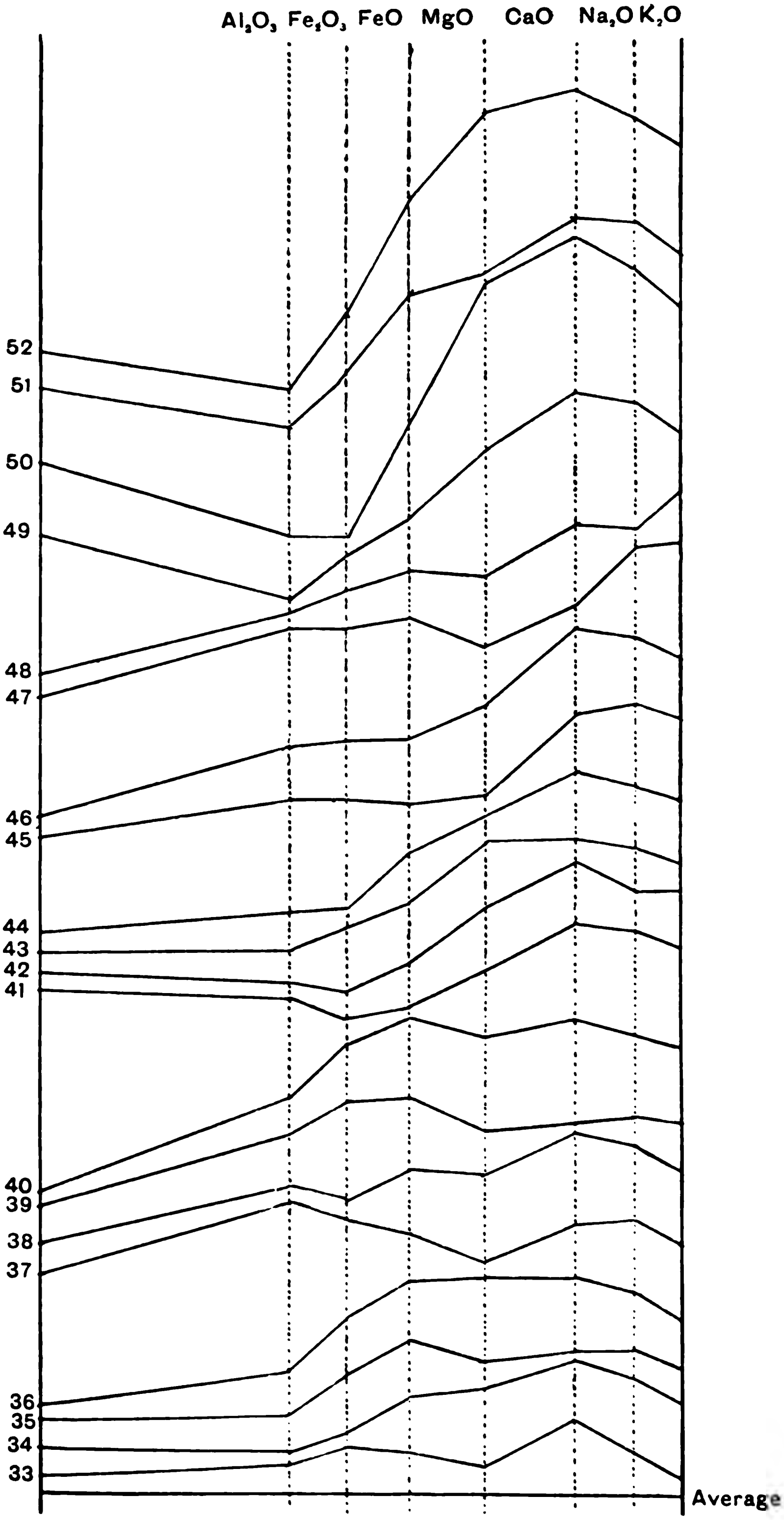
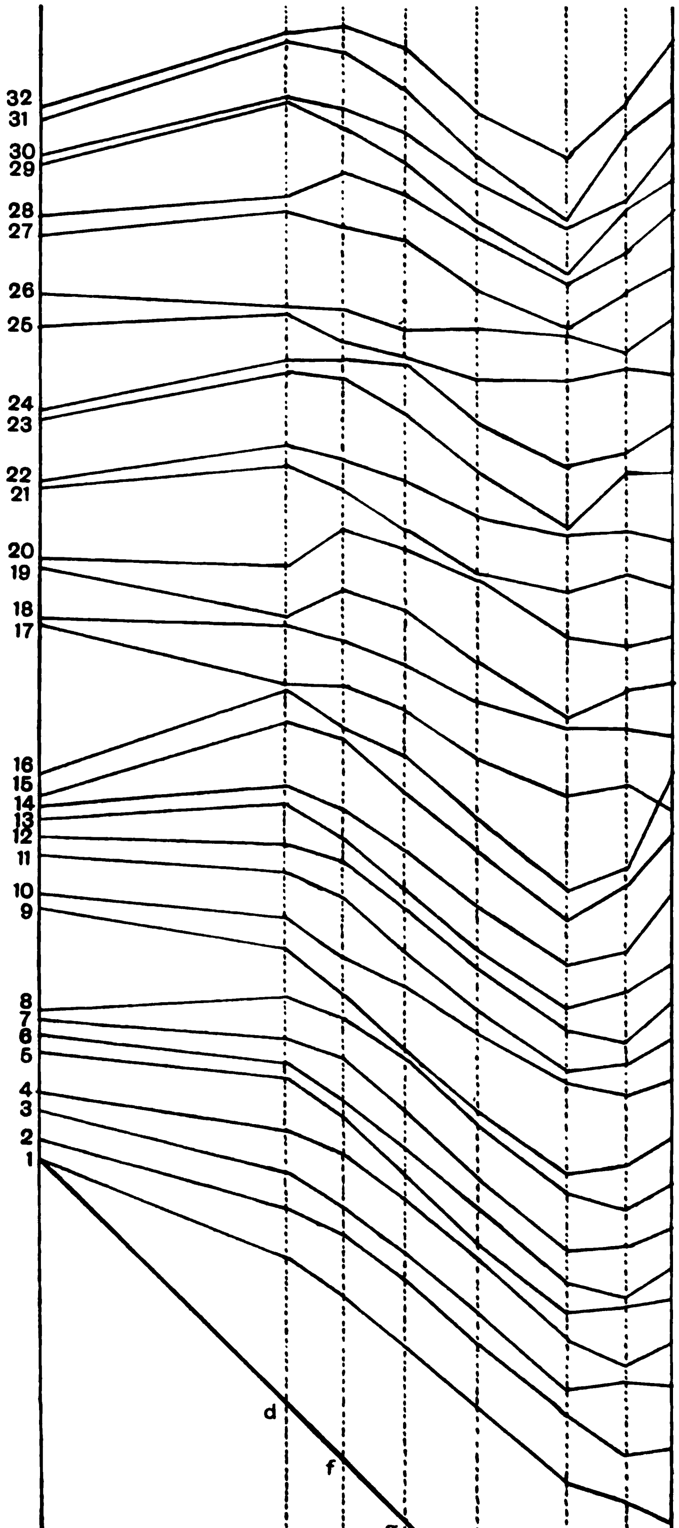


Diagram of
sixty-four Rock Groups.
Scale
10%
20%
64
63
62
61
60
59
58
57
56
55
54
53



Al_2O_3 Fe_2O_3 FeO MgO CaO Na_2O K_2O
52
51
50
49
48
47
46
45
44
43
42
41
40
39
38
37
36
35
34
33
Average


32
31
30
29
28
27
26
25
24
23
22
21
20
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
d
f
g

INDEX.

GROUPS.

GROUPS.

GROUPS.

GROUPS.

(*Issued separately February* 1, 1908.)

VIII.—A Note on the Roman Numerals. By James A. S. Barrett, M.A. (*Communicated by* J. Sutherland Black, M.A., LL.D.)

(MS. received October 2, 1907. Read November 4, 1907.)

CONTENTS.

Mathematics accepts the Roman numerals as symbols for definite numerical values, without regard to the question of their origin and early history; but this paper is concerned primarily with the form and origin of these symbols, and only incidentally with their numerical values. Yet, though our subject is only indirectly related to mathematics, a survey of some of the hypotheses that have been put forward to account for the forms of these symbols cannot but prove interesting to mathematicians, as the symbols were in general use until the sixteenth century and are still current everywhere, on the dials of clocks, the title-pages of books, and in printed references to authorities.

Our problem may be regarded as a single item of that vast subject, palæography or the science of ancient writing, which is concerned with the decipherment and history of the signs that man has employed to denote and record his thoughts and experiences. It may justly be termed a vast subject, for it embraces a study of languages and histories, customs and countries—ancient peoples with all their diverse modes of thought and expression.

As a result of the zeal and success with which archæological research has been prosecuted during recent years, this branch of science has undergone great extension and development. Curtain after curtain has been lifted, disclosing new departments to be studied, other and more remote periods to be investigated. Traces of an early civilisation have been discovered in Crete and the Ægean; and the Egyptian characters are now no longer regarded as the parent-alphabet. The spade of the expert has been busy in many lands, here removing the dust of centuries from some engraved tablet, there uncovering a scroll that was folded when the world was young. The characters, whether engraved or written, are, it may be, so old that they are again new. But their designs were born of mind, and still possess a magic vitality—some strange affinity with, and significance for, mind—as if through them the ancient world would whisper its long-hoarded secret. And when the daylight once again falls on the old tracings, the brooding mind discovers that they have a message: it is as if from the still undeciphered characters

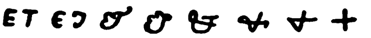
"Thought leapt out to wed with Thought
Ere Thought could wed itself with Speech."

If, by some chance discovery or the labour of many years, a clue to the interpretation of the old designs be found, then, it may be, the literature or history of an epoch that had passed out of human memory becomes again legible. For the alphabet, the mere ABC—like an *Iliad* in a nutshell—contains the record of man's early existence. Through these little symbols the intellectual history of mankind may be traced backwards towards its obscure beginnings, just as by following the course of some river one may explore a continent. But how mighty that sacred river of language, that broad vehicle of thought! How remote and inaccessible its source! Through how many unmapped deserts and past how many different civilisations has it flowed; ever swelling, ever changing; gleaming now in the sunlight of history, or again hidden in the mists of hypothesis and uncertainty! Rome and Greece were on its banks; it flowed past Egypt and Phœnicia; important tributaries reached it from that great Ægean, or Mycenæan, empire; and here and there along its course some stream of newly acquired letters or methods of writing was merged in its waters. And now, in modern times, this great river is still flowing, noiselessly carrying down to us on its broad bosom the literary flotsam and jetsam of civilisations that have perished; till, at its mouth—that ever-changing boundary which we name "to-day"—the quiet of its far-travelled waters is disturbed by the multitudinous waves of the spoken tongue. That sacred river of learning, of language—the alphabet! When one thinks of its measureless course, and of those empires, so stately, so transitory, whose intellectual commerce was borne on its waters, one pictures the whole as existing in some magic dreamland, like that which the poet saw in his vision:

"In Xanadu did Kubla Khan
A stately pleasure-dome decree:
Where Alph, the sacred river, ran
Through caverns measureless to man
Down to a sunless sea."

These introductory remarks may serve to remind us that our subject is complex, and that explanations or hypotheses may be sought in many directions and in different epochs. The purpose of this paper is, however, a strictly limited one: not a survey of that vast river of language, or of any one of its main tributaries, but merely a question regarding the probable explanation of the seven little signs called the Roman numerals, which have floated down to us on the great current. What do they signify? Whence did they come? What do scholars tell us regarding the origin of their forms?

The difficulty of the problem will be made evident by the following considerations:—(1) the remoteness of the period at which the symbols began to be employed; (2) the silence of the ancients regarding the origin or source of the signs—for nations, like individuals, do not chronicle their early beginnings; (3) the frequent modification of some of the forms, due to (*a*) the particular instruments or materials that were used for recording them, or to (*b*) their gradual approximation towards, or divergence from, other cognate signs, or to (*c*) the natural tendency towards the abbreviation of all signs and the avoidance of unnecessary labour in using them; (4) the survival and contemporaneous employment of divergent forms of the same symbol, so that among the various forms the original sign can hardly be ascertained with certainty.

To prepare us for the modification and variation of the forms which represented the Roman numerals, Nos. 3 and 4 of the above considerations may be briefly illustrated by some popular instances of alphabetic or other signs that have undergone modification or evolution. Thus we have  (Taylor, *The Alphabet*, i. 8), the well-known ampersand and the algebraical sign *plus*; the comma (,), which is the attenuated remnant of the original hair-line (/) that was used to mark a pause; the i and the j, which, like u and v, were originally one letter; the i, which acquired the accent or dot as a mark of distinction from u, m, and n—for example, ín; the long s (ſ), which has dropped out of use, except in the ligatures ſs and the *Fraktur* ß, because it was inconveniently like f; the ʒ in the abbreviation *viz.* (*videlicet*), which was originally only the sign of contraction, the semi-colon (;), written rapidly without lifting the pen, so that "vi;" became "viʒ," and, having acquired a second mark of contraction, is now printed as "viz."

The various hypotheses that have been put forward in explanation of the form of the Roman numerals may conveniently be grouped according to their subject-matter, as follows: (1) the alphabetic form; (2) the decussating principle; (3) the pictographic method.

I. First, then, of the hypothesis according to which the Roman numerals are explained by reference to independent alphabetic forms. According to the acrologic method, the Roman numerals should not only correspond to alphabetic forms, but also represent the initial letter of their several names. As an example of this method, at least for the purpose of illustration, the numerical signs employed by the Greeks in their Herodian system may be adduced, thus: Π(έντε) = 5, Δ(έκα) = 10, Η (the aspirate sign in Ηεκατόν) = 100, Χ(ίλιοι) = 1000, Μ(ύριοι) = 10,000. But, obviously, the acrologic method may be applied in two ways, according as we assume the priority of the name or of the sign. If the sign for a number be similar to the initial of that number's name, we may still ask whether it be not more probable that the sign gave rise to the name, than that the name gave its initial as a sign for the number; for, although calculation by means of numbers is a late development, it does not follow that primitive man did not employ signs to represent numbers at a date antecedent to the use of alphabetic characters. This statement might be expanded and illustrated in various ways. But to do so is unnecessary; for it is clear that the acrologic method cannot afford a clue to the origin of the Roman numerals, as only two of the forms admit of comparison with the initials of their names, viz. C (*centum*) and M (*mille*). It remains to inquire whether the source of the Roman numerals can be found in alphabetic forms, apart from the acrologic method.

According to Professor Zangemeister, the first scientific explanation was given by Professor Mommsen, the historian (see *Die unteritalischen Dialekte*, Leipzig, 1850, S. 19 & 33; and *Römische Geschichte*, Buch i. Kap. 14). But, in this connection, we shall quote the statement of Canon Isaac Taylor, who, in his great work *The Alphabet* (London, 1883, vol. i. p. 6, and vol. ii. p. 139), epitomises the alphabetic hypothesis which he had derived from Ritschl's "Zur Geschichte des lateinischen Alphabets" (*Reinisches Museum für Philologie*, 1869). Canon Taylor accepts Ritschl's theory so far as concerns the numerals L, C, D, M, while he dissents from Ritschl with regard to the numerals V, X, preferring to hold, with Mommsen and Grotefend, that these had an ideographic origin.

Canon Taylor writes as follows:—"The aspirated mutes, *phi*, *chi*, *theta*, were also retained in Etruscan, but not being required in Latin as phonetic symbols were utilised as numerals. For 50 the Romans used the Chalcidian *chi* ⩛, which assumed the less difficult lapidary type ⊥, and was then easily assimilated to L; while *theta* ⊙, which was employed to denote 100, was assimilated to C, doubtless because this letter was the initial of *centum*. For 1000 they used *phi* ↀ, which was written CIↃ, a sign afterwards confounded with ⋒ or M, the initial of *mille*; and the half of the primitive symbol ↀ, also assimilated to a familiar form D, was employed to denote 500."

The half-hearted approval with which this explanation was received was due, not so much to any inherent adequacy or excellence in the hypothesis itself, as to the remarkable lack of other hypotheses to contest its claims. The Roman sign for 1000, which assumed various forms (*e.g.* <I> ⋈ ∞ CXↃ CIↃ and the late ↀ, with the lapidary forms ᛗ ᛦ), constitutes the main difficulty in all alphabetic hypotheses. The explanation given by Canon Taylor appears to be based on the approximate similarity of one of its forms to the *phi* ↀ. The way is thus opened to add the easy but far-reaching suggestion that I> or IↃ or D is the half of ↀ, just as 500 is the half of 1000. *Theta* (though the elimination of the inscribed mark from that variously formed mute, ⊗, ⊙, Θ, need not be regarded as constituting an objection) is then contorted or "assimilated" to the form C, and the reason of this transformation is alleged to be that C is the initial of *centum*. How *centum* or its initial came into (*ex hypothesi*) independent existence is not explained; nor is any difficulty found in the fact that ↀ was not the early form of the sign for 1000, and that the half of it, denoting 500, was assimilated to D, although that was the initial of *decem*. It may also be pointed out that the Etruscans, who did retain the three letters *theta*, *phi*, *chi* in their alphabet, used numerical symbols that were almost identical with the Roman numerals, viz. I = 1, Λ = 5, X = 10, ↑ = 50, ⊗ = ? 100.

This hypothesis seems to involve the astounding assumption that a nation should have the conception of definite, high values without possessing any symbols to denote these values. Also it proceeds on a method which is entirely arbitrary. It discloses no common principle in the derivation of the forms; for some are admittedly pictographic, while others are referred to an alphabetic origin. It suggests no motive for the selection of the three particular alphabetic characters by the Romans, nor any reason why non-native signs should have been arbitrarily adopted at all—adopted, only to be thereafter transformed into Latin characters. The fact that the Greeks, in their later and alphabetic notation, employed the signs ϛ *stigma*, ϙ *koppa*, and ϡ *sampi*, to denote the values 6, 90, and 900 respectively, appears to afford some evidence in favour of this hypothesis. But the later Greek system of alphabetic numeration was unquestionably artificial (whether it was suggested by the ultimate acrology of the Herodian system, or was due to other circumstances, does not concern us here), and its purposes were definite and obvious; while, on the other hand, it will hardly be maintained that the cumbrous and unadaptable Roman notation, beginning (as Canon Taylor himself admits) with the primitive ideographs I, V, X, was an artificial system; or that, had it been such, the Romans required to go beyond their own alphabet in order to obtain the few signs that were necessary for the completion of their system. It is not, of course, denied that a system whose lower values are represented by pictographic signs may be arbitrarily extended by the adoption of alphabetic or other signs to denote the higher values—just as natives in various parts of the world have extended their own numerical systems by the arbitrary adoption of the English word *thousand*. But the acceptance of the pictographic method in explanation of the signs that represent the lower values in any system would at least suggest the question: Can the whole system not be similarly accounted for on the pictographic method?

In regard to the higher values of the Roman system, the alphabetic explanation is confronted by a dilemma. For, if it be held that *mille* was the highest denomination of that system, one may inquire why additional alphabetic forms were not adopted to denote higher values, as might readily have been done if the notation were merely alphabetic and arbitrary, and as indeed had been done in the Greek Herodian system, where M(ύριοι) = 10,000. If, on the other hand, it be held that the Roman system did not stop at *mille*, one may ask why, on the alphabetic hypothesis, the higher values are variously represented by the cumbrous duplication of CIↃ (as CIↃ CIↃ CIↃ = 3000; CCIↃↃ = 10,000; etc.), or by the convenient but later devices V̄, X̄, M̄, instead of by a supplementary alphabetic character. In fact, the rigid and unadaptable Roman system,*

* Next to the obscurity of its origin, the most remarkable thing about the Roman notation is its continuance and persistence without undergoing curtailment in respect of brevity, or evolution in respect of greater adaptability. The Arabian, Assyrian, and later Greek systems adopted the device of place-value (*e.g.* 4000; 𒌋𒁹𒈨 = 10 × 100, 𒐉𒌋𒁹 𒈨 = 4 × 1000; ρνγ́ = 153); even the Herodian system had its ingenious device of the circumscribing Π (𐅄 = 50, 𐅅 = 500, 𐅆 = 5000); while, on the other hand, the Roman system which existed contemporaneously and coextensively with the Empire is, if we may except the later and restricted employment of such forms as $\overline{XV}$ = 15,000, as inflexible and cumbrous as when first it appeared on the page of history. For mathematical purposes its

with its pictographs for the lower values, with its limited use of place-value, and with its morphological termination at CIↃ, points to another origin than the alphabetic, and would suggest rather a progressive evolution according to a definite principle or method. These considerations are non-technical and general, but they have this useful characteristic: they can be urged against any hypothesis that would account for the Roman numerals by means of mere alphabetic forms. "The system," writes Sir E. M. Thompson (*Greek and Latin Palæography*, 1906, p. 105), "was not an alphabetical one, for, although C (100) has been said to be the first letter of *centum* and M (1000) the first letter of *mille*, both these signs had a different derivation, and by a natural process only took the forms of the letters which they resembled most nearly."

II. The decussating principle, *i.e.* the hypothesis which explains the Roman numerals as formed by means of an additional decussation, or cross-line, for each tenth power.

Professor John Leslie, in his *Philosophy of Arithmetic* (Edinburgh, 1820, pp. 8–9), attempted to account for the symbols by means of this hypothesis, somewhat as follows:—Perpendicular lines, each denoting *one*, were repeated until *ten* was reached. A dash across the common unit, as X, then signified *ten*. The addition of a third stroke, or the mere drawing of three connected strokes, as ⊏ (=C), would then denote $1 \times 10 \times 10$ (=100); and four combined strokes formed M, "the utmost length to which the Romans first proceeded by direct notation." "But the division of these marks afterwards furnished characters for the intermediate numbers, and thence greatly shortened the repetition of the lower ones." Thus the half of X is V; ⊏ was divided as L; and "the four combined strokes M, which originally formed the character for *a thousand*, came afterwards, in the progress of the arts, to assume a round shape (ᛰ), frequently expressed thus CIↃ," of which the half, or IↃ, was employed to represent 500.

This ingenious, but curiously naïve, presentation of the problem need not be seriously considered. It is here referred to, and briefly described, only in order to assert Professor Leslie's title to be considered a forerunner of Professor Karl Zangemeister, the learned editor of the Pompeian tablets, whose explanation of the numerals by means of the same decussating principle is regarded by certain palæographers as affording a plausible solution of the problem. Professor Zangemeister's paper, entitled

disadvantages are obvious and great. Its only rule is that the larger number precedes the smaller—with this exception, that fours and nines, of whatever rank, may be written in terms of fives and tens (as IV, XL, CD; IX, XC, CM).

"Entstehung der römischen Zahlzeichen," was published in the *Sitzb. der k. Preussischen Akademie* (1887, pp. 1011–28). His account is complicated by incidental discussion regarding the relative priority of the different forms of the symbols; but his essential argument admits of brief and simple statement. He mentions that the words *decussis*, *decussare*, *decussatio*, *decussatim* occur in the works of Cicero, Vitruvius, Columella, and others; and he affirms that striking evidence for the existence of a principle of decussation is furnished by the following forms ("Einen schlagenden Beweis dafür, dass dies Princip des decussare wirklich bestand, bieten die folgenden Formen, welche sich bis in die römische Kaiserzeit hinein finden"): ꟾX = 20, ꟾꟾX = 30, ꟾꟾꟾX = 40. When the principle of decussation has been thus asserted, the next step in the logical, though not in the numerical, process is to apply it to the interpretation of the sign for 1000; because, as already stated, that sign is the crux of all hypotheses.

Now M, as used by the Romans, was merely an abbreviation for *mille* or *millia*, and had no connection with the *sign* for 1000 (although, after the invention of printing, the M came to represent the numerical sign direct; because the sign CIↃ, written hurriedly as (ꟾ), was, by the early printers, represented by the approximately similar form of the Gothic m, viz. 𝔐, and thus later by M). Zangemeister holds that the original sign was ∞ or CXↃ, and that the various other forms, ∞ and the later ↀ, were derived therefrom. Without being able to point to intermediate forms in its previous evolution, he considers that this sign ∞ arose from an earlier combination of decussated strokes, as ꟾ⋈ꟾ (= 1 × 10 × 10 × 10) = 1000. The half of this sign ∞, or ⋈, gives V, which gradually became assimilated to D (= 500); in any case, he thinks, the sign D cannot be derived from ↀ (= 1000), as the latter is relatively later. He asserts that the uncommon sign X̸ may, with great probability, be taken as denoting 100; and (on the analogy of ꟾ⋈ꟾ = 1000) he interprets this sign X̸ as being ꟾX (= 1 × 10 × 10), and supposes that the X gradually vanished, while the upright dash obligingly bowed to the receding X, until it finally became recognised as the C for which he was searching. The reason why the form X̸ did not continue in use, but gave place to the C, was possibly the fact that an X (= 10) could, by the addition of a single upright stroke, be falsified or altered to X̸ (= 100) ("Warum das X̸ sich nicht dauernd im Gebrauch erhielt, sondern dem andern Zeichen weichen musste, dafür lässt sich der Grund denken, dass ein X = 10 leicht durch Zufügung eines senkrechten Striches zu X̸ = 100 gefälscht werden konnte"). From that doubtful and ultimately discarded form, however, he derives the sign for 50, by the simple process of halving it, as V; or again, as an alternative origin, he suggests that this sign for 50 (ψ) may be merely a decussated V (5 × 10 = 50). Ten is, of course, the decussated I (X), and the half of that form is V (= 5). The fact that the signs of the smaller values I, V, X had "Doppelgänger" in the alphabet may, he thinks, have assisted the process by which, at a later date, the signs for 50, 100, 500 approximated to the letters L, C, D. Further, for numbers above 1000, in order to avoid unsymmetrical forms like CCXↃ or CXↃↃ, which could readily be falsified, the sign for 500 was taken as the base: thus

I> or IↃ = 500
IↃↃ = 5000; doubled as CCIↃↃ = 10,000
IↃↃↃ = 50,000; doubled as CCCIↃↃↃ = 100,000;

and, later, one finds the forms X̄V̄ = XV millia, and |X̄| = 1 million.

As *decussis*, *decussare*, and the other forms are merely mentioned in a note, without reference or amplification, it may be inferred that Zangemeister had little expectation of deriving effective support for his hypothesis by means of quotation from the Roman authors. *Decussis* (= *decem asses*) denoted ten aces, or units, whether in length, value, weight, or capacity; then the number *ten* in general, as also the sign X which represented it. *Decussare* signified "to divide, or cross, like the letter X." There is no suggestion that the word originally had, or ever acquired, a general mathematical meaning, as "to multiply by ten," or even "to reckon by tens." Thus the word had not the significance or value of, say, the Greek πεμπάζειν. Nor is it evident why such importance should be attached to the forms ᚺ ᛏᚺ ᛏᚺᚺ etc., or how they are to be held as furnishing "striking evidence" for the existence of a decussating principle. Are they, it may be asked, not merely a device for representing a series of X's? If that be so, then we get ᚺ = XX, which furnishes no stronger proof of the existence of a decussating principle than does a single X.

Further, if we turn to the other forms, we find that ↀ, or CXↃ, alone is not problematic. For the doubtful and discarded sign X is regarded as standing for 100 only with "great probability"; while any intermediate form in the supposed evolution of C, as *e.g.* ⚹, is confessedly hypothetic ("Diese Form ist von mir nur erschlossen, sie findet aber ihre Bestätigung in dem . . . Zeichen für 1000," usw.).

Thus, though the hypothesis in question is interesting, because it suggests not only a possible source of the several forms but also a principle connecting them, there is no direct evidence to prove the existence of such a principle. The forms X and ⋈, indeed, lend themselves conveniently to explanation by means of the alleged decussating principle. But even the most ingenious explanation of one or two forms requires to be corroborated by independent evidence, or shown to be historically probable, before it can be accepted in regard to those other forms to which it does not directly apply. In fact, the method of comparison and analogy is the only safe guide where variation in form has obscured the source of the symbols. Also, it may be pointed out that, if the hypothesis of Zangemeister be accepted, the Roman numerals represent a denary system; for the forms V, ⩛, D are regarded as mere arbitrary symbols, adopted from X, ⨯̄, and ⋈ respectively, not independently evolved. Is it to be believed that the Roman system was not originally quinary like the Herodian system of the Greeks (*cf.* *πεμπάζειν*), which so closely resembles it in method and in structure, though not in the form of the signs themselves? If the decussating principle be the true explanation, then the Roman numerals were the creation of scholars, and not the natural product of a people; and they were formed according to a principle which has left no clear record of itself in Roman literature, and not much trace even in the signs themselves.

III. The pictographic method. In our examination of the two preceding hypotheses, attention has necessarily been centred on the forms of the numerical signs, while practically no corroborative evidence has been adduced in favour of either hypothesis from Roman literature or from a consideration of the numerical systems of other nations. In contrast with such procedure, an attempt will be made in treating of this third hypothesis to adduce corroborative evidence from a consideration of the acknowledged origin of all alphabetic forms, from the chance allusions of Roman authors, and from a few examples of numerical signs used by other nations. In I. our question was: Can the Roman numerals be explained by reference to alphabetic characters? In II. we inquired whether an examination of the several signs would enable us to recognise in each an exhibition of the principle of decussation. In III. we shall ask: Can the Roman numerals be plausibly accounted for by the pictographic method, on the supposition that their origin was analogous to that of all alphabetic characters?

"If the history of any one of our alphabetic symbols be traced backwards," writes Canon Taylor (*op. cit.*, vol. i. pp. 8–9), "it will be found to resolve itself ultimately into the conventionalised picture of some object." Whatever problems remain in regard to the evolution of individual characters or the relative age and influence of particular alphabets, this view is regarded as axiomatic. On it is based the interpretation of Egyptian hieroglyphs, Hittite inscriptions, Cretan pictographs, the rude drawings of the North American Indians, and the rock-paintings of Australia.

Incidentally, too, it may be interesting to note that this method of interpretation, or rather this origin of alphabetic characters, was known to the Romans; thus Tacitus (*Ann.*, xi. 14) writes: "Primi per figuras animalium Aegyptii sensus mentis effingebant—ea antiquissima monimenta memoriae humanae inpressa saxis cernuntur—et litterarum semet inventores perhibent."

An examination of such systems, however, shows that not only animals and natural objects were represented in the primitive signs, but that the human body and its parts—the eyes, the arms and hands, the legs and feet —were rudely sketched and made to represent or express some act, motion, or feeling of primitive man.

Nor was it for alphabetic characters alone that the human body was thus used as model.* For measurements of all kinds it formed a convenient standard, sufficiently exact for practical purposes, capable of being applied to everything and employed by everyone. Did one wish to measure the size of an object? He had only to compare that object with a part, or parts, of his body. Did one wish to estimate distance? He had but to measure that distance with his own pair of compasses—recording the result in the number of strides taken. Thus the Romans called "handbreadths," *palmi* (palms: "hands," as one says in horse-measurement); "finger-breadths," *digiti*; "length from elbow to tip of middle finger," *cubitus* (cubit); "length from shoulder to wrist or finger," *ulna* (ell); "foot-length," *pes* ("foot"); "thumb-joint length," *uncia* (inch); and a "thousand paces," *mille passuum* (mile). To these might be added other examples, as, the ancient measure of weight, "dram," or "drachma" (Greek δραχμή, "what the hand can grasp"), a "handful"; "palm," for which the Egyptian symbol was the open hand; "fathom," the measure of length (A.-S. *fæthm*, "embrace"), like the Greek ὄργυια, the "length of the outstretched arms." Vitruvius, the learned and leisured architect to Augustus Cæsar, or the later writer who adopted that architect's name in the fourth century A.D., refers to this practice of body-measurement (*De Architectura*, III., i.): ". . . mensurarum rationes, quae in omnibus operibus videntur necessariae esse, ex corporis membris collegerunt, uti digitum palmum pedem cubitum." For "man," in a very literal sense, "was the measure of all things."

* A curious use of the body is described by N. W. Thomas in his *Natives of Australia* (London, 1906), p. 27: ". . . They touch various parts of the body in succession, the wrist, the arm, the head, etc., each standing for a particular day, until the intended date is reached. The two or more parties to the arrangement can then keep count of the flight of time by this ingenious system of mnemonics, and meet on the appointed day with as much certainty as if they noted their engagement in a diary."

But, if man used his body as a standard of measurement, could he not employ it also in the process of counting? Did not Nature provide him with a ready-made abacus—his fingers? Were the objects before him, say, five or ten; then one or both of his hands formed an unwritten record of the simple calculation. In this way the *mnemonic* stage was reached: he could at any time "count" the number of his few possessions by comparing them with his fingers. Did he wish to record the number for other persons, a rude sketch of the fingers was all that was necessary. And thus the *pictographic* stage was reached: he could now denote the number by means of a picture or drawing. When the picture became commonly used and understood, so that only a conventional symbol, and not an actual drawing, of the hand or fingers was required, then the final or *ideographic* stage was reached: man had invented idea-drawings or symbols to represent numbers, and could calculate.

If we would see the old discovery or invention taking place to-day, we have only to glance at the practice of some uncivilised race. Thus, Mr Charles Partridge, Assistant District Commissioner in Southern Nigeria, describes (*Cross River Natives*, London, 1905, pp. 244–5) the method of enumeration employed in his district:—"The natives count in fives. *One hand means five*; *two hands, ten.* When yams are counted, they are arranged in little heaps, each containing five. Brass rods are counted in bundles of twenty each, and large quantities in heaps of five bundles each —a hundred rods in a heap. They have adopted our word "thousand" to express ten of these heaps. The Ikwe [district of S. Nigeria] numbers are as follows: 1 olu, 2 abaw, 3 ataw, 4 anaw, 5 iso, 6 isi, 7 essa, 8 essataw, 9 tulu, 10 ili."

The following four extracts are quoted from Lubbock's *Origin of Civilisation* (2nd ed., London, 1870):—

"All over the world the fingers are used as counters, and although the numerals of most races are so worn down by use that we can no longer detect their original meaning, there are many savage tribes in which the words used are merely the verbal expressions of the signs used in counting with the fingers. . . . In Labrador 'tallek,' a hand, means also 'five,' and the term for twenty means hands and feet together. So also [Crantz, *Hist. of Greenland*, i. 225] the Esquimaux of Greenland for twenty say 'a man, that is, as many fingers and toes as a man has; . . . instead of 100 they say five men'" (p. 336).

"'The Aht Indians [Sproat, *Scenes and Studies of Savage Life*, pp. 121–2] count upon their fingers. They always count . . . by raising the hands with the palms upwards, and extending all the fingers, and bending down each finger as it is used for enumeration. They begin with the little finger. This little finger, then, is one. Now six is five (that is, one whole hand) and one more. We can easily see then why their word for six comprehends the word for one. . . . Again, when they have bent down the eighth finger, the most noticeable feature of the hand is that two fingers, that is, a finger and a thumb, remain extended. Now the Aht word for eight comprehends "atlah," the word for two. . . . Their word for nine comprehends "tsowwauk," the word for one. Nine is ten (or two whole hands) wanting one'" (p. 337).

"The Zamuca and Muysca Indians [Humboldt, *Personal Researches*, ii. 117] have a cumbrous but interesting system of numeration. For five they say, 'hand finished.' For six, 'one of the other hand,' . . . for ten they say, 'two hands finished,' . . . twenty is the feet finished; or in other words 'man,' because a man has ten fingers and ten toes, thus making twenty" (p. 337).

"Speaking of the Guiana natives, Mr Brett [Brett, *Indian Tribes of Guiana*, p. 417] observes . . . 'forty-five is laboriously expressed by . . . "two men and one hand upon it"'" (p. 338).

That the Romans, too, had some method of counting by the fingers is shown by their use of the word *digiti* ("digits," as we still term the lower numbers), in such phrases as: "in digitos rem redire," "in digitos mittere," "in digitis constituere," "digitos tollere." Vitruvius (*loc. cit.*) refers to the body as the source or origin of numbers: "Si autem in utrisque palmis ex articulis ab natura decem sunt perfecti," and "Ergo si convenit ex articulis hominis numerum inventum esse"; and Ovid (*Fasti*, iii. 121–3) has this allusion to what appears to have been the contemporary practice of digital numeration:

"Annus erat, decimum cum luna receperat orbem.
Hic numerus magno tunc in honore fuit.
Seu quia tot digiti, per quos numerare solemus."

And Juvenal (*Satires*, x. 240–1) implies that the system was not restricted to the lower numbers:

"Felix nimirum, qui tot per saecula mortem
Distulit atque suos jam dextra computat annos."

It is also evident that the first ten numerals admit of being interpreted and explained as successive steps in a digital numeration, though their forms have been conventionalised and even assimilated to alphabetic characters:

A.—I, II, III, IIII, V.
B.—VI, VII, VIII, VIIII, X.

We observe that in series A the numbers one to four are denoted by symbols, long ago conventionalised, of the finger, or fingers, of one hand; while the sign for five would originally be an image of the whole hand ("hand finished," as the Indians say; see p. 173), with the thumb projecting from the fingers. In series B the numbers six to nine are denoted by the addition of one finger or more of the second hand to the sign of the complete single hand; while ten is represented by a cross, or *decussis*, signifying the crossed arms, *i.e.* the combination or summation of the two hands. In fact, these signs seem to represent pictorially a digital numeration like that which, as we have seen above, is practised by uncivilised peoples at the present day. The parallel is so clear that another interpretation need hardly be sought. That the *four* and *nine* were occasionally, though not invariably, written as IV and IX instead of IIII * and VIIII need not occasion any difficulty. The former signs may have been independently formed from the method of early numeration (see p. 173), or more probably they would be adopted on the analogy of VI and XI.

In the absence of direct proof—which, indeed, is hardly to be looked for—this hypothesis might be supported by considerations derived from (1) the etymology of numerical terms, (2) a comparative study of the symbols of other numerical systems, (3) the accounts of gesture-language in ancient or modern times, (4) ideographs that may be relevant and significant though not incorporated in any numerical system. The limits of this paper permit of only a line or two in illustration of some of these headings.

"Let Father Gumilla, one of the early Jesuit missionaries in South America" (Tylor, *Primitive Culture*, 4th ed., London, 1903, i. p. 245, quoting from Gumilla, *Historia del Orenoco*, vol. iii. ch. xlv.), "describe for us the relation of gesture to speech in counting. . . . 'They [the Indians] say, for instance, "give me one pair of scissors," and forthwith they raise one finger; "give me two," and at once they raise two, and so on.† *They would never say "five" without showing a hand, never "ten" without holding out both. . . .*'" "The Zulu (*ibid.*, p. 251) counting on his fingers begins in general with the little finger of his left hand. When he comes to five, this he may call *edesanta*, 'finish hand'; then he goes on to the thumb of the right hand, and so the word *tatisitupa*, 'taking the thumb,' becomes a numeral for six. Then the verb *komba*, 'to point,' indicating the forefinger, or 'pointer,' makes the next numeral, seven. . . . This curious way of using the numeral verb is shown in such an example as 'amahasi *akombile*,'

* The form IIII is still commonly used on watches and clocks.

† It is interesting to recollect that in the "Latin" form of benediction the thumb, index, and middle finger are extended to symbolise the Trinity.

'the horses have pointed,' *i.e.* 'there were seven of them.' . . . *At the completion of each ten the two hands with open fingers are clapped together.*"

Turning now to ideographs, we find that a hand-symbol occurs in Egyptian, Hittite, Greek, Mexican, and other records. The Akkadian cuneiform for the word hand, ⋿, which later was represented as ⋿, and the Greek hieroglyph for δραχμή ("what the hand can grasp"), ⊢ ⊾ ⊾ (Gardthausen, *Griechische Palaeographie*, p. 259), may be adduced as

A North American Indian making the sign for *six*.
From the *First Annual Report of the Bureau of Ethnology . . . Smithsonian Institution, 1879–80* (Washington, 1881), p. 487.

illustrating the transition to an angular or "V" shape. Similarly, we can point to an ideograph of the arms-gesture. An illustration of it is contained in Mr Evans's "Primitive Pictographs . . . from Crete" (*J.H.S.*, vol. xiv. p. 303), and a reproduction of that illustration will be found in Mr Edward Clodd's *Story of the Alphabet* (London, 1900, p. 168). The form is

and Mr Evans describes it as "another ideograph taken from gesture-language. The sign may have indicated 'ten' or [being repeated ?] any multiple of ten: thus any great number."

Let us now return to the Roman numerals and examine their further development. We have seen that the earlier signs in the series may be regarded as ideographs of gesture-language: that the signs of the numbers one to four were originally rude drawings of the fingers or digits; that V, the symbol of five, represented the open hand; and that thereafter digits from the second hand were successively added until VIIII was reached, ten being denoted by a symbol representing the crossed arms, *i.e.* the summation of the two hands. We note that the fifth character is a new symbol, and that it is equivalent to the summation of five of the preceding symbols, for V = IIIII. Let us assume that the remaining symbols were likewise derived from ideographs of gesture-language, and that the principles governing their evolution were the following:—

1. *That no gesture shall be repeated more than five times in any one series.* This natural limit may be regarded as due to the fact that the eye cannot easily count more than five similar gestures or signs; or, more probably, it may be attributed to the influence of the hand (*i.e.* the five digits) in forming a habit of counting by fives.

2. *That the completion of a series of five similar gestures shall be signified by a position or gesture.* This concluding position or gesture (analogous to a mark of punctuation in writing) would be required to facilitate the summation of the preceding gestures. But it would inevitably acquire the value of the preceding series of five similar gestures, and thus it would ultimately supersede that series.

These two principles (which are involved in the pictographic formula, "IIIII, the five digits = V, the hand-gesture") provide the requisite clue for the construction of the later gestures, and thus of the numerical signs for the higher values. For, if these principles be granted, with the human body for instrument, and the smallest complement of instinct and faculty—a wish to count and the ability to draw—then signs corresponding to the Roman numerals will be the result, although their exact forms will be modified according as they are assimilated to this or that alphabet, and according to the other fluctuating conditions of time and place.

The actual development of the gesture-notation, governed by the two principles just stated, is not to be regarded as having taken place on a preconceived plan. Rather must we view it as the slow and gradual evolution of an experimental method, new gestures being added to the system only as the general conception of numerical values extended, and as the gestures representing them became fixed. Each new gesture, when accepted and codified, would in turn become the basis of, and means to, a further development. And in order that it might become a means to such further development, it required to be, so far as was practicable, not a mere position but a definite movement—a real gesture—capable of being repeated without ambiguity and without uncertainty. This tendency—the transition from mere stationary positions to actual gestures—is exemplified most clearly by the sign X, where the position of the two separate and extended hands has been superseded by the actual gesture of the crossed arms. Another tendency, namely, that a greater portion of the body is required to exhibit a greater value, is shown by even the first step in the progression: one hand means five; two hands, ten.

I. (1–5). First came the manipulation of the fingers or digits until the natural limit (p. 176) was reached, as: I, II, III, IIII, IIIII. The last position, viz. IIIII, originally the successive extension of the fingers, would, after a long interval of time, be superseded by the relatively complete gesture, the mere opening of the hand. This is represented by V, the sign of the open hand, and ultimately by the approximately similar alphabetic character V.

II. (5–10). The previous gesture, repeated with additional digits of the other hand until the natural limit is reached, gives V, VI, VII, VIII, VIIII, VIIIII or VV. But this last position, the mere extension of both hands, was incapable of being repeated without ambiguity; and thus it was superseded by the crossed-arms gesture, which signified the completion of the preceding series of five similar gestures, and which could be clearly repeated. We see this particular gesture in the pictograph on p. 175. It denoted ten, or, being repeated, any multiple of ten. It was represented by the conventional ideograph X, and ultimately by the corresponding letter X.

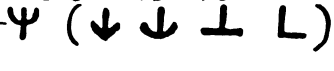
III. (10–50). The arms are successively crossed, as in the above gesture, until the natural limit of five similar gestures is reached, as X, XX, XXX, XXXX, XXXXX. The simplest method of signifying the completion of this series of five similar gestures would be to raise the arms slightly. This "gesture," as it did not move within definite limits, remained a vague position, and could not be repeated without ambiguity; but it did not need to be repeated. It served the purpose of marking the completion of the preceding series, and readily lent itself to representation by means of a pictograph of *the extended arms, with an upright stroke to denote the body, or a part of it.* By the time that this gesture acquired a definite numerical value, the signs of the lower values had doubtless become mere symbols; and thus the inference may be justified, that the representation of this gesture did not long continue in the pictographic stage, but quickly acquired a merely symbolic expression. That, however, we are here on the track of an actual gesture may be deduced from the pictographs of the North American Indians, no less than from the Egyptian hieroglyphs. This pictograph  represents the Indian gesture-sign for "many" (Clodd, p. 72), as if the man would signify that the amount which he desired to indicate exceeded the limits of his primitive notation—for, like the Greek *μύριοι*, the Latin *centum* and *mille*, and the English "million," the higher terms of all notations have denoted an indefinite number.* As regards the gesture, we may refer to the Egyptian hieroglyph , which likewise denoted a numerical value; and, as regards the form, we may perhaps compare some of the Cretan and Ægean signs given in Mr Evans's paper (p. 348)—for example, . At a very early period, and possibly under the influence of the Chalcidian character  (Gardthausen, Taf. i.; Thompson, p. 10; Taylor, ii. 59, 67, 90–1), the symbol which the Romans employed to designate fifty emerges in the form ∨. The attenuation which, on the above hypothesis, it has undergone in transition from the pictographic to the symbolic stage is not more remarkable than that of other symbols (*cf.* the negation signs:† Californian Indian , Maya , Egyptian : Clodd, p. 122); nor is it greater than the modification which, as a matter of fact and not of inference, it afterwards suffered in the process of assimilation to the form of the Latin "ell," L.‡ With that earlier process of attenuation or modification we may compare the later dismemberment which this letter, and with it the henceforth identical numerical sign, underwent, as shown in the uncial form L and the minuscule l, so that fifty is now represented by L and l alike. Other characters have had a similar history, as Ff and Tt; and the progressive attenuation of the Greek aspirate may likewise be compared: ⊢, L, ʻ (Thompson, pp. 71–72; Taylor, ii. 86).

This gesture of the raised arms, then, according to our second principle, signified the completion of the preceding series of five similar gestures (viz. XXXXX), and would thus acquire the value of that series and ultimately supersede it. We have discussed the

* "A curious feature of the native languages is that few have any numerals above three or four. . . . Anything above the highest numeral is 'many.'"—N. W. Thomas, *op. cit.*, p. 27.

"Spix and Martius [*Reise in Brazilien*, p. 387] say of the low tribes of Brazil, 'They count commonly by their finger joints, so up to three only. Any larger number they express by the word "many."'"—Tylor, *Primitive Culture*, 4th ed., London, 1903, i. 242.

† See *First Annual Report of the Bureau of Ethnology . . . Smithsonian Institution, 1879–80* (Washington, 1881), pp. 355–6.

‡ Mommsen (*Die unteritalischen Dialekte*, Leipzig, 1850, p. 33) gives the form and evolution of the Roman sign for fifty as follows:— 

graphic sign for this gesture in considerable detail, on account of the obscurity of its origin and the varied history of its form. The evolution of the remaining signs may be treated more briefly.

IV. (50–100). Again we proceed as before, by successive repetitions from the earlier series of gestures, until the natural limit of the new progression is reached, as L, LX, LXX, LXXX, LXXXX, LXXXXX. Now, the extended-arms position cannot be employed to signify the completion of this series of five similar gestures, for that position has already acquired a particular significance and value. The only available gesture, therefore, is that of the right arm moving into (if the expression be allowed) the "teapot-handle" position; this movement, being a true gesture, is capable of being repeated. The conventional symbol for it would be approximately <,* and, as such, would readily be assimilated to, and represented by, the early form of the Latin C. The gesture which this C represented signified the completion of the preceding series of five similar gestures (viz. the gestures corresponding to the signs LXXXXX), and thus it would inevitably acquire the value of that series and ultimately supersede it. In this way <, *i.e.* C, acquired the value of 100.

V. (100–500). Again, C, being a true gesture, can be repeated until the natural limit is reached, as C, CC, CCC, CCCC, CCCCC. No further movement of the right arm is available, as a value has already been allotted to each possible gesture; nor can the left arm be anew extended, crossed, or raised, for a similar reason. The only method of signifying the completion of this series of five similar gestures is, therefore, to put the left arm in the same "teapot-handle" position. As this was only a position, and did not undergo transition to a movement or gesture, it would be *denoted as a position*, *i.e.* the upright stroke for the body (*cf.* the upright wedge 𒁹 which precedes all names of men in Assyrian cuneiforms) would be included in the symbol. This gives us the form I>, which, as all the earlier symbols acquired alphabetic representation, would readily approximate to ▷,

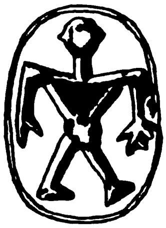
* It may be inferred that, at this elementary stage, when the values 500 and 1000 had not been evolved nor had any signs been allotted to them, the highest known denomination, viz. 100, would monopolise the total symbol for "man" (see p. 181). This inference is corroborated by the fact that the sign for 100 occurs in the cognate Etruscan system as ⊗; and one may assume that not until the higher values 500 and 1000 were evolved was the Roman sign for 100 differentiated from the sign for "man," which was thus set free to represent 1000. An analogy may be found in the fact that, while many tribes use the term "man" to denote 20 (*i.e.* the fingers and toes), the Tasmanians actually reach the limit of "man" with one hand (*i.e.* 5). See Tylor, *Primitive Culture*, 4th ed., London, 1903, i. 242–264.

the early form of the Latin letter, and would thereafter share its evolution to the form D, and the current D. A survival of the old form, viz. IↃ, was, however, used on the title-pages of printed books until comparatively recent times.

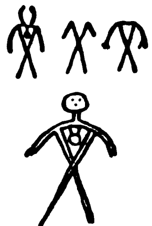
VI. (500–1000). Proceeding again by successive additions of the earlier series of gestures until the natural limit is reached, we get IↃ, IↃC, IↃCC, IↃCCC, IↃCCCC, IↃCCCCC. The only possible method of signifying the completion of this series is to bring both arms simultaneously to rest in the "teapot-handle" position, as CIↃ. Thus the gesture and sign CIↃ acquired the value of 1000 and superseded the gestures and signs IↃCCCCC.

It may be asked: "If CIↃ represents both arms in the 'teapot-handle' position, how can the several forms IↃC, IↃCC, etc., be also held as representing both arms in that position?" The explanation is that CIↃ represents a simultaneous gesture or position, while, like all the earlier series, the forms IↃC, IↃCC, etc., represent successive gestures in a progressive order.

It will doubtless be urged that the form ∞ or CXↃ is, in all probability, much older than CIↃ and could not be derived therefrom. But, while we have employed the form CIↃ for convenience' sake, we have not stated that it was the original sign. According to our present hypothesis, indeed, the probability is entirely the other way; and the answer to the above objection presents a striking corroboration of the pictographic hypothesis. For, if palæography shows that ∞ or CXↃ must be regarded as the earlier form, that is entirely what the pictographic method would lead us to infer. No elaborate pictograph for the gesture-sign of 1000 has been adduced, and, indeed, it is quite possible that no such pictograph ever existed; for, as the earlier gesture-signs reached the conventional and symbolic stage of representation, no detailed picture would need to be employed to denote the larger and late-acquired gesture-sign for 1000. The representation of the gesture corresponding to that number would, in short, be a conventional symbol, formed on the lines of the pictograph for "man," with omission of the details that were unessential in the representation of this gesture, and with emphasis or exaggerated representation of those details that were essential. Now, whether we look at the pictographs of Crete, or at the rude drawings of the North American Indians, we find that the symbol for "man" represents the body in the form of a cross:



I.



II.*

Thus, just as primitive races, employing a very rudimentary body-notation, stopped at twenty and called the result "man" (p. 172),† and for a hundred said "five men" (p. 172), so the inventors of this system, employing a highly developed body-notation, reached one thousand and graphically denoted the same by a modified form of the symbol for "man."

The divergent forms which the sign for 1000 assumed have been exemplified above, as <I> ⋈ ∞ ∈X∋ CIↃ ↀ ♏ ⋔ (p. 165; for the transition to M see p. 168), though, indeed, they were innumerable.‡ And, while they exhibit a suggestive resemblance to the "man" sign, their very number and variety show that originally they were not copies of a precise design, whether an alphabetic character or a particular combination of decussating lines, but conventional ideographs of a familiar object which admitted of varied representation—the human body.

An interesting point may here be stated. The sudden morphological termination of the Roman system at the sign for 1000 has not been, and cannot be, accounted for on either of the preceding hypotheses. The pictographic method, on the other hand, provides an obvious explanation of this surprising fact. *The system, as regards its form, terminated with the sign for 1000, because no further gesture was practicable.* Its further extension, ingenious as regards its construction, but cumbrous and impracticable as regards its form, represents merely the consistent employment of the signs and methods derived from the original body-notation. Those

* I. Evans, *op. cit.*, p. 338; *cf.* pp. 297, 302, 341. II. Clodd, *op. cit.*, pp. 59, 70; *cf.* pp. 36, 66, 67, 72.

† The following passage supports our hypothesis in a remarkable way:—"If we pass from the rude Greenlanders to the comparatively civilised Aztecs, we shall find on the northern as on the southern continent traces of early finger-numeration surviving among higher races. The Mexican names for the first four numerals are as obscure in etymology as our own. But when we come to 5 we find this expressed by *macuilli*; and as *ma* (ma-itl) means 'hand,' and *cuiloa* 'to paint or depict,' it is likely that the word for 5 may have meant something like 'hand-depicting.' In 10, *matlactli*, the word *ma*, 'hand,' appears again, while *tlactli* means half, and is represented *in the Mexican picture-writings by the figure of half a man from the waist upward*; thus it appears that the Aztec 10 means the 'hand-half' of a man, just as among the Towka Indians of South America 10 is expressed as 'half a man,' a whole man being 20. When the Aztecs reach 20 they call it *cempoalli*, 'one counting,' with evidently the same meaning as elsewhere—one whole man, fingers and toes."—Tylor, *Primitive Culture*, 4th ed., London, 1903, i. p. 249.

‡ ". . . Das römische [Zeichen für 1000], das in unzähligen Variationen die ausgebreitetste Anwendung gefunden hat."—Mommsen, *Die unteritalischen Dialekte*, p. 34.

who extended the system were using the signs of the higher values as symbols to which no series of actual gestures could possibly correspond—just as we use the terms "five thousand miles" and "forty thousand horse-power" without implying that the primitive bases for these values (the human stride and the drawing power of a horse) have been employed in the computation.

For, just as each new gesture in the series which we have considered becomes the basis of, and means to, a further development, so also the gesture stage as a whole gives place to the ideographic or symbolic stage; and that again gives us, on the one hand, alphabetic characters, and, on the other, numerical figures. A further stage is attained when the written or spoken word becomes the basis and expresses some abstract idea that could never be represented by any of the preceding symbols; or, in the case of numerical signs, when, as in algebra, the figures themselves are in turn superseded by letters or arbitrary symbols, denoting some wholly abstract conception.

(*Issued separately February* 1, 1908.)

IX.—Dr Edward Sang's Logarithmic, Trigonometrical, and Astronomical Tables.

At the Council Meeting of 5th July 1907, the following communication was received from the Misses Sang, daughters of the late Dr Edward Sang:—

"We, the daughters of the late Dr Edward Sang, LL.D., F.R.S.E., owners under his will of his collection of MS. Calculations in Trigonometry and Astronomy, having by letter of gift of date 12th February 1906 given the above collection to the President and Council of the Royal Society of Edinburgh, and having, by the cancelling on the 24th May 1907 of their acceptance thereof, received back the collection from the President and Council of the Royal Society of Edinburgh, do hereby give the said collection to the British Nation, and do hereby appoint the President and Council of the Royal Society of Edinburgh custodiers of the said collection, in trust for the British Nation, with power to publish such parts as may be judged useful to the scientific world.

"We do also hereby give into the custody of the President and Council of the Royal Society of Edinburgh, in trust for the British Nation, the duplicate Electrotype Plates of Dr Sang's 1871 New Seven-Place Table of Logarithms to 200,000, with power to use them for reproducing new editions, or publishing extended tables of seven-place logarithms.

"We would express the hope that Dr Sang's idea and plan for reproducing an authoritative and accurate Logarithmic Table, as explained in the last paragraph (p. 6 of the preface to the 1871 New Table of Seven-Place Logarithms), will be borne in mind, and given effect to.

"(Signed) Anna Wilkie Sang.
"(„) Flora Chalmers Sang.

"Oakdale, Broadstone Park,
Inverness, *1st July* 1907."

The manuscript volumes number forty-seven in all, the contents of thirty-three of which are in transfer duplicate. Volumes 1 to 3 contain the details of the steps of the calculations on which the results contained in the next thirty-six volumes are based.

Volume 4 contains the logarithms, calculated to 28 figures, of the prime numbers up to 10,000, and a few beyond.

Volumes 5 and 6 contain the logarithms to 28 figures of all numbers up to 20,000.

From these the succeeding thirty-two volumes are constructed, giving the logarithms to 15 places of all numbers from 100,000 to 370,000.

This colossal work must ever remain of the greatest value to computers of logarithmic tables. It is a great national possession.

The other Tables in the collection are trigonometrical and astronomical. Of special interest are the Tables of Sines and Tangents calculated according to the centesimal division of the quadrant.

It is hoped that ere long some of these Tables may be published in some form, so as to make them more immediately accessible to computers. They are the foundation of Dr Sang's published book of seven-place logarithms to 200,000, undoubtedly the most perfect of its kind ever printed. By placing the duplicate electrotype plates of this book along with the manuscript volumes in the custody of the Royal Society, with power to publish, the Misses Sang have given to the Nation every facility for publishing a new or even an extended edition of their father's work.

The complete account of the various tables follows, and the attention of the scientific world is now drawn to the importance of the collection in the custody of the Society.

In the name of the British Nation, the Royal Society of Edinburgh now publicly thank the Misses Sang for their valuable gift, and, as custodiers of these manuscript volumes, undertake to do all in their power to make them of real use to the scientific world.

The above statement was read by the Chairman at the First Ordinary Meeting of the Society, held on 4th November 1907.

The following general account was drawn up in November 1890 by Dr Edward Sang himself:—

"These computations were designed and undertaken with the view to the change from the ancient subdivision of the quadrant to the decimal system, a change long desired, and destined inevitably to be made. One hundred years ago it was on the very point of being completed. Mathematicians were then engaged in the introduction of the decimal system into every branch of calculation and measurement; but for the introduction of this new system into the measurement of angles, it was necessary to have a new trigonometrical canon. The French Government deputed M. Prony, with a large army of computers, to compile this new canon, and astronomers awaited with impatience the advent of this indispensable preparative. Laplace had, in anticipation, reduced all his data in the *Mécanique Céleste* to the new system, and instruments had been graduated suitably.

"We can hardly doubt but that if this new canon had then been published, the decimal graduation of the quadrant would have been very generally adopted even at the beginning of the present century; by the end of the first decade of this century it might indeed have been universally adopted. But the new trigonometrical tables, though magniloquently described, never made their appearance; and thus for something like seventy years the progress of the sciences thereon depending has been impeded.

"Very few are old enough to remember the disappointment felt throughout the scientific world. About 1815, in our school, the boys were exercised in computing short tables of logarithms and of sines and tangents, in order to gain the right to use Hutton's seven-place tables; and well do I recollect the almost awe with which we listened to descriptions of the extent and value of the renowned Cadastre Tables.

"In 1819 the British Government, at the instigation of Gilbert Davies, M.P., approached the French Government with a proposal to share the expense of publishing the Cadastre Tables, and a commission was appointed to consider the matter. The negotiations, however, fell through, for reasons which were never very publicly made known—but in the session 1820–21 the rumour was current amongst us students of mathematics in the University of Edinburgh, that the English Commissioners were dissatisfied of the soundness of the calculations—and so it was that the idea of an entire recalculation came into my mind.

"In the year 1848, encouraged by the acquisition of a copy of that admirable work, *Burckhardt's Table des Diviseurs* up to three million, the idea took a concrete shape in my mind, and I resolved to systematise the work which before I had carried on in a desultory way. Necessarily the first step was to construct a table of logarithms sufficiently extensive to satisfy all the wants of computers in trigonometry and astronomy; and having many times felt the inconvenience of the loss of the details of the calculations made on separate papers, I resolved to record from the very beginning every important step. This plan of operation has many conveniences—it enables us to retrace and examine every case of doubt, and also to take advantage, in new calculations, of anything in the previous work which may happen to be applicable.

"For all the ordinary operations of surveying and practical astronomy five-place logarithms, as M. Lalande has stated, are perfectly sufficient; and for the higher branches of astronomy and geodetics the usual seven-place tables are enough. But for the purpose of constructing new working tables it becomes necessary to carry the actual work further, both in the extent of the arguments and in the number of decimal places, and therefore I determined on the formation of a table of logarithms to nine places for all numbers up to one million. But again, in order that such a table be true to the ninth place, the actual calculation must be carried still further —and to meet the cases in which the doubtful figures from say 4997 to 5003 might occur in one million of cases, it became prudent to carry the accuracy even to the fifteenth place. And this limit of accuracy was further defined by the circumstance that there the differences of the third order just disappear. Even then it may happen that the doubt as to the figures which are to be rejected may not be cleared up, and it follows that a still more minute criterion should be at hand for use, and therefore the order of the work came to be as follows.

"In the first place, the computations of the logarithms of all numbers up to ten thousand, to twenty-eight (for twenty-five) places, was undertaken. At the outset, each logarithm of a prime number was computed twice, but as the work proceeded, it was judged advisable to have three distinct computations of each. The whole of this work is distinctly recorded and indexed, so that every step in reference to any given number can at once be traced out.

"The idea was entertained of this work being ultimately extended to one hundred thousand, and the logarithms of the composite numbers from ten to twenty thousand were computed, spaces being left for those of intermediate prime numbers.

"By the addition of the logarithms thus obtained, those of the great majority of composite numbers from the limit one hundred thousand to one hundred and fifty thousand were computed, and the intervals were filled up by help of second differences. In this part of the work I was aided by my daughters. But, in all such separate additions, we are liable to sporadic errors, and in order to guard against these the whole of this work was redone by the use of the last two figures of the second differences; and thereafter the calculations were made by short interpolations of second differences all the way to three hundred and seventy thousand. Necessarily, on account of the occurrence of the minute final errors, the last, or fifteenth, figures cannot be trusted to within one or two units; and after a very severe examination of the whole, it was found that in a very few instances this accumulation of last place inaccuracy extended even to five units; and thus we are warranted in expecting that no last place error will be found reaching so far as to unit in the fourteenth place—a degree of accuracy far, very far, beyond what can ever be required in any practical matter.

"In the compilation of the trigonometrical canon the same precautions were taken for securing the accuracy of the results. In the usual way, by means of the extraction of the square root, the quadrant was divided into ten equal parts, and the sines of these computed to thirty-three, for thirty places. These again were bisected thrice, thus giving the sine of each eightieth part of the quadrant; all the steps of the process being recorded.

"The quinquesection of these parts was effected by help of the method of the solution of equations of all orders, published by me in 1829; and the computation of the multiples of those parts was effected by the use of the usual formula for second differences. A table of the multiples of 2 ver. 00^c 25′ was made to facilitate the work, and the sines, first differences, and second differences were recorded in such a way as to enable one instantly to examine the accuracy. The same method of quinquesection was again repeated, and the computation of the canon to each fifth minute was effected by help of a table of one thousand multiples of 2 ver. 00^c 05′, the record being given to thirty-three places, the verification being examined at every fifth place. In this work there is no likelihood of a single error having escaped notice.

"For the third time this method of quinquesection was applied in order to obtain the sines of arcs to a single minute. A table of one thousand multiples of 2 ver. 00^c 01′ was computed to thirty-three places, but in the actual canon it was judged proper to curtail these, and the calculations were restricted to eighteen decimals on the scroll paper. In the actual canon as transcribed, only fifteen places are given. In all cases the function, its first difference, and its second difference are given in position ready for instantaneous examination; and the whole is expected to be free of error excepting in the rare cases where the rejected figures are 500—these cases being duly noted.

"For the computation of the canon of logarithmic sines the obvious process is to compute each one of its terms from the actual sine, by help of the table of logarithms; but this process does not possess the great advantage of self-verification, and attempts have been made to obtain a better one. Formulæ indeed have been given for the computation of the logarithmic sine without the intervention of the sine itself, but when we come to apply these formulæ to actual business we find that they imply a much greater amount of labour than the natural process does; and after all, they are only applicable to the separate individual cases.

"Nepair, as is well known, arranged his computations of the logarithms from the actual sines in such a way as to lessen by one-half the amount of the labour. Nepair's arrangement was therefore followed, and the work was begun from the sine of 100^c down to 50^c. The calculations were made by help of the fifteen-place table of logarithms from 100,000 to 370,000. If this table had been continued up to the whole million, the labour would have been greatly diminished, but we had to bring the numbers to within the actual range of our table by halving or doubling as the case might be. The results were then tested by first, second, and third differences, and in not a few cases the computation had to be redone, for the sake of some minute difference among the last figures. The log sines for the other half of the quadrant, that is from 50^c to 0^c, were deduced from the preceding by the use of first differences alone. The log tangents from 50^c down to 0^c were also deduced directly by help of the first differences alone. In this way the series of fundamental tables needed for the new system has been completed, so far as the limit of minutes goes.

"While that work was in progress, a circumstance occurred which temporarily changed the order of procedure. Kepler's celebrated problem has ever since his time exercised mathematicians, and, sharing the ambition of many others, I also sought often, and in vain, for an easy solution of it. Accident brought it again before me, and this time, considering not the relations of the lines connected with it, but the relations of the areas concerned, an exceedingly simple solution was found. In order to give effect to this method it was necessary to compute a table of the areas of circular segments in terms of the whole area of the circle. That again rendered it necessary to calculate the sines measured in parts of the quadrant as a unit, instead of in parts of the radius, as usual. This computation was effected by using the multiples of twice the versed sine formerly employed. From this again the canon of circular segments for each minute of the whole circumference was readily deduced. The mean anomaly of a planet may be deduced from its angle of position, or as it is generally called, its excentric anomaly, by simple additions and subtractions of these circular segments. The converse problem is very easily resolved, particularly when the first estimate is a tolerably close one. In order to be able promptly to make this first estimate sufficiently near in every possible case, a table of mean anomalies from degree to degree of the angular position, and also from degree to degree of the angle of excentricity of the orbit, has been computed according to the decimal system.

"The change to this system is inevitable. Each new discovery, each improvement in the art of observing, intensifies the need for the change, at the same time that each augmentation of our stock of data arranged in the ancient way adds to the difficulties. How much the change is needed may be estimated by an inspection of the Nautical Almanac. Every page in it cries out aloud in distress, 'Give us decimals.' For the Sun's meridian passage, the usual difference columns are suppressed, and those titled var. in 1 hour are substituted; and similarly for the Moon's hourly place a column titled var. in 10^{m} is given; while for the interpolation of lunar distances, proportional logarithms of the difference are given. While artisans and physicists are using the ten-millionth part of the Earth's quadrant as their unit of linear measure, astronomers are still subdividing the quadrant into 90, 60, 60, and 100 parts. The labour of interpolation is unnecessarily doubled at the very least, and that heavy burden is laid on the shoulders of all the daily users of the ephemeris. The trouble attending the reduction of observations tends to lead the navigator to shun the making of observations. The matter is not merely of national, it is of cosmopolitan interest—and this continuous waste of labour has much need to be ended.

"The collection of computations above described contains all that is essentially needed for the change of system, as far as the trigonometrical department is concerned; the great desideratum being the Canon of Logarithmic Sines and Tangents. In addition to the results being accurate to a degree far beyond what can ever be needed in practical matters, it contains what no work of the kind has contained before, a complete and clear record of all the steps by which those results were reached. Thus we are enabled at once to verify, or if necessary, to correct the record, so making it a standard for all time.

"For these reasons it is proposed that the entire collection be acquired by, and preserved in, some official library, so as to be accessible to all interested in such matters; so that future computers may be enabled to extend the work without the need of recomputing what has been already done; and also so that those extracts which are judged to be expedient may be published.

"Seeing that the Logarithmic Canon is useful in all manner of calculations, the printing of the table of nine-place logarithms might be advantageously proceeded with at once. The publication of the corresponding Canon of Logarithmic Sines and Tangents would only be advisable in the expectation of its early adoption by astronomers.

"But land-surveyors, when transporting the theodolite from one station to another, have to compute the new azimuth from the previously observed one. This is easily done by adding or subtracting 180°; yet in the hurry of business this occasionally gives rise to mistakes. On the other hand, with 400^c on the azimuth circle, we should only have to add or subtract 200^c, thus almost obviating the chance of a mistake. Hence the surveyor would be greatly benefited by the immediate publication of a five-place trigonometrical canon, arranged in the decimal way."

List of Logarithmic, Trigonometrical, and Astronomical Calculations, in Manuscript, by Edward Sang.

Nos. 1 *and* 2. *Logarithms I., II. Construction.*

These two volumes contain a complete record of the articulate steps of the calculations for the logarithms, to 28 places, of all prime numbers up to 10,000, with those of other large primes which happen in the course of the work.

No. 3. *Logarithms III. Revision.*

This third volume contains the calculation, in revision, for all those primes whose logarithms had not been computed thrice. This record is accompanied by an Index of all the Divisors used in the work, and of the primes themselves and the divisors with which they have been connected. In this revision no deviation exceeding 10 units in the 28th place was allowed to pass.

By this registration, a future computer is enabled to lessen his labour when he happens to have to do with a divisor which had occurred before, or when any easy multiple or sub-multiple may occur.

No. 4. *Logarithms. Primes.*

This is a list of the first 10,000 Prime Numbers (up to 104,759), with the logarithms, to 28 places, of those which have been computed (continuously up to 10,037, with occasional ones beyond), and with references to the pages of the construction in which they have been given. (The logarithms of the remaining primes are given to 15 places.)

No. 5. *Logarithms* 0.

Contains the logarithms, to 28 places, of all numbers up to 10,000; those of the composites having been got by the addition of those contained in No. 4.

No. 6. *Logarithms I.*

Contains the logarithms, to 28 places, of all composite numbers from 10,000 to 20,000, with those of primes incidentally found.

Nos. 7, 8, 9, 10, 11. *Logarithms* 10, 11, 12, 13, 14
(*Nos.* 100,000 to 150,000).

The logarithms given in these five volumes are restricted to 15 places. Those of the majority of the composite numbers were got by addition from vols. 0 and 1; the intermediates having been filled in by interpolation of second differences. This work had been done on scroll paper, and thence copied on the actual pages.

Nos. 12, 13, 14, 15, 16. *Logarithms* 10, 11, 12, 13, 14
(*Nos.* 100,000 to 150,000).

In order to remove the risk of detached errors in copying, the last two figures of the second differences were alone copied into their places from the previous volumes, and from these the complete second differences, the first differences, and the logarithms were re-computed by integration. (Also in transfer duplicate.)*

Nos. 17, 18, 19, 20, 21. *Logarithms* 15, 16, 17, 18, 19
(*Nos.* 150,000 to 200,000).

The logarithms in these five volumes were got by interpolating two terms between the even numbers of the preceding volumes, adding the logarithm of 1·5. The interpolation was done on paper-aside, using only the last two figures of the second differences. These last two figures were then copied into their places on the actual pages, and the work finished by integration. (Also in transfer duplicate.)

Nos. 22–38. *Logarithms* 20–36 (*Nos.* 200,000 to 370,000).

In these seventeen volumes, the logarithms have been found by interpolating one term between the terms of the preceding volumes from 10, adding the logarithm of 2; the work having been done by integration as before, and the results tested by addition at least twice in each decade. (Also in transfer duplicate.)

No. 39. *Logarithms. Auxiliary Table.*

This volume shows the last 10 figures of the logarithms of numbers from 1 00000 0000 to 1 00000 9999, and from 1 00000 0000 to 99999 0000, which are used for computing the logarithms of numbers consisting of more than six effective places. (Also in transfer duplicate.)

No. 40. *Sines.*

This is the record of all the articulate steps in the calculation, to 33 places, of the sines of arcs differing by the 2000th part of the quadrant.

* The volumes in transfer duplicate have been placed in the library of the University of Edinburgh.

By the extraction of the square root and repeated bisections, the quadrant was divided into 80 parts, and the sines of the multiples of $\cdot 01^c\ 25'$ were computed.

Thereafter the sines and cosines of $00^c\ 25'$ and of $01^c\ 25'$ were got by the direct resolution of the appropriate equations of the fifth degree, and were compared with those which had been got in computing the recurring functions of submultiples of π, the steps of which are copied into this record.

By help of 100 multiples of 2 ver. 25′, and of 1000 multiples of 2 ver. 5′, a table of sines of arcs differing by 25′, and thereafter one of arcs differing by 5′, were computed on the actual pages.

Although these have the appearance of being interpolations, they are truly independent computations, the use of the preceding work preventing mistakes, as well as the accumulation of the minute errors due to the rejection of figures beyond the 33rd place.

Nos. 41, 42. *Canon of Sines, Parts I., II.*

These volumes contain the sines to 15 places of arcs differing by 1′ (centesimal division) with their first and second differences, the computation having been facilitated by a table of 1000 multiples of 2 ver. 1′.

The table has been bound in two parts, for the convenience of referring to the sine and to the cosine of an arc. (Also in transfer duplicate.)

No. 43. *Log Sines and Tangents.*

The log sines from $100^c\ 00'$ down to $50^c\ 00'$ are here given to 15 places, with their first, second, and third differences. They were computed directly from the Canon of Sines by the 15-place table of logarithms from 100 000 to 370 000, and by use of the auxiliary table.

The log sines from $50^c\ 00'$ to $0^c\ 00'$ were derived from the preceding, according to the formula—

$$\sin a = \tfrac{1}{2} \sin 2a \,.\, \sec a \ ,$$

using the first differences only.

The log tangents from $50^c\ 00'$ to $0^c\ 00'$ were obtained from the preceding log sines, using only the first differences.

Upwards of two million eight hundred thousand figures were written for the completion of this volume. (Also in transfer duplicate.)

No. 44. *Sines in Degrees.*

This volume contains the values of the sines measured, not in parts of the radius, but in parts of the quadrant, and given to the ten-thousandth part of the degree. These sines were computed directly from degree to degree, then for each quarter of a degree, using the multiples of 2 ver. 25′, then to each 20th of a degree, and lastly to each minute. The work thus represents three independent computations.

No. 45. *Circular Segments.*

These circular segments are measured in parts of the surface of the circle as divided into 400 degrees of surface, and these subdivided into 1 0000 0000 parts. They have been computed by the integration of the second differences of the sines measured in degrees, and are carried round the entire 400 degrees of the circumference.

This table is intended to facilitate calculations concerning the elliptic motions of the planets; it gives us the mean anomaly when the planet's position is given, from the formula—

$$\text{Mean anomaly} = \tfrac{1}{2}\{\text{segm}\,(p+e) + \text{segm}\,(p-e)\},$$

in which p is the angle of position and e the angle of eccentricity of the orbit. (Also in transfer duplicate.)

No. 46. *Mean Anomalies* (*A*).

These are the mean anomalies in orbits of each degree of eccentricity from $e = 0^c$ to $e = 100^c$, given for each arc of position from $p = 0^c$ to $p = 200^c$, and carried to the eighth decimal place of the degree.

No. 47. *Mean Anomalies* (*B*).

In this volume the anomalies are given only to the nearest second, but the differences for a change of 1^c of position, and the variations for a change of 1^c in ellipticity, are filled in; and thus, of the three—the eccentricity, the position, the anomaly—any one may be determined from the others. (Also in transfer duplicate.)

31 Mayfield Road, Edinburgh,
July 1890.

Explanatory Statement from Miss Flora Chalmers Sang.

"12 Marchmont Street,
"Edinburgh, 20*th December* 1907.

"I desire to supplement the above documents with the following personal explanation.

"On the evening on which my father first brought his MS. Calculations before the Royal Soeiety of Edinburgh, when I learnt that he intended to associate my sister Jane's name and my own with his in the calculation of the 15-place Table of Logarithms, I objected so strongly to having my name brought before the public that he was obliged to yield to me.

"Here I acknowledge that I was wrong in my refusal, since by it I placed my father in a false position. It is in order to redress this error of mine that I write this explanation.

"In November 1890—the month before his death—when in his sick-room he was drawing up his "Account," he said to me, he did not see why he should not acknowledge the assistance we had given him, and I at once consented.

"My father had inscribed the volumes calculated and written by himself with his own name. In those calculated and written by my sister and by myself, he had pencilled our respective initials. He requested me, as he felt able, to bring our volumes to him in order that he might write in our names in full. His strength, however, failed him before the task was accomplished. Some months later I myself completed it.

"FLORA CHALMERS SANG."

The following scheme shows the exact extent of the assistance rendered by Dr Sang's daughters, Miss Jane Nicol Sang and Miss Flora Chalmers Sang, in preparing the logarithmic tables. The information was obtained from a careful inspection of the volumes.

Number of Volume.	Calculator.		
1 to 6	Dr Sang.	...	...
7	...	Miss J. N. Sang.	...
8 to 11	...	...	Miss F. C. Sang.
12 to 19	Dr Sang.	...	...
20 to 27	...	...	Miss F. C. Sang.
28 to 30	Dr Sang.	...	...
31	...	...	Miss F. C. Sang.
32 to 33	...	Miss J. N. Sang.	...
34 to 35	...	...	Miss F. C. Sang.
36 to 37	...	Miss J. N. Sang.	...
38	...	...	Miss F. C. Sang.
39 to 47	Dr Sang.	...	...
Number due to each,	26	5	16

The Committee appointed by the Council in 1905 to consider and report on the value of Dr Sang's Manuscript Volumes, which have now been gifted by his daughters to the Nation, asked the opinion of eminent calculators abroad. Extracts from the replies received from Dr Bauschinger and Dr Ristenpart are appended.

Extracts from Letter (27*th May* 1905) *received from Dr F. Bauschinger.*

"I can only express my highest admiration regarding this gigantic work, which I could never have believed it possible for a single man to accomplish. The whole plan I regard as exemplary, and of great and lasting scientific value. I believe the whole calculating world will welcome with joy the realisation of your endeavour to make this work of fundamental significance easily accessible. A complete printing is impossible; but it is not necessary if the following plan is adopted. Let the whole work be purchased and deposited in a safe place, where it may be seen on inquiry by anyone interested. Let vol. 4 (log primes) be photographed, and a small number of prints distributed among several institutions. Let vols. 7–11, 17–21, 39, 41, 42, 43 be printed and offered for sale; their significance is immense, and will increase year by year. For the work which I am at present planning it would be of the greatest service to me if vols. 7–11 and 17–21 were already accessible. I regard the printing of these volumes, and also of Nos. 41 and 42, to be the most pressing."

Extracts from Letter (27*th May* 1907) *received from Dr Ristenpart, Director of the Astronomical Bureau of the Prussian Academy.*

"There is indeed no question that the work under consideration (its accuracy being assumed) possesses a high scientific value. So little has been done to calculate with logarithmic and trigonometrical functions to more than 7, 8, or 10 places, and the works with 10 figures are in the last places so frequently unreliable (as, for example, in the celebrated *Opus Palatinum*), that the possibility of a more severe control of existing works and an extension of their tables to still more significant figures would be of exceptional value.

"The preservation of the Tables by a public institution appears to me to be absolutely required, but it is not sufficient. An Index of the Tables should be published in the Transactions of this institution, and a revised copy of those Tables which are not to be printed should be preserved in a second institution to guard against the possible loss of the one copy through fire or other catastrophe.

"Certain parts should undoubtedly be printed, namely:—the logarithms of the primes to 28 figures, in a much-read, easily accessible Journal, but best in several, such as the *Astronomische Nachrichten, Crelle's Journal für die reine und angewandte Mathematik, Philosophical Transactions,* etc.; the contents of Tables 7–38 should be published as an independent work of logarithmic tables to 15 figures. It is specially to be desired at the same time that the tabulation, which at present reaches to 370,000, should be continued by analogous operations to 1,000,000. Table 39 would form an appendix to this publication."

(*Issued separately February* 1, 1908.)

X.—The Theory of Compound Determinants in the Historical Order of its Development up to 1860. By Thomas Muir, LL.D.

(MS. received August 20, 1907. Read November 4, 1907.)

Determinants whose elements are themselves determinants made their appearance at a very early stage in the history of the subject, the first foreshadowing of them being contained in Lagrange's "équation identique et très remarquable" of 1773, namely,

$$\xi\eta'\zeta'' + \eta\zeta'\xi'' + \zeta\xi'\eta'' - \xi\zeta'\eta'' - \eta\xi'\zeta'' - \zeta\eta'\xi'' = (xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x'')^2,$$

where

$$\xi, \quad \eta, \quad \zeta, \ldots = y'z'' - y''z', \quad z'x'' - z''x', \quad x'y'' - x''y', \ldots$$

This, viewed as a result in determinants, is a case of Cauchy's theorem of 1812 regarding the adjugate, and the adjugate of course is an instance of the special form to which we have now come. Jacobi's theorem regarding any minor of the adjugate has a like history and may be similarly classified. Passing from the case of the adjugate, where each element is a primary minor of the original determinant, Cauchy also considered the determinants of other "systèmes dérivés," that is to say, the determinants whose elements are the secondary, ternary, . . . minors of the original, and gave the theorem that the product of the determinants of two "complementary derived systems" is a power of the original determinant, the index of the power being

$$n(n-1)(n-2)\ldots(n-p+1)/1.2.3\ldots p,$$

where n is the order of the original determinant and p the order of each element of one of the "derived systems." He also in the same memoir established the theorem that *the determinant of any "derived system" of a product-determinant is equal to the product of the determinants* of the *corresponding "derived systems" of the two factors.*

Those are all the general results that fall to be noted prior to the middle of the nineteenth century; and, as is readily seen, they all concern what at a later date came to be called the "compounds" of $|a_{1n}|$. With one exception they are due to Cauchy.*

The fact has also to be recalled, however, that compound determinants

* They are numbered xx., xxi., xli., xlii. in my *History*.

of a *special* type were considered by Jacobi in 1841, namely, those whose elements are *functional* determinants, his main theorem being

$$\sum \pm J_1^{(1)} J_2^{(2)} \cdots J_m^{(m)} = \left\{ \sum \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} \right\}^{m-1} \cdot \sum \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_{n+m}}{\partial x_{n+m}},$$

where $f_1, f_2, \ldots, f_{n+m}$ are functions of $x_1, x_2, \ldots, x_{n+m}$, and

$$J_r^{(s)} = \sum \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} \cdot \frac{\partial f_{n+s}}{\partial x_{n+r}}.$$

This theorem and certain deductions therefrom have been already dealt with in another connection (see pp. 381–385 of *History*).

SYLVESTER (1850).

[On the intersections, contacts, and other correlations of two conics expressed by indeterminate co-ordinates. *Cambridge and Dub. Math. Journ.*, v. pp. 262–282: or *Collected Math. Papers*, i. pp. 119–137.]

In a footnote to this paper the name "*Compound* Determinants" first appears. The passage is (p. 270): ". . . a theorem given by M. Cauchy, and which is included as a particular case in a theorem of my own relating to Compound Determinants, *i.e.* Determinants of Determinants, which will take its place as an immediate consequence of my fundamental theorem given in a memoir about to appear. The well-known rule for the Multiplication of Determinants is also a direct and simple consequence from my theorem on Compound Determinants, which indeed comprises, I believe, in one glance all the heretofore existing doctrine of determinants."

It will be of interest as we advance to try to identify the theorems of this perfervid statement, namely, (*a*) Sylvester's "fundamental theorem"; (*b*) his widely general "theorem on compound determinants" deduced therefrom, and including as a particular case a theorem of Cauchy's, and giving rise to the multiplication-theorem and many others as corollaries.

SYLVESTER (1851, March).

[On the relation between the minor determinants of linearly equivalent quadratic functions. *Philos. Magazine* (4), i. pp. 295–305, 415: or *Collected Math. Papers*, i. pp. 241–250, 251.]

As we have already had occasion to note,* there is here given, by way of illustrating the power of the umbral notation, a theorem regarding a

* *Proc. Roy. Soc. Edinburgh*, xxv., p. 929.

compound determinant, namely, the theorem which Sylvester writes in the form

$$\left\{\begin{matrix} \overline{a_1\, a_2 \ldots a_r\, a_{r+1}} & \overline{a_1\, a_2 \ldots a_r\, a_{r+2}} & \ldots & \overline{a_1\, a_2 \ldots a_r\, a_{r+s}} \\ a_1\, a_2 \ldots a_r\, a_{r+1} & a_1\, a_2 \ldots a_r\, a_{r+2} & \ldots & a_1\, a_2 \ldots a_r\, a_{r+s} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} a_1\, a_2 \ldots a_r \\ a_1\, a_2 \ldots a_r \end{matrix}\right\}^{s-1} \times \left\{\begin{matrix} a_1\, a_2 \ldots a_r\, a_{r+1}\, a_{r+2} \ldots a_{r+s} \\ a_1\, a_2 \ldots a_r\, a_{r+1}\, a_{r+2} \ldots a_{r+s} \end{matrix}\right\},$$

but which would now be better understood in the slightly modified form

$$\left\| \begin{vmatrix} a_1\, a_2 \ldots a_r\, a_{r+1} \\ b_1\, b_2 \ldots b_r\, b_{r+1} \end{vmatrix} \begin{vmatrix} a_1\, a_2 \ldots a_r\, a_{r+2} \\ b_1\, b_2 \ldots b_r\, b_{r+2} \end{vmatrix} \ldots \begin{vmatrix} a_1\, a_2 \ldots a_r\, a_{r+s} \\ b_1\, b_2 \ldots b_r\, b_{r+s} \end{vmatrix} \right\|$$

$$= \begin{vmatrix} a_1\, a_2 \ldots a_r \\ b_1\, b_2 \ldots b_r \end{vmatrix}^{s-1} \cdot \begin{vmatrix} a_1\, a_2 \ldots a_{r+s} \\ b_1\, b_2 \ldots b_{r+s} \end{vmatrix}.$$

No proof of it is given. At a later date it would have been viewed as the "extensional" of the manifest identity

$$\begin{vmatrix} \begin{matrix} a_{r+1} & a_{r+1} & \ldots & a_{r+1} \\ b_{r+1} & b_{r+2} & & b_{r+s} \end{matrix} \\ \\ \begin{matrix} a_{r+2} & a_{r+2} & \ldots & a_{r+2} \\ b_{r+1} & b_{r+2} & & b_{r+s} \end{matrix} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \begin{matrix} a_{r+s} & a_{r+s} & \ldots & a_{r+s} \\ b_{r+1} & b_{r+2} & & b_{r+s} \end{matrix} \end{vmatrix} = \begin{vmatrix} a_{r+1} & a_{r+2} & \ldots & a_{r+s} \\ b_{r+1} & b_{r+2} & & b_{r+s} \end{vmatrix}.$$

Later on in the same paper Sylvester gives for a particular purpose what he calls an "important generalisation." His words are (p. 304): "Suppose two sets of umbræ

$$\begin{matrix} a_1 & a_2 & \ldots & a_{m+n} \\ b_1 & b_2 & \ldots & b_{m+n}, \end{matrix}$$

and let r be any number less than n, and let any r-ary combination of the m numbers $1, 2, 3, \ldots, m$ be expressed by ${}^q\theta_1,\ \theta_2, \ldots, \theta_m$, where q goes through all the values intermediate between 1 and μ, μ being

$$\frac{m(m-1)\ \ldots\ (m-r+1)}{1\cdot 2\ \ \ldots\ \ r};$$

then I say that the compound determinant

$$\begin{matrix} \overline{a_{1_{\theta_1}}\, a_{1_{\theta_2}} \ldots a_{1_{\theta_m}}\, a_{m+1}\, a_{m+2} \ldots a_{m+n}} & \overline{a_{2_{\theta_1}}\, a_{2_{\theta_2}} \ldots a_{2_{\theta_m}}\, a_{m+1}\, a_{m+2} \ldots a_{m+n}} \\ b_{1_{\theta_1}}\, b_{1_{\theta_2}} \ldots b_{1_{\theta_m}}\, b_{m+1}\, b_{m+2} \ldots b_{m+n} & b_{2_{\theta_1}}\, b_{2_{\theta_2}} \ldots b_{2_{\theta_m}}\, b_{m+1}\, b_{m+2} \ldots b_{m+n} \\ \ldots\ldots\ldots\ldots & \overline{a_{\mu_{\theta_1}}\, a_{\mu_{\theta_2}} \ldots a_{\mu_{\theta_m}}\, a_{m+1}\, a_{m+2} \ldots a_{m+n}} \\ & b_{\mu_{\theta_1}}\, b_{\mu_{\theta_2}} \ldots b_{\mu_{\theta_m}}\, b_{m+1}\, b_{m+2} \ldots b_{m+n} \end{matrix}$$

is equal to the following product:

$$\begin{vmatrix} a_{m+1} & a_{m+2} & \dots & a_{m+n} \\ b_{m+1} & b_{m+2} & \dots & b_{m+n} \end{vmatrix}^{\mu'} \cdot \begin{vmatrix} a_1 & a_2 & \dots & a_{m+n} \\ b_1 & b_2 & \dots & b_{m+n} \end{vmatrix}^{\mu''},$$

where

$$\mu'' = \frac{(m-1)(m-2)\dots(m-r+1)}{1\cdot 2 \quad \dots \quad (r-1)},$$

and

$$\mu' = \frac{(m-1)(m-2)\dots(m-r)}{1\cdot 2 \quad \dots \quad r}."$$

In reference to this one must remark at the outset on the inappropriateness of the notation

$${}^q\theta_1, {}^q\theta_2, \dots, {}^q\theta_m$$

for the q^{th} combination of r integers taken from $1, 2, \dots, m$. Manifestly

$${}^q\theta_1, {}^q\theta_2, \dots, {}^q\theta_r,$$

though equally awkward, would have been less misleading. Indeed, as there is one clear misprint in the enunciation, namely, "less than n" for "less than m"; and as Sylvester is known to have been inaccurate in the correction of proofs, we might suspect θ_m to be a misprint for θ_r, were it not that θ_m occurs four times in the short passage and θ_r not once.* For ${}^q\theta_p$ it would have been much more convenient to write pq, which would thus have stood for "the p^{th} integer of the q^{th} combination"; and Sylvester's theorem might then have been written

$$\left\| \begin{vmatrix} a_{11} & a_{21} & \dots & a_{r1} & a_{m+1} & a_{m+2} & \dots & a_{m+n} \\ b_{11} & b_{21} & \dots & b_{r1} & b_{m+1} & b_{m+2} & \dots & b_{m+n} \end{vmatrix} \begin{vmatrix} a_{12} & a_{22} & \dots & a_{r2} & a_{m+1} & a_{m+2} & \dots & a_{m+n} \\ b_{12} & b_{22} & \dots & b_{r2} & b_{m+1} & b_{m+2} & \dots & b_{m+n} \end{vmatrix} \dots \begin{vmatrix} a_{1\mu} & a_{2\mu} & \dots & a_{r\mu} & a_{m+1} & a_{m+2} & \dots & a_{m+n} \\ b_{1\mu} & b_{2\mu} & \dots & b_{r\mu} & b_{m+1} & b_{m+2} & \dots & b_{m+n} \end{vmatrix} \right\|$$

$$= \begin{vmatrix} a_{m+1} & a_{m+2} & \dots & a_{m+n} \\ b_{m+1} & b_{m+2} & \dots & b_{m+n} \end{vmatrix}^{C_{m-1,r}} \cdot \begin{vmatrix} a_1 & a_2 & \dots & a_{m+n} \\ b_1 & b_2 & \dots & b_{m+n} \end{vmatrix}^{C_{m-1,r-1}},$$

μ being used as before for $C_{m,r}$. To help towards clearness let us illustrate by means of the case where $m=4$, $n=3$, $r=2$. We then have $\mu=6$; each set of θ's equal to a binary combination of the first four integers, that is to say,

$${}^1\theta_1{}^1\theta_2, \quad {}^2\theta_1{}^2\theta_2, \quad {}^3\theta_1{}^3\theta_2, \quad \dots, \quad {}^6\theta_1{}^6\theta_2$$

equal to

12, 13, 14, 23, 24, 34:

* θ_m was actually a misprint. Sylvester himself had to draw attention to it a year later in the *Cambridge and Dub. Math. Journ.*, viii. p. 61.

and the theorem stands

$$\left\| \begin{matrix} \begin{vmatrix} a_1 & a_2 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_5 & b_6 & b_7 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_5 & a_6 & a_7 \\ b_1 & b_3 & b_5 & b_6 & b_7 \end{vmatrix} & \cdots\cdots & \begin{vmatrix} a_1 & a_2 & a_5 & a_6 & a_7 \\ b_3 & b_4 & b_5 & b_6 & b_7 \end{vmatrix} \\ \begin{vmatrix} a_1 & a_3 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_5 & b_6 & b_7 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 & a_5 & a_6 & a_7 \\ b_1 & b_3 & b_5 & b_6 & b_7 \end{vmatrix} & \cdots\cdots & \begin{vmatrix} a_1 & a_3 & a_5 & a_6 & a_7 \\ b_3 & b_4 & b_5 & b_6 & b_7 \end{vmatrix} \\ \cdots & \cdots & \cdots & \cdots \\ \begin{vmatrix} a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_5 & b_6 & b_7 \end{vmatrix} & \begin{vmatrix} a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_3 & b_5 & b_6 & b_7 \end{vmatrix} & \cdots\cdots & \begin{vmatrix} a_3 & a_4 & a_5 & a_6 & a_7 \\ b_3 & b_4 & b_5 & b_6 & b_7 \end{vmatrix} \end{matrix} \right\|$$

$$= \begin{vmatrix} a_5 & a_6 & a_7 \\ b_5 & b_6 & b_7 \end{vmatrix}^3 \cdot \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \end{vmatrix}^3 .$$

No proof is given by Sylvester: attention is merely drawn by him to the fact that when r is put equal to 1 we obtain the theorem with which his paper commences. It is rather remarkable that he should not have singled out the case where $n = 0$. For then the theorem becomes

$$\left\| \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{r1} \\ b_{11} & b_{21} & \cdots & b_{r1} \end{vmatrix} \begin{vmatrix} a_{12} & a_{22} & \cdots & a_{r2} \\ b_{12} & b_{22} & \cdots & b_{r2} \end{vmatrix} \cdots\cdots \begin{vmatrix} a_{1\mu} & a_{2\mu} & \cdots & a_{r\mu} \\ b_{1\mu} & b_{2\mu} & \cdots & b_{r\mu} \end{vmatrix} \right\|$$

$$= \begin{vmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{vmatrix}^{C_{m-1,r-1}},$$

where, as before, the subscript pq denotes the p^{th} integer of the q^{th} set of r integers taken from 1, 2, . . . , m, and μ stands for $C_{m,n}$; and this is the theorem well known at a later date in the form: *The* rth *compound of a determinant of the* mth *order is a power of the said determinant, the index of the power being* $C_{m-1,r-1}$. Cauchy, it will be remembered, only got the length of a similar theorem in reference to the *product* of two complementary compounds. Now, since the complementary of the r^{th} compound is the $(m-r)^{th}$ compound, the product of the two must be that power of the original determinant whose index is

$$C_{m-1,r-1} + C_{m-1,m-r-1},$$

$$i.e. \quad C_{m-1,r-1} + C_{m-1,r},$$

$$i.e. \quad C_{m,r},$$

which agrees of course with Cauchy's result.

We thus learn that Sylvester's general result may be accurately described in later phraseology as *the "extensional" of the theorem regarding the* rth *compound of a determinant*, and that the discovery of both the said theorem and of its "extensional" is almost certainly due to him. At the same time it is hard to believe that this "extensional" is the all-embracing theorem referred to by him in a previous paper: for by no stretch of imagination could we see comprised in it "all the heretofore existing doctrine of determinants." His last words thereanent are: "This very general theorem is itself several degrees removed from my still unpublished Fundamental Theorem, which is a theorem for the expansion of products of determinants."

SYLVESTER (1852 Dec.).

[On a theorem concerning the combination of determinants. *Cambridge and Dub. Math. Journ.*, viii. pp. 60–62: or *Collected Math. Papers*, i. pp. 399–401.]

The statement of the theorem referred to in the title unfortunately shows want of proper care,* with the result that it is unnecessarily lengthy. It may be recast as follows:—

If from the array

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn}, \end{matrix} \qquad \textit{or A, say,}$$

we form every possible array of r *rows* ($r \ngtr m < n$), *calling the said arrays* $A_1, A_2, \ldots, A$, *where of course* $\mu = C_{m,r}$; *and if the corresponding arrays formed from*

$$\begin{matrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{m1} & b_{m2} & \dots & b_{mn}, \end{matrix} \qquad \textit{or B, say,}$$

be denoted by $B_1, B_2, \ldots, B_\mu$; *then*

$$\begin{vmatrix} A_1 \cdot B_1 & A_1 \cdot B_2 & \dots & A_1 \cdot B_\mu \\ A_2 \cdot B_1 & A_2 \cdot B_2 & \dots & A_2 \cdot B_\mu \\ \cdot & \cdot & \cdot & \cdot \\ A_\mu \cdot A_1 & A_\mu \cdot B_2 & \dots & A_\mu \cdot B_\mu \end{vmatrix} = (A \cdot B)^{C_{m-1,r-1}}.$$

By way of proof, Sylvester merely states that it is obtainable from his general theorem of March 1851, "by making

$$\left.\begin{matrix} a_{m+1} & a_{m+2} & \dots & a_{m+n} \\ b_{m+1} & b_{m+2} & \dots & b_{m+n} \end{matrix}\right\}$$

represent a determinant all whose terms (*i.e.* elements) are zeros except those which lie in one of the diagonals, these latter being all units."

His only other remark is that when $r=1$ and when $r=m$ the right-

* See especially line 8 from bottom of p. 61, where in every case m should be $m-1$.

hand members are identical, and that the equating of the two left-hand members which is thus legitimised gives Cauchy's extended multiplication-theorem.

The former remark, one regrets to note, is another instance of inaccuracy. The specialisation given therein is only one of two which are needed, the other being that every element of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ b_1 & b_2 & b_3 & \dots & b_m \end{vmatrix}$$

be made 0. To make the matter clear, let us take the case of the general theorem where $m=4$, $n=5$, $r=2$, and perform the requisite specialisations. The general theorem then is

$$\left\| \begin{matrix} \begin{vmatrix} a_1 a_2 a_5 a_6 \dots a_9 \\ b_1 b_2 b_5 b_6 \dots b_9 \end{vmatrix} & \begin{vmatrix} a_1 a_2 a_5 a_6 \dots a_9 \\ b_1 b_3 b_5 b_6 \dots b_9 \end{vmatrix} & \dots & \begin{vmatrix} a_1 a_2 a_5 a_6 \dots a_9 \\ b_3 b_4 b_5 b_6 \dots b_9 \end{vmatrix} \\ \begin{vmatrix} a_1 a_3 a_5 a_6 \dots a_9 \\ b_1 b_2 b_5 b_6 \dots b_9 \end{vmatrix} & \begin{vmatrix} a_1 a_3 a_5 a_6 \dots a_9 \\ b_1 b_3 b_5 b_6 \dots b_9 \end{vmatrix} & \dots & \begin{vmatrix} a_1 a_3 a_5 a_6 \dots a_9 \\ b_3 b_4 b_5 b_6 \dots b_9 \end{vmatrix} \\ \cdots & \cdots & \cdots & \cdots \\ \begin{vmatrix} a_3 a_4 a_5 a_6 \dots a_9 \\ b_1 b_2 b_5 b_6 \dots b_9 \end{vmatrix} & \begin{vmatrix} a_3 a_4 a_5 a_6 \dots a_9 \\ b_1 b_3 b_5 b_6 \dots b_9 \end{vmatrix} & \dots & \begin{vmatrix} a_3 a_4 a_5 a_6 \dots a_9 \\ b_3 b_4 b_5 b_6 \dots b_9 \end{vmatrix} \end{matrix} \right\|$$

$$= \begin{vmatrix} a_5 & a_6 & \dots & a_9 \\ b_5 & b_6 & \dots & b_9 \end{vmatrix}^3 \cdot \begin{vmatrix} a_1 & a_2 & \dots & a_9 \\ b_1 & b_2 & \dots & b_9 \end{vmatrix}^3 .$$

Changing now the matrix of the determinant

$$\begin{vmatrix} a_5 & a_6 & \dots & a_9 \\ b_5 & b_6 & \dots & b_9 \end{vmatrix}$$

into matrix unity, and the matrix of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix}$$

into matrix zero, the first element of the compound determinant becomes, if we write $a_r b_s$ in place of $\frac{a_r}{b_s}$,

$$\begin{vmatrix} \cdot & \cdot & a_1b_5 & a_1b_6 & \dots & a_1b_9 \\ \cdot & \cdot & a_2b_5 & a_2b_6 & \dots & a_2b_9 \\ a_5b_1 & a_5b_2 & 1 & \cdot & \dots & \cdot \\ a_6b_1 & a_6b_2 & \cdot & 1 & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_9b_1 & a_9b_2 & \cdot & \cdot & \dots & 1 \end{vmatrix}$$

which from Laplace's expansion-theorem we know to be equal to

$$\sum \begin{vmatrix} a_1b_5 & a_1b_6 \\ a_2b_5 & a_2b_6 \end{vmatrix} \cdot \begin{vmatrix} a_5b_1 & a_6b_1 \\ a_5b_2 & a_6b_2 \end{vmatrix}$$

or

$$\begin{vmatrix} a_1b_5 & a_1b_6 & \dots & a_1b_9 \\ a_2b_5 & a_2b_6 & \dots & a_2b_9 \end{vmatrix} \cdot \begin{vmatrix} a_5b_1 & a_6b_1 & \dots & a_9b_1 \\ a_5b_2 & a_6b_2 & \dots & a_9b_2 \end{vmatrix}.$$

Further, it is seen that the other elements of the compound determinant take like forms, and that in fact the said determinant is

$$\begin{vmatrix} A_1\cdot B_1 & A_1\cdot B_2 & \dots & A_1\cdot B_6 \\ A_2\cdot B_1 & A_2\cdot B_2 & \dots & A_2\cdot B_6 \\ \cdot & \cdot & \cdot & \cdot \\ A_6\cdot B_1 & A_6\cdot B_2 & \dots & A_6\cdot B_6 \end{vmatrix}$$

if A and B be taken to denote the arrays

$$\begin{matrix} a_1b_5 & a_1b_6 & \dots & a_1b_9 \\ a_2b_5 & a_2b_6 & \dots & a_2b_9 \\ a_3b_5 & a_3b_6 & \dots & a_3b_9 \\ a_4b_5 & a_4b_6 & \dots & a_4b_9 , \end{matrix} \qquad \begin{matrix} a_5b_1 & a_6b_1 & \dots & a_9b_1 \\ a_5b_2 & a_6b_2 & \dots & a_9b_2 \\ a_5b_3 & a_6b_3 & \dots & a_9b_3 \\ a_5b_4 & a_6b_4 & \dots & a_9b_4 . \end{matrix}$$

As for the right-hand member of the general identity, the first determinant in it becomes 1, and the second becomes

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & a_1b_5 & a_1b_6 & \dots & a_1b_9 \\ \cdot & \cdot & \cdot & \cdot & a_2b_5 & a_2b_6 & \dots & a_2b_9 \\ \cdot & \cdot & \cdot & \cdot & a_3b_5 & a_3b_6 & \dots & a_3b_9 \\ \cdot & \cdot & \cdot & \cdot & a_4b_5 & a_4b_6 & \dots & a_4b_9 \\ a_5b_1 & a_5b_2 & a_5b_3 & a_5b_4 & 1 & \cdot & \dots & \cdot \\ a_6b_1 & a_6b_2 & a_6b_3 & a_6b_4 & \cdot & 1 & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_9b_1 & a_9b_2 & a_9b_3 & a_9b_4 & \cdot & \cdot & \dots & 1 \end{vmatrix},$$

which equals $A \cdot B$; so that the member in question becomes

$$(A \cdot B)^2$$

as it ought.

The important thing to note in connection with the deduction here made is the fact that Sylvester must at this date have known how to express the product of two m-by-n arrays as a determinant of the $(m+n)^{\text{th}}$ order.*

SPOTTISWOODE (1853).

[Elementary theorems relating to determinants. Second edition, rewritten and much enlarged by the author. *Crelle's Journal*, li. pp. 209–271, 328–381.]

In his first edition Spottiswoode had a section (§ vi.) headed *On Inverse Systems and Determinants of Determinants*; but in it, as we have seen, he dealt merely with the adjugate determinant and its minors. Now, this is supplanted by a section (§ x.) of considerably greater extent (pp. 350–372) with the short Sylvestrian title *On Compound Determinants.*

Although it is the original definition of a compound determinant which is given on starting, the name afterwards seems to be unconsciously limited to compound determinants whose elements are minors of a given determinant of the n^{th} order. This limitation leads to the introduction of the word *class* in connection with compound determinants to indicate "the degree of minority of the constituents" (*i.e.* elements), a compound determinant of the i^{th} class being one whose elements are minors of the $(n-i)^{\text{th}}$ order; thus, the adjugate determinant is a compound determinant of the n^{th} order and 1^{st} class. If a compound determinant of the i^{th} class be of the highest possible order, namely, the

$$\frac{n(n-1)\ldots(n-i+1)}{1\cdot 2\quad\ldots\quad i}\text{th},$$

—that is to say, contains *all* the minors of the $(n-i)^{\text{th}}$ order—Spottiswoode

* And knowing this he *might* have indicated another mode of proving Cauchy's extended multiplication-theorem. For example:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\cdot\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\cdot\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\cdot\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}$$

$$= \begin{vmatrix} \cdot & \cdot & a_1 & a_2 & a_3 \\ \cdot & \cdot & b_1 & b_2 & b_3 \\ x_1 & y_1 & 1 & \cdot & \cdot \\ x_2 & y_2 & \cdot & 1 & \cdot \\ x_3 & y_3 & \cdot & \cdot & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -a_1x_1-a_2x_2-a_3x_3 & -a_1y_1-a_2y_2-a_3y_3 & \cdot & \cdot & \cdot \\ -b_1x_1-b_2x_2-b_3x_3 & -b_1y_1-b_2y_2-b_3y_3 & \cdot & \cdot & \cdot \\ x_1 & y_1 & 1 & \cdot & \cdot \\ x_2 & y_2 & \cdot & 1 & \cdot \\ x_3 & y_3 & \cdot & \cdot & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x_1+a_2x_2+a_3x_3 & a_1y_1+a_2y_2+a_3y_3 \\ b_1x_1+b_2x_2+b_3x_3 & b_1y_1+b_2y_2+b_3y_3 \end{vmatrix}.$$

calls it the "*complete* determinant of the i^{th} class," a name equivalent therefore to the more modern "$(n-i)^{th}$ compound."

One of the notations employed is essentially the same as Sylvester's—that is to say, he uses

$$\left\{ \overline{\begin{matrix} 1_1 & 2_1 & \dots & u_1 \\ 1_1 & 2_1 & \dots & u_1 \end{matrix}} \quad \overline{\begin{matrix} 1_2 & 2_2 & \dots & u_2 \\ 1_2 & 2_2 & \dots & u_2 \end{matrix}} \quad \dots\dots \quad \overline{\begin{matrix} 1_n & 2_n & \dots & u_n \\ 1_n & 2_n & \dots & u_n \end{matrix}} \right\}$$

for what at a later date would have been written

$$\left\| \begin{vmatrix} 1_1 & 2_1 & \dots & u_1 \\ 1_1 & 2_1 & \dots & u_1 \end{vmatrix} \quad \begin{vmatrix} 1_2 & 2_2 & \dots & u_2 \\ 1_2 & 2_2 & \dots & u_2 \end{vmatrix} \quad \dots\dots \quad \begin{vmatrix} 1_n & 2_n & \dots & u_n \\ 1_n & 2_n & \dots & u_n \end{vmatrix} \right\|.$$

The other notation is his own, and is worthy of careful note. It differs from Sylvester's in making use of the row-numbers and column-numbers not of the retained elements but of the elements omitted, the said numbers being enclosed in brackets for the purpose of recalling this difference. "Thus," he says (p. 352), "the complete compound determinant of the first class may be written

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdots\cdots \begin{pmatrix} n \\ n \end{pmatrix} \right\};$$

that of the second class

$$\left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \cdots\cdots \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \cdots\cdots \right\};$$

and generally that of the i^{th} class

$$\left\{ \begin{pmatrix} 1_1 & 1_2 & \dots & 1_i \\ 1_1 & 1_2 & \dots & 1_i \end{pmatrix} \begin{pmatrix} 2_1 & 2_2 & \dots & 2_i \\ 2_1 & 2_2 & \dots & 2_i \end{pmatrix} \dots\dots \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_i \\ \mu_1 & \mu_2 & \dots & \mu_i \end{pmatrix} \right\},$$

where

$$\mu = \frac{n(n-1)\dots(n-i+1)}{1 \cdot 2 \quad \dots \quad i};"$$

and where, it should have been added, r_s stands for the s^{th} integer in the r^{th} combination of i integers taken from $1, 2, \dots, n$.

These preliminaries having been attended to, a discussion of the properties follows. The first five pages (pp. 353–358) and two later pages (pp. 366–368) are mainly concerned with compound determinants of the first class (that is to say, with the adjugate determinant), and they do not break fresh ground. The same, however, cannot be said with reference to the next two pages (pp. 358–360), which concern those of the second class. The result first reached is that the complete determinant of this class, namely,

$$\left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \cdots\cdots \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \cdots\cdots \right\}, \text{ or } \Delta_2, \text{ say,}$$

$$=\begin{Bmatrix}1 & 2 & \ldots & n\\1 & 2 & \ldots & n\end{Bmatrix}^{\nu},$$

where $\nu=\frac{1}{2}(n-1)(n-2)$. Although there is a semblance of reasoning, no real proof is given. Passing then to any first minor of Δ_2, say the first minor got by leaving out $\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$ from the detailed symbol for Δ_2, he finds that

$$\left\{\begin{pmatrix}1 & 3\\1 & 3\end{pmatrix}\ \begin{pmatrix}1 & 4\\1 & 4\end{pmatrix}\cdots\cdots\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix}\cdots\cdots\right\} = \begin{Bmatrix}1 & 2\\1 & 2\end{Bmatrix}\begin{Bmatrix}1 & 2 & \ldots\ldots & n\\1 & 2 & \ldots\ldots & n\end{Bmatrix}^{\nu-1}.$$

It is next pointed out that if we proceed to the second minors of Δ_2 it becomes necessary to distinguish two cases,—to distinguish, for example, the case where we leave out $\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 3\\1 & 3\end{pmatrix}$ from the case where we leave out $\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$ $\begin{pmatrix}3 & 4\\3 & 4\end{pmatrix}$. In the latter case the row-numbers 1, 2, 3, 4 are all different, and the result is

$$\left\{\begin{pmatrix}5 & 6\\5 & 6\end{pmatrix}\ \begin{pmatrix}7 & 8\\7 & 8\end{pmatrix}\cdots\cdots\begin{pmatrix}1 & 3\\1 & 3\end{pmatrix}\ \begin{pmatrix}2 & 4\\2 & 4\end{pmatrix}\cdots\cdots\right\} = \begin{Bmatrix}1 & 2 & 3 & 4\\1 & 2 & 3 & 4\end{Bmatrix}\begin{Bmatrix}1 & 2 & \ldots & n\\1 & 2 & \ldots & n\end{Bmatrix}^{\nu-2};$$

in the former case the row-number 1 occurs twice, and the result is

$$\left\{\begin{pmatrix}1 & 4\\1 & 4\end{pmatrix}\ \begin{pmatrix}1 & 5\\1 & 5\end{pmatrix}\cdots\cdots\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix}\cdots\cdots\right\} = \begin{Bmatrix}1\\1\end{Bmatrix}\begin{Bmatrix}1 & 2 & 3\\1 & 2 & 3\end{Bmatrix}\begin{Bmatrix}1 & 2 & \ldots & n\\1 & 2 & \ldots & n\end{Bmatrix}^{\nu-2}.$$

Generally, if $\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$, $\begin{pmatrix}3 & 4\\3 & 4\end{pmatrix}$, $\begin{pmatrix}5 & 6\\5 & 6\end{pmatrix}$, ..., $\begin{pmatrix}2i-1 & 2i\\2i-1 & 2i\end{pmatrix}$ be left out, we have

$$\left\{\begin{pmatrix}2i+1 & 2i+2\\2i+1 & 2i+2\end{pmatrix}\begin{pmatrix}2i+3 & 2i+4\\2i+3 & 2i+4\end{pmatrix}\cdots\cdots\begin{pmatrix}1 & 3\\1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\\2 & 4\end{pmatrix}\cdots\cdots\right\} = \begin{Bmatrix}1 & 2 & \ldots & 2i\\1 & 2 & \ldots & 2i\end{Bmatrix}\begin{Bmatrix}1 & 2 & \ldots & n\\1 & 2 & \ldots & n\end{Bmatrix}^{\nu-i};$$

and if $\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 3\\1 & 3\end{pmatrix}$, ..., $\begin{pmatrix}1 & i\\1 & i\end{pmatrix}$ be left out, we have

$$\left\{\begin{pmatrix}1 & i+1\\1 & i+1\end{pmatrix}\begin{pmatrix}1 & i+2\\1 & i+2\end{pmatrix}\cdots\cdots\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix}\cdots\cdots\right\} = \begin{Bmatrix}1\\1\end{Bmatrix}^{i-2}\begin{Bmatrix}1 & 2 & \ldots & i\\1 & 2 & \ldots & i\end{Bmatrix}\begin{Bmatrix}1 & 2 & \ldots & n\\1 & 2 & \ldots & n\end{Bmatrix}^{\nu-i+1}.$$

Again, a fresh variety of minor may be got by leaving out $\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 3\\1 & 3\end{pmatrix}$, $\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix}$ the row-numbers being then two 1's, two 2's, and two 3's, and the result is

$$\left\{\begin{pmatrix}1 & 4\\1 & 4\end{pmatrix}\ \begin{pmatrix}1 & 5\\1 & 5\end{pmatrix}\cdots\cdots\begin{pmatrix}2 & 4\\2 & 4\end{pmatrix}\cdots\cdots\right\} = \begin{Bmatrix}1 & 2 & 3\\1 & 2 & 3\end{Bmatrix}\begin{Bmatrix}1 & 2 & \ldots & n\\1 & 2 & \ldots & n\end{Bmatrix}^{\nu-3},$$

"and so on."

From these cases of compound determinants of the second class, the author passes to those of the i^{th} class, but contents himself with stating only two results. The first is that the complete determinant of this class, denoted as above by

$$\left\{\begin{pmatrix} 1_1 & 1_2 & \dots & 1_i \\ 1_1 & 1_2 & \dots & 1_i \end{pmatrix} \begin{pmatrix} 2_1 & 2_2 & \dots & 2_i \\ 2_1 & 2_2 & \dots & 2_i \end{pmatrix} \dots\dots \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_i \\ \mu_1 & \mu_2 & \dots & \mu_i \end{pmatrix}\right\}, \text{ or } \Delta_i, \text{ say,}$$
$$= \left\{\begin{matrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{matrix}\right\}^{\nu},$$

where $\nu = \dfrac{(n-1)(n-2)\dots(n-i+1)}{1\cdot 2\dots i-1}$,—a result which agrees with that obtained on putting $n=0$ in Sylvester's general theorem of March 1851; and the other is that any first minor of Δ_i, say the minor

$$\left\{\begin{pmatrix} 2_1 & 2_2 & \dots & 2_i \\ 2_1 & 2_2 & \dots & 2_i \end{pmatrix} \begin{pmatrix} 3_1 & 3_2 & \dots & 3_i \\ 3_1 & 3_2 & \dots & 3_i \end{pmatrix} \dots\dots \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_i \\ \mu_1 & \mu_2 & \dots & \mu_i \end{pmatrix}\right\}.$$
$$= \left\{\begin{matrix} 1_1 & 1_2 & \dots & 1_i \\ 1_1 & 1_2 & \dots & 1_i \end{matrix}\right\} \left\{\begin{matrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{matrix}\right\}^{\nu-1}.$$

He appends, however, the words "and so on," and tells us that "other formulæ may be written as required."

Two theorems of Sylvester's are next given, the one being the general theorem just alluded to, and the other that contained in the paper of 16th December 1852. In the case of the former he varies the notation, and probably by reason of the above-mentioned serious misprint of an m for an r in the original he misses Sylvester's meaning, and makes an incorrect statement. In the case of the other no risk of this kind is incurred, because he takes the unusual course of reproducing Sylvester's words letter for letter to the extent of almost two pages (pp. 361–363). The original two pages, however, being, as we have seen, not without evidence of Sylvester's carelessness, this course also was unsafe.

Lastly, he shows how the multiplication-theorem may be deduced from Sylvester's first theorem of March 1851 regarding compound determinants, by taking as an example of the latter the identity

$$\left\| \begin{matrix} \begin{vmatrix} 1 & \cdot & a \\ \cdot & 1 & a' \\ \alpha & \beta & \cdot \end{vmatrix} & \begin{vmatrix} 1 & \cdot & b \\ \cdot & 1 & b' \\ \alpha & \beta & \cdot \end{vmatrix} \\ \begin{vmatrix} 1 & \cdot & a \\ \cdot & 1 & a' \\ \alpha' & \beta' & \cdot \end{vmatrix} & \begin{vmatrix} 1 & \cdot & b \\ \cdot & 1 & b' \\ \alpha' & \beta' & \cdot \end{vmatrix} \end{matrix} \right\| = \begin{vmatrix} 1 & \cdot \\ \cdot & 1 \end{vmatrix}^{2-1} \cdot \begin{vmatrix} 1 & \cdot & a & b \\ \cdot & 1 & a' & b' \\ \alpha & \beta & \cdot & \cdot \\ \alpha' & \beta' & \cdot & \cdot \end{vmatrix},$$

and pointing out that this is evidently the same as saying

$$\begin{vmatrix} a\alpha + a'\beta & b\alpha + b'\beta \\ a\alpha' + a'\beta' & b\alpha' + b'\beta' \end{vmatrix} = \begin{vmatrix} \alpha & \beta \\ \alpha' & \beta' \end{vmatrix} \cdot \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} .$$

In later phraseology, it may be said that the specialisation necessary for the purpose is

$$s = r ,$$

$$\text{matrix of} \begin{vmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{vmatrix} = 1, \quad \text{matrix of} \begin{vmatrix} a_{r+1} & a_{r+2} & \dots & a_{2r} \\ b_{r+1} & b_{r+2} & \dots & b_{2r} \end{vmatrix} = 0 .*$$

BRIOSCHI (1854, March).

[LA TEORICA DEI DETERMINANTI, E LE SUE PRINCIPALI APPLICAZIONI; del Dr Francesco Brioschi; viii+116 pp.; Pavia. Translation into French, by Combescure; ix+216 pp.; Paris, 1856. Translation into German, by Schellbach; vii+102 pp.; Berlin, 1856.]

After proving (p. 100) Jacobi's theorem, above referred to, regarding a compound determinant whose elements are functional determinants, Brioschi bids the reader make the functions $f_1, f_2, \dots$ linear, say

$$f_r = a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{r,n+m}x_{n+m} ,$$

and note that the outcome is

$$\sum \pm A^{(1)}A_2^{(2)} \dots A_m^{(m)} = \left\{ \sum \pm a_{11}a_{22} \cdots a_{nn} \right\}^{m-1} \cdot \sum \pm a_{11}a_{22} \cdots a_{n+m,n+m} ,$$

where

$$A_r^{(s)} = \sum \pm a_{11}a_{22} \cdots a_{nn}a_{n+s,n+r}$$

—that is to say, is Sylvester's first result of March 1851.

The same course is followed by Bellavitis in his *Sposizione* of 1857 (see §§ 71, 72, p. 56).

* Some of the pages of Spottiswoode dealt with in the foregoing are, by reason of misprints and other neglects, not easy reading. On p. 360 there are at least *nine* misprints.

(*Issued separately, April* 8, 1908.)

XI.—The Product of the Primary Minors of an n-by-$(n+1)$ Array. By Thomas Muir, LL.D.

(MS. received October 7, 1907. Read November 4, 1907.)

(1) In a paper by A. Scholtz entitled "Sechs Punkte eines Kegelschnittes" (*Archiv d. Math. u. Phys.*, lxii. pp. 317–324, year 1878) there appears a statement which, after correction of two misprints, runs thus:—

$$\begin{vmatrix} y_i y_n & z_i z_n & y_i z_n + y_n z_i \\ y_n y_i & z_n z_i & y_n z_i + y_i z_n \\ y_i y_l & z_i z_l & y_i z_l + y_l z_i \end{vmatrix} = -\xi_{in} \cdot \xi_{ni} \cdot \xi_{li}$$

where $\xi_{in} = y_i z_n - y_n z_i$. A year or so later there was published a paper by Hunyady with the title "Beitrag zur Theorie des Flächen zweiten Grades" (*Crelle's Journ.*, lxxxix. pp. 47–69), in which it is asserted, again without proof, that

$$\begin{vmatrix} y_1y_2 & z_1z_2 & p_1p_2 & y_1z_2+y_2z_1 & y_1p_2+y_2p_1 & z_1p_2+z_2p_1 \\ y_1y_3 & z_1z_3 & p_1p_3 & y_1z_3+y_3z_1 & y_1p_3+y_3p_1 & z_1p_3+z_3p_1 \\ y_1y_4 & z_1z_4 & p_1p_4 & y_1z_4+y_4z_1 & y_1p_4+y_4p_1 & z_1p_4+z_4p_1 \\ y_2y_3 & z_2z_3 & p_2p_3 & y_2z_3+y_3z_2 & y_2p_3+y_3p_2 & z_2p_3+z_3p_2 \\ y_2y_4 & z_2z_4 & p_2p_4 & y_2z_4+y_4z_2 & y_2p_4+y_4p_2 & z_2p_4+z_4p_2 \\ y_3y_4 & z_3z_4 & p_3p_4 & y_3z_4+y_4z_3 & y_3p_4+y_4p_3 & z_3p_4+z_4p_3 \end{vmatrix} = -\xi_{234}\xi_{341}\xi_{412}\xi_{123}.$$

The object of the present note is to formulate and prove a general theorem of which these are cases, and to draw certain deductions therefrom.

(2) The theorem somewhat imperfectly enunciated is—*The product of the* n*-line determinants formable from an array of* n *rows and* n+1 *columns is expressible as a determinant of the order* $\frac{1}{2}$n(n+1). By way of proof let us consider the case where $n=4$, and where, therefore, the array may be written

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 . \end{matrix}$$

The determinant of the 10th order which has to be proved equal to the product

$$|a_1b_2c_3d_4| \cdot |a_1b_2c_3d_5| \cdot |a_1b_2c_4d_5| \cdot |a_1b_3c_4d_5| \cdot |a_2b_3c_4d_5$$

is then

$$\begin{vmatrix} a_1a_2 & \dots & d_1d_2 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 & \dots & c_1d_2+c_2d_1 \\ a_1a_3 & \dots & d_1d_3 & a_1b_3+a_3b_1 & a_1c_3+a_3c_1 & \dots & c_1d_3+c_3d_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_4a_5 & \dots & d_4d_5 & a_4b_5+a_5b_4 & a_4c_5+a_5c_4 & \dots & c_4d_5+c_5d_4 \end{vmatrix}.$$

By performing on this the set of operations

$$\begin{aligned} &\text{col}_5 - \frac{b_1}{a_1}\text{col}_1 - \frac{a_1}{b_1}\text{col}_2, \\ &\text{col}_6 - \frac{c_1}{a_1}\text{col}_1 - \frac{a_1}{c_1}\text{col}_3, \\ &\text{col}_7 - \frac{d_1}{a_1}\text{col}_1 - \frac{a_1}{d_1}\text{col}_4, \\ &\text{col}_8 - \frac{c_1}{b_1}\text{col}_2 - \frac{b_1}{c_1}\text{col}_3, \\ &\text{col}_9 - \frac{d_1}{b_1}\text{col}_2 - \frac{b_1}{d_1}\text{col}_4, \\ &\text{col}_{10} - \frac{d_1}{c_1}\text{col}_3 - \frac{c_1}{d_1}\text{col}_4, \end{aligned}$$

we reduce to zero all the binomial elements in which 1 appears as a suffix, and replace every other binomial element by a fraction of the product of two determinants of the 2nd order, the reason being that

$$\begin{aligned} (hk+mn) - \frac{x}{y}hm - \frac{y}{x}nk &= \frac{(hx-ny)(ky-mx)}{xy}, \\ &= 0, \quad \text{when} \quad x, y = n, h \quad \text{or} \quad k, m. \end{aligned}$$

The determinant can thus be resolved into two factors, the first of which is

$$\begin{vmatrix} a_1a_2 & b_1b_2 & c_1c_2 & d_1d_2 \\ a_1a_3 & b_1b_3 & c_1c_3 & d_1d_3 \\ a_1a_4 & b_1b_4 & c_1c_4 & d_1d_4 \\ a_1a_5 & b_1b_5 & c_1c_5 & d_1d_5 \end{vmatrix} \quad \text{or} \quad a_1b_1c_1d_1 \cdot |\, a_2b_3c_4d_5 \,|;$$

and the second

$$\frac{1}{(a_1b_1c_1d_1)^3} \cdot \begin{vmatrix} |a_2b_1|\cdot|a_3b_1| & |a_2c_1|\cdot|a_3c_1| & \cdots & |c_2d_1|\cdot|c_3d_1| \\ |a_2b_1|\cdot|a_4b_1| & |a_2c_1|\cdot|a_4c_1| & \cdots & |c_2d_1|\cdot|c_4d_1| \\ \cdot & \cdot & \cdot & \cdot \\ |a_4b_1|\cdot|a_5b_1| & |a_4c_1|\cdot|a_5c_1| & \cdots & |c_4d_1|\cdot|c_5d_1| \end{vmatrix}.$$

By establishing the fact that $|\, a_2b_3c_4d_5 \,|$ is a factor of the original determinant, the former of these is the all-important: for we see that, if in our set of operations the multipliers of the columns had all had the suffix 2, we should similarly have proved $|a_1b_3c_4d_5|$ to be a factor; that if the suffix had been 3, the factor reached would have been $|a_1b_2c_4d_5|$; and so on. Now, as the product of the five factors thus obtained is of the 20th degree in the elements, and the original determinant is also of the 20th degree, we have only to ascertain the connecting factor of degree 0. To do this, we note that in the special case where the matrix is

$$\begin{matrix} a_1 & a_2 & \cdot & \cdot & \cdot \\ b_1 & \cdot & b_3 & \cdot & \cdot \\ c_1 & \cdot & \cdot & c_4 & \cdot \\ d_1 & \cdot & \cdot & \cdot & d_5 \end{matrix}$$

the original determinant reduces to its diagonal term, but that, although each of the set of five determinants reduces to one term, this term cannot be made the diagonal term in every case without effecting certain row-transpositions which necessitate $1+2+3$ changes of sign. The connecting sign-factor is thus seen to be $(-1)^6$ when $n=4$, and generally to be

$$(-1)^{\frac{1}{2}n(n-1)} \qquad \text{(I.)}$$

(3) The fact that the determinant under consideration in § 2 was resolved into two factors, and that only one of the two was utilised for evaluation purposes, enables us to give an interesting theorem in reference to the second factor. For, the result of the first resolution being, say,

$$\Delta = a_1b_1c_1d_1 \cdot |a_2b_3c_4d_5| \cdot \frac{1}{(a_1b_1c_1d_1)^3}W,$$

and the final result being

$$\Delta = |a_1b_2c_3d_4| \cdot |a_1b_2c_3d_5| \cdot |a_1b_2c_4d_5| \cdot |a_1b_3c_4d_5| \cdot |a_2b_3c_4d_5|,$$

it follows that

$$W = (a_1b_1c_1d_1)^2 \cdot |a_1b_2c_3d_4| \cdot |a_1b_2c_3d_5| \cdot |a_1b_2c_4d_5| \cdot |a_1b_3c_4d_5| \qquad \text{(II.)}$$

The formation of the compound determinant W is peculiar. If the originating array consist of n rows and $n+1$ columns, the order of W is $\frac{1}{2}n(n-1)$. The two-line determinants which go to form it are all the two-line minors of the array which have their first column coincident with the first column of the array: they therefore themselves form an array of $\frac{1}{2}n(n-1)$ rows and n columns. By combining in pairs the n elements of each of these $\frac{1}{2}n(n-1)$ rows we obtain $\frac{1}{2}n(n-1)$ products to be the elements of the corresponding row of W. It is this want of similarity in the mode of equalising the length and breadth of the W matrix which accounts for the absence of symmetry in the result.

(4) A direct proof of theorem (II.) is not easily formulated: in fact, determinants of the kind which it concerns, namely, those whose elements are *products of determinants*, seem to have been little studied. The following case, in which the factors of the said elements are the *principal minors of a given determinant*, is, therefore, of interest.

The given determinant being $|a_1b_2c_3d_4|$, or Δ say, and its adjugate being denoted by $|A_1B_2C_3D_4|$, the determinant for investigation is

$$\begin{vmatrix} A_1A_2 & A_2A_3 & A_3A_4 & A_4A_1 \\ B_1B_2 & B_2B_3 & B_3B_4 & B_4B_1 \\ C_1C_2 & C_2C_3 & C_3C_4 & C_4C_1 \\ D_1D_2 & D_2D_3 & D_3D_4 & D_4D_1 \end{vmatrix}.$$

Expanding it in terms of minors formed from the first two columns and minors formed from the last two columns, we obtain

$$\begin{aligned} &A_2B_2\,|A_1B_3|\cdot C_4D_4\,|C_3D_1| - A_2C_2\,|A_1C_3|\cdot B_4D_4\,|B_3D_1| + A_2D_2\,|A_1D_3|\cdot B_4C_4\,|B_3C_1| \\ &+ B_2C_2\,|B_1C_3|\cdot A_4D_4\,|A_3D_1| - B_2D_2\,|B_1D_3|\cdot A_4C_4\,|A_3C_1| + C_2D_2\,|C_1D_3|\cdot A_4B_4\,|A_3B_1|. \end{aligned}$$

But as $|A_1B_3| = -|c_2d_4|\Delta$, $|C_1D_3| = -|a_2b_4|\Delta$, ... this is transformable into

$$\Delta^2\cdot\begin{bmatrix} - A_2B_2\,|c_2d_4|\cdot C_4D_4\,|a_2b_4| + A_2C_2\,|b_2d_4|\cdot B_4D_4\,|a_2c_4| \\ - B_2C_2\,|a_2d_4|\cdot A_4D_4\,|b_2c_4| + B_2D_2\,|a_2c_4|\cdot A_4C_4\,|b_2d_4| \\ \qquad - A_2D_2\,|b_2c_4|\cdot B_4C_4\,|a_2d_4| \\ \qquad - C_2D_2\,|a_2b_4|\cdot A_4B_4\,|b_2c_4| \end{bmatrix};$$

and as the expression here bracketed is the similar expansion of another four-line determinant, we have the interesting identity

$$\begin{vmatrix} A_1A_2 & A_2A_3 & A_3A_4 & A_4A_1 \\ B_1B_2 & B_2B_3 & B_3B_4 & B_4B_1 \\ C_1C_2 & C_2C_3 & C_3C_4 & C_4C_1 \\ D_1D_2 & D_2D_3 & D_3D_4 & D_4D_1 \end{vmatrix} = \Delta^2 \begin{vmatrix} a_2A_2 & a_4A_2 & a_4A_4 & a_2A_4 \\ b_2B_2 & b_4B_2 & b_4B_4 & b_2B_4 \\ c_2C_2 & c_4C_2 & c_4C_4 & c_2C_4 \\ d_2D_2 & d_4D_2 & d_4D_4 & d_2D_4 \end{vmatrix}, \qquad \text{(III.)}$$

where the determinant on the right is got from that on the left by putting a_2, b_2, c_2, d_2 for A_1, B_1, C_1, D_1, and a_4, b_4, c_4, d_4 for A_3, B_3, C_3, D_3. Taking the complementary of the identity (III.), we have

$$\begin{vmatrix} a_1a_2 & a_2a_3 & a_3a_4 & a_4a_1 \\ b_1b_2 & b_2b_3 & b_3b_4 & b_4b_1 \\ c_1c_2 & c_2c_3 & c_3c_4 & c_4c_1 \\ d_1d_2 & d_2d_3 & d_3d_4 & d_4d_1 \end{vmatrix}\cdot\Delta^2 = \begin{vmatrix} a_2A_2 & a_2A_4 & a_4A_4 & a_4A_2 \\ b_2B_2 & b_2B_4 & b_4B_4 & b_4B_2 \\ c_2C_2 & c_2C_4 & c_4C_4 & c_4C_2 \\ d_2D_2 & d_2D_4 & d_4D_4 & d_4D_2 \end{vmatrix} \qquad \text{(IV.)}$$

where the determinant on the right differs only in sign from the last determinant in (III.). By combination of (III.) and (IV.) there thus follows

$$| A_1A_2 \quad B_2B_3 \quad C_3C_4 \quad D_4D_1 | = -\Delta^2 | a_1a_2 \quad b_2b_3 \quad c_3c_4 \quad d_4d_1 | . \qquad \text{(V.)}$$

(5) If before operating on $| A_1B_2 , B_2B_3 , C_3C_4 , D_4D_1 |$ we had made its 2nd, 3rd, 4th columns the 3rd, 4th, 2nd respectively, we should have obtained on the right of (III.)

$$\Delta^2 . \begin{vmatrix} a_1A_1 & a_3A_1 & a_1A_3 & a_3A_3 \\ b_1B_1 & b_3B_1 & b_1B_3 & b_3B_3 \\ c_1C_1 & c_3C_1 & c_1C_3 & c_3C_3 \\ d_1D_1 & d_3D_1 & d_1D_3 & d_3D_3 \end{vmatrix} .$$

There thus results the curious identity,

$$\begin{vmatrix} a_2A_2 & a_2A_4 & a_4A_4 & a_4A_2 \\ b_2B_2 & b_2B_4 & b_4B_4 & b_4B_2 \\ c^2C_2 & c_2C_4 & c_4C_4 & c_4C_2 \\ d_2D_2 & d_2D_4 & d_4D_4 & d_4D_2 \end{vmatrix} = - \begin{vmatrix} a_1A_1 & a_3A_1 & a_1A_3 & a_3A_3 \\ b_1B_1 & b_3B_1 & b_1B_3 & b_3B_3 \\ c_1C_1 & c_3C_1 & c_1C_3 & c_3C_3 \\ d_1D_1 & d_3D_1 & d_1D_3 & d_3D_3 \end{vmatrix} ; \qquad \text{(VI.)}$$

or, in words, the determinant obtained from $| A_1A_2 , B_2B_3 , C_3C_4 , D_4D_1 |$ by changing $A_2 , B_2 , C_2 , D_2 , A_4 , B_4 , C_4 , D_4$ into $a_1 , b_1 , c_1 , d_1 , a_3 , b_3 , c_3 , d_3$, differs only in sign from that obtained by changing $A_1 , B_1 , C_1 , D_1 , A_3 , B_3 , C_3 , D_3$ into $a_2 , b_2 , c_2 , d_2 , a_4 , b_4 , c_4 , d_4$.

It will be found to be a direct consequence of the fact that the interchanges

$$\begin{pmatrix} a_1 , b_1 , c_1 , d_1 \\ a_2 , b_2 , c_2 , d_2 \end{pmatrix} , \quad \begin{pmatrix} a_3 , b_3 , c_3 , d_3 \\ a_4 , b_4 , c_4 , d_4 \end{pmatrix}$$

entail the interchanges

$$\begin{pmatrix} A_1 , B_1 , C_1 , D_1 \\ A_2 , B_2 , C_2 , D_2 \end{pmatrix} , \quad \begin{pmatrix} A_3 , B_3 , C_3 , D_3 \\ A_4 , B_4 , C_4 , D_4 \end{pmatrix} ,$$

and that the said interchanges made in (III.) do not alter Δ and alter $| A_1A_2 \quad B_2B_3 \quad C_3C_4 \quad D_4D_1 |$ merely in sign.

(6) Denoting the determinants of (VI.) by M and N, we have

$$N = \begin{vmatrix} . & A_1a_3 & . & A_3a_3 \\ B_1|a_3b_1| & B_1b_3 & B_3|a_3b_1| & B_3b_3 \\ C_1|a_3c_1| & C_1c_3 & C_3|a_3c_1| & C_3c_3 \\ D_1|a_3d_1| & D_1d_3 & D_3|a_3d_1| & D_3d_3 \end{vmatrix} \div a_3^2 ,$$

and, therefore, by Laplace's expansion theorem,

$$\begin{aligned} N &= \frac{1}{a_3}\Big\{ -b_3 \; |A_1B_3| \; |a_3c_1| \; |a_3d_1| \; |C_1D_3| \; + \; c_3 |A_1C_3| \; |a_3b_1| \; |a_3d_1| \; |B_1D_3| \\ &\qquad\qquad - \; d_3 |A_1D_3| \; |a_3b_1| \; |a_3c_1| \; |B_1C_3| \Big\}, \\ &= \frac{\Delta^2}{a_3}\Big\{ -b_3 |c_2d_4| \; |a_1c_3| \; |a_1d_3| \; |a_2b_4| \; + \; c_3 |b_2d_4| \; |a_1b_3| \; |a_1d_3| \; |a_2c_4| \\ &\qquad\qquad - \; d^3 |b_2c_4| \; |a_1b_3| \; |a_1c_3| \; |a_2d_4| \Big\}. \text{(VII.)} \end{aligned}$$

Similar expressions are got by the like treatment of M, or simply by interchange of suffixes in (VII.); for example, by interchanging 1 and 3.

Again, by performing on $| a_1a_2 \;\; b_2b_3 \;\; c_3c_4 \;\; d_4d_1 |$ the operations

$$a_3\,\text{col}_4 - a_1\,\text{col}_3, \quad a_2\,\text{col}_3 - a_4\,\text{col}_2, \quad a_1\,\text{col}_2 - a_3\,\text{col}_1,$$

we obtain

$$\begin{vmatrix} b_2 & |a_1b_3| & b_3 & |a_2b_4| & b_4 & |a_3b_1| \\ c_2 & |a_1c_3| & c_3 & |a_2c_4| & c_4 & |a_3c_1| \\ d_2 & |a_1d_3| & d_3 & |a_2d_4| & d_4 & |a_3d_1| \end{vmatrix} \div a_3,$$

which, by expansion in terms of the elements of the second column and their co-factors, gives us

$$\begin{aligned} | a_1a_2 \; b_2b_3 \; c_3c_4 \; d_4d_1 | = \frac{1}{a_3}\Big\{ &+ b_3|a_2b_4| \; |a_1c_3| \; |a_1d_3| \; |c_2d_4| - c_3|a_2c_4| \; |a_1b_3| \; |a_1d_3| \; |b_2d_4| \\ &+ \; d_3|a_2d_4| \; |a_1b_3| \; |a_1c_3| \; |b_2c_4| \Big\}. \qquad \text{(VIII.)} \end{aligned}$$

By the combination of those two results we are led to

$$N = -\Delta^2 \cdot | a_1a_2 \;\; b_2b_3 \;\; c_3c_4 \;\; d_4d_1 |,$$

in agreement with (IV.) and (VI.).

(7) There is another determinant which may be treated in the same fashion as $| A_1A_2 \;\; B_2B_3 \;\; C_3C_4 \;\; D_4D_1 |$ in § 4, namely,

$$\begin{vmatrix} A_2A_3A_4 & A_1A_3A_4 & A_1A_2A_4 & A_1A_2A_3 \\ B_2B_3B_4 & B_1B_3B_4 & B_1B_2B_4 & B_1B_2B_3 \\ C_2C_3C_4 & C_1C_3C_4 & C_1C_2C_4 & C_1C_2C_3 \\ D_2D_3D_4 & D_1D_3D_4 & D_1D_2D_4 & D_1D_2D_3 \end{vmatrix}.$$

The first result is

$$| A_2A_3A_4 \;\; B_3B_4B_1 \;\; C_4C_1C_2 \;\; D_1D_2D_3 | = \Delta^2 \begin{vmatrix} a_2A_3A_4 & a_1A_3A_4 & A_1A_2a_3 & A_1A_2a_3 \\ b_2B_3B_4 & b_1B_3B_4 & B_1B_2b_4 & B_1B_2b_3 \\ c_2C_3C_4 & c_1C_3C_4 & C_1C_2c_4 & C_1C_2c_3 \\ d_2D_3D_4 & d_1D_3D_4 & D_1D_2d_4 & D_1D_2D_3 \end{vmatrix}; \quad \text{(IX.)}$$

and the second, which is the complementary of the first, is

$$\Delta^2 \cdot | a_2a_3a_4 \quad b_3b_4b_1 \quad c_4c_1c_2 \quad d_1d_2d_3 | = \begin{vmatrix} A_2a_3a_4 & A_1a_3a_4 & a_1a_2A_4 & a_1a_2A_3 \\ B_2b_3b_4 & B_1b_3b_4 & b_1b_2B_4 & b_1b_2B_3 \\ C_2c_3c_4 & C_1c_3c_4 & c_1c_2C_4 & c_1c_2C_3 \\ D_2d_3d_4 & D_1d_3d_4 & d_1d_2D_4 & d_1d_2D_3 \end{vmatrix} . \qquad \text{(X.)}$$

A further result, however, is clearly necessary in order to establish an identity analogous to (V.). *

* Prof. Nanson in the *Educ. Times* for 1902, pp. 515–516, gives

$$| A_1A_2 \quad B_2B_3 \quad C_3C_1 | = -\Delta^2 . | a_1a_2 \quad b_2b_3 \quad c_3c_1 | ,$$

where Δ is $| a_1b_2c_3 |$; and it is not difficult to show also that

$$| A_1^2 B_2^2 C_3^2 | - | A_1A_2 \quad B_2B_3 \quad C_3C_1 | = \Delta^2 \left\{ | a_1^2 b_2^2 c_3^2 | - | a_1a_2 \quad b_2b_3 \quad c_3c_1 | \right\} .$$

(*Issued separately April* 8, 1908.)

XII.—Seismic Radiations. By Professor C. G. Knott, D.Sc.

(Read December 2, 1907; and January 20, 1908.)

When a large earthquake occurs in any part of the earth, its tremors can be recorded on suitable instruments all over the surface of the globe. These instruments are of various types, the most familiar being the horizontal pendulum, the evolution of which we owe to the labours of a small band of enthusiasts who were engaged by the Japanese Government in the late seventies and early eighties to teach the students of Japan the scientific methods of the West. The most conspicuous of these is undoubtedly Professor John Milne, who in his seismological laboratory established in the Isle of Wight continues to study the mysterious movements of the earth. Prompted by him, the British Association has installed some fifty instruments in various parts of the world; and from the accumulating records furnished by these instruments, Milne pursues his seismological studies. In addition to these fifty British Association stations, there have grown up in recent years many seismological laboratories in Europe, Asia, and America; and the data supplied from all these sources place us in a much better position than ever before to draw sure conclusions from the character of the records.

In 1898 I drew a rough sketch of the probable form of wave-fronts and rays of seismic disturbance, basing my calculations upon the early results obtained by Milne as to the times of transit of the different types of motion which characterise the distant earthquake record. This was published as part of the Seismological Committee's Report to the British Association in 1899. A little earlier, Rudzki * and V. Kövesligethy † had independently worked out some brachistochronic problems suggested by the earthquake phenomena; but their mathematical assumptions were made so as to get a soluble case, and had no dynamical basis in harmony with known seismological facts.

In a second paper published in 1905, V. Kövesligethy † works out his original theory more fully, and shows how it may be harmonised with certain of these facts. But his fundamental assumption that the speed of propagation of the seismic waves through the earth diminishes as the depth

* *Beiträge zur Geophysik*, iii., 1898.

† *Mathem. u. Naturw. Berichte aus Ungarn*, xiii., 1897; xxiii., 1905.

increases, is not one which can be readily admitted in these days. The evidence seems to be all the other way.

In 1905 and 1906, Benndorf * published two important papers, in the second of which he works out a law connecting speed of propagation with depth, starting from the angles of emergence as measured by Schlüter.

Before I knew of these papers of Benndorf, I had myself returned to the problem of nine years ago, and after some trials had obtained a form of solution which seemed to satisfy all the known phenomena. It is this which forms the subject of the present paper.

It is recognised by all seismologists that the records obtained at stations outside the destructive area of a large earthquake consist of two well-marked portions—called by Milne the "preliminary tremors" and the "large waves." In all well-developed seismograms two distinct phases of the preliminary tremors can be distinguished, a fact first clearly discussed by Oldham; and Omori, the Professor of Seismology in Tokyo, further divides the large waves into five distinct phases.

Here I consider only the two phases of the preliminary tremors, regarding the nature of which there has been a good deal of controversy. The most obvious view regarding these is that they are elastic waves propagated from the earthquake source through the body of the earth. This is the view I have always held along with Milne and Oldham; but it is not the view favoured by the Japanese seismologists. Benndorf's second paper, already referred to, strongly supports the view now to be presented.

When an earthquake occurs it sends forth in all directions seismic disturbances apparently of three distinct kinds. Two of these, the phases, namely, of the preliminary tremors, are transmitted by brachistochronic paths through the earth with speeds of propagation which are determined by the elastic constants and the density of the earth. The large waves, on the other hand, pass round the circumference as surface waves, taking fully two hours to reach the antipodal point after the epoch of the earthquake. This is proved by the fact that the time of transit from the epicentre to any station is proportional to the arcual distance of the station. But the time of transit of the preliminary tremors does not follow this simple law. The times of transit for various distances are given below when the results of the calculation are compared with the results of observation.

In the following investigation I assume—

1. That the source of disturbance is close to the surface of the globe.

* *Mitt. d. Erdbeben-Kommission* (Kaiserl. Akad. der Wissens., Vienna), Nos. xxix., xxxi.

2. That the speed of propagation depends only on the distance from the centre.

Let v be the speed at distance r and T the time along any path. Then by Hamilton's general method applied to brachistochronic problems we have

$$\left(\frac{\partial T}{\partial r}\right)^2 + \left(\frac{\partial T}{r\partial \theta}\right)^2 = \frac{1}{v^2} \quad . \quad . \quad . \quad . \quad (1)$$

where r, θ are the polar co-ordinates of a ray passing in any chosen diametral plane including the origin of the disturbance.

Put $r = x\mathrm{R}$, where R (= 6370 kilometres) is the radius of the globe, and $\mathrm{T} = \mathrm{SR}$. Equation (1) then becomes

$$x^2\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial \theta}\right)^2 = \frac{x^2}{v^2} \quad . \quad . \quad . \quad . \quad (1')$$

We get a solution of this by putting $\partial S/\partial\theta = a$, a quantity independent of x and θ, and then integrating the equation

$$x\frac{\partial S}{\partial x} = \sqrt{\frac{x^2}{v^2} - a^2}$$

between appropriate limits.

All rays are supposed to begin at the source $x = 1$, $\theta = 0$. Hence

$$\frac{T}{R} = S = a\theta + \int_1^x \frac{dx}{x}\sqrt{x^2/v^2 - a^2} \quad . \quad . \quad . \quad (2)$$

an equation which gives the time of transit from the origin to any point x, θ.

The equation of the path is obtained by equating to an arbitrary constant the result of differentiating S with respect to the constant a. Hence with the same limits any ray is represented by

$$0 = \theta - a\int_1^x \frac{dx}{x\sqrt{x^2/v^2 - a^2}} \quad . \quad . \quad . \quad (3)$$

each particular ray corresponding to a definite value of the parameter a. If ψ is the angle at which the radius meets the ray at any point, we have

$$\mathrm{cotan}\,\psi = \frac{\partial r}{x\partial\theta} = \sqrt{\frac{x^2}{a^2v^2} - 1} \quad . \quad . \quad . \quad (4)$$

At the surface where $x = 1$, this quantity becomes the tangent of the angle of emergence of the ray. Representing the angle of emergence by e, and writing U for the value of v at the surface, we get

$$\tan e = \sqrt{\frac{1}{a^2U^2} - 1} \quad \text{or} \quad \cos e = aU \quad . \quad . \quad . \quad (4')$$

When a wave-front impinges on the earth's surface, the speed of propagation of the surface trace of the disturbance is obviously greater than the true speed of the wave in the ratio of unity to the cosine of the angle of emergence. Hence a is the reciprocal of the fictive speed of propagation of the surface trace. This relation, which can be established almost intuitively by consideration of the manner in which an impinging wave runs along a boundary, must be as old as Fresnel, and is made use of explicitly by Green and many later writers on the laws of reflexion and refraction of light and elastic waves.[1] It gives, indeed, the most lucid explanation of the phenomenon of total reflexion. Nevertheless. both V. Kövesligethy and Benndorf think it necessary to prove the relation analytically, and to enunciate it as something altogether new in the theory of wave motion.

It is clear that a knowledge of the angle of emergence can give no information regarding the speeds of propagation in the deeper parts of the earth. Benndorf has, however, by certain plausible assumptions as to the limiting values of the quantities involved, obtained a solution which satisfies the values of the angles of emergence determined experimentally by Schlüter. The experimental determination of the angle of emergence depends on the comparison of the vertical displacement of the ground with the simultaneous maximum horizontal displacement, and involves a mathematical reduction. The displacements were measured on appropriate forms of seismometer. They were very small and subject to large errors. Taken as a whole, Schlüter's values for the angles of emergence at one locality due to earthquake tremors coming from sources at different distances are in accordance with what would be expected from the nature of the problem; but the data are too meagre to establish any peculiarity of detail such as seems to be indicated.

I purpose to work out a definite case, assuming for the speed v the expression

$$v^2 = V^2 - \mu^2 x^2 = \mu^2(a^2 - x^2).$$

To determine V and μ we have the two conditions, (1) the value of v at the surface, (2) the time of transit across a diameter.

After some trials I chose the following expression as satisfying these two conditions fairly well, namely,

$$v = \mu \sqrt{a^2 - x^2} = 13{\cdot}6 \sqrt{1{\cdot}2 - x^2}.$$

Putting $\phi = \mu a$, we get equations (2) and (3) in the somewhat simpler forms

[1] See, for example, my paper of 1888 on Earthquakes and Earthquake Sounds, etc., *Trans. Seism. Soc. of Japan*, vol. xii., p. 123; also *Phil. Mag.*, July 1899, p. 71.

$$\frac{\mu T}{R} = \phi\theta + \int_1^x \frac{dx}{x}\sqrt{\frac{x^2}{a^2 - x^2} - \phi^2} \quad . \quad . \quad . \quad (2)$$

$$0 = \theta - \phi \int_1^x \frac{dx}{x\sqrt{\frac{x^2}{a^2 - x^2} - \phi^2}} \quad . \quad . \quad . \quad (3)$$

The integrals of these are

$$\begin{aligned}\frac{\mu T}{R} = \phi\theta + \sqrt{1+\phi^2}&\left\{\sin^{-1}\sqrt{\frac{x^2}{a^2}(1+\phi^2) - \phi^2} - \sin^{-1}\sqrt{\frac{1}{a^2}(1+\phi^2) - \phi^2}\right\} \\ + \quad \phi&\left\{\sin^{-1}\phi\sqrt{\frac{a^2}{x^2} - 1} \qquad - \sin^{-1}\phi\sqrt{a^2 - 1}\right\}\end{aligned} \quad . \quad (4)$$

$$\begin{aligned}0 = \theta + \frac{\phi}{\sqrt{1+\phi^2}}&\left\{\sin^{-1}\sqrt{\frac{x^2}{a^2}(1+\phi^2) - \phi^2} - \sin^{-1}\sqrt{\frac{1}{a^2}(1+\phi^2) - \phi^2}\right\} \\ + \quad &\left\{\sin^{-1}\phi\sqrt{\frac{a^2}{x^2} - 1} \qquad - \sin^{-1}\phi\sqrt{a^2 - 1}\right\}\end{aligned} \quad . \quad (5)$$

Multiply (5) by ϕ and subtract from (4) and we get a very simple expression for the time, namely,

$$\frac{\mu T}{R} = \frac{1}{\sqrt{1+\phi^2}}\left\{\sin^{-1}\sqrt{\frac{x^2}{a^2}(1+\phi^2) - \phi^2} - \sin^{-1}\sqrt{\frac{1}{a^2}(1+\phi^2) - \phi^2}\right\} \quad . \quad (6)$$

For each possible value of ϕ equation (5) gives a ray, and the least value of x for this ray is determined by the condition

$$x^2 = \frac{a^2\phi^2}{1+\phi^2} = \frac{1 \cdot 2\phi^2}{1+\phi^2} \quad . \quad . \quad . \quad . \quad (7)$$

Also, since $a^2\phi^2$ cannot exceed $1+\phi^2$, we find for the greatest possible value of ϕ^2 the expression

$$\phi^2 = \frac{1}{a^2 - 1}.$$

In the case to be discussed, $a^2 = 1 \cdot 2$; hence ϕ^2 cannot exceed 5, and ϕ cannot exceed 2·236.

When x^2 has the limiting value just given, equation (6) becomes

$$\frac{T}{574} = \frac{\mu T}{R} = \frac{1}{\sqrt{1+\phi^2}}\left\{\sin^{-1} 0 - \sin^{-1}\sqrt{\frac{5-\phi^2}{6}}\right\} \quad . \quad . \quad (8)$$

and gives the time of describing half the ray.

Again, to find half the arc subtended at the centre by a complete ray, we put $x^2(1+\phi^2) = a^2\phi^2$ in equation (5) and obtain

$$\begin{aligned}0 = \theta + \frac{\phi}{\sqrt{1+\phi^2}}&\left(\sin^{-1} 0 - \sin^{-1}\sqrt{\frac{5-\phi^2}{6}}\right) \\ + \quad &\sin^{-1} 1 - \sin^{-1}\phi\sqrt{0 \cdot 2}\end{aligned} \quad . \quad . \quad (9)$$

Finally, we have by equation (4′) for the angle of emergence

$$e = \cos^{-1}\frac{\phi U}{\mu} = \cos^{-1}\phi\sqrt{1\cdot 2} = \cos^{-1}(1\cdot 095\phi) \quad . \quad . \quad (10)$$

From these four equations (7), (8), (9), and (10) we can quickly calculate for each assumed value of ϕ between zero and 2·236 the important characteristics of various rays. The results are given in the following table:—

CASE I. $v = 13{\cdot}6\sqrt{1{\cdot}2 - x^2}$ throughout the globe.

Parameter ϕ.	Arc $2\theta°$.	Transit time 2T min.	Minimum radius x_1.	Emergence angle $e°$.	Energy distribution over surface defined by arc.
2·2	5·03	1·6	0·998	10·3	3·26
2	10·1	2·9	·98	26·6	20·04
1·8	15·6	4·3	·958	36·4	35·26
1·5	24·9	6·5	·913	47·9	54·99
1·2	37·7	8·8	·841	57·6	71·24
1	49·6	10·6	·775	63·4	80·03
0·8	65	12·4	·684	69·0	87·22
·5	98·7	14·9	·492	73·4	95·02
·2	144·2	17·5	·214	84·9	99·21
·1	161·7	17·8	·109	87·4	99·82
·05	170·9	17·9	·055	88·7	99·97
·01	178·2	18	·011	89·8	99·99
0	180	18	0	90	100

As we shall see immediately, the above table gives for arcs smaller than 100° values of times of transit distinctly too high; but for arcs greater than 100° the values agree fairly well. This shows that the speed of propagation must begin to increase more rapidly with depth than is indicated by the assumed formula for v, attaining an almost constant value through a large proportion of the inner parts of the earth.

Let us assume, then, a second case in which the formula

$$v^2 = V^2 - \mu^2 x^2$$

applies from the surface to a depth of only one-tenth of the radius. For greater depths the speed is the same throughout.

After a few trials I chose the expression

$$v = 24{\cdot}45\sqrt{1{\cdot}06 - x^2}$$

as holding true from $x = 1$ to $x = 0{\cdot}9$. At the surface the speed is 6 kilometres per second; and at all points in the globe up to value $x = 0{\cdot}9$ the speed is 12·23 kilometres per second.

The ray will be wholly curved when it does not penetrate deeper than $x = 0{\cdot}9$; but when it penetrates deeper than this limit, the middle portion of the ray will be straight.

The position of the straight part and the angle which it subtends at the centre are obtained by consideration of the angle with which the outer curved part of the ray meets the spherical surface of radius 0·9. This angle might be called the *angle of immergence* into the nucleus of constant speed of propagation. For the rays which penetrate this nucleus we use the formulæ already given between the limits $x=1$ and $x=0{\cdot}9$; and then for the straight parts we use the simple formula for constant velocity.

The important data are given in sufficient detail in the following table, which differs from the former table only because of the necessity of discriminating between the curved and straight parts of the rays when these exist. The immergence angle has the meaning already defined, and quantities which have to do with the curved and straight parts are marked with suffixes 1 and 2 respectively.

CASE II. $v=24{\cdot}45\sqrt{1{\cdot}06-x^2}$ through the outer shell of one-tenth the radius.

Parameter ϕ.	Arc. $2\theta_1°$.	Arc. $2\theta_2°$.	Transit time. $2T_1$.	Transit time. $2T_2$.	Minimum radius x_1.	Emergence angle $e°$.	Immergence angle $i°$.	Energy distribution over surface defined by arc.
4	1·7	...	0·4	...	0·999	11·5	...	8·03
3	6	...	2·0	...	·977	42·4	...	45·96
2	17·9	...	3·9	...	·922	60·7	...	76·10
1·8	21·6	...	4·5	...	·9	63·8	...	80·10
1·7	14·2	38·5	3·5	5·1	·85	65·4	19·2	82·65
1·5	10·5	67·1	3	8·6	·75	68·4	33·6	86·49
1	6·7	111·4	2·5	12·9	·5	75·8	68·6	94
0·5	2·6	146·2	2·4	15·3	·25	83·0	73·9	98·5
0	...	...	2·3	15·6	...	90	90	100

The whole arc in each case is $2\theta_1+2\theta_2$, and the time of transit $2T_1+2T_2$.

Drawing the time-graphs for these two cases and picking out the values for every 30° of arc, we get the following abridged table, in which Milne's corrected values are added for the sake of comparison:—

TRANSIT TIMES.

Arc.	Case I.	Case II.	Milne.
30°	7·5	5·7	5·2
60°	11·9	9·5	9·8
90°	14·5	13	13·1
120°	16·4	15·6	15·3
150°	17·6	17·6	17
180°	18	17·9	18

The values for Case II. agree very well with Milne's corrected values. That is to say:—

The observed facts of seismic radiation can be co-ordinated on the assumption that throughout all but a comparatively thin crust of the earth the elastic waves of highest speed are transmitted with a speed of 12·23 kilometres per second, and that within this crust, of thickness equal to one-tenth the radius, the speed diminishes from value 12·23 kilometres per second at the inner surface to 6 kilometres per second at the outer surface.

If this wave of highest speed be a compressible wave with longitudinal vibrations, the *cosine* of the angle of emergence of the ray will give the ratio of the magnitude of the horizontal motion to the whole amplitude. Now, most of the observations have been obtained with instruments recording horizontal motion only, and comparatively few with instruments recording vertical motion. But at great arcual distances from the epicentre the angle of emergence increases towards 90°, with corresponding diminution in the value of the *cosine*, and the horizontal component of the displacement will be very small, ultimately vanishing at the antipodal point. At such great distances there will consequently be a tendency for the preliminary tremors, assumed to be mainly compressional, to be retarded in their arrival. As a matter of fact, it is extremely difficult at times to determine the exact moment at which the tremors begin on records which have been obtained at stations further distant than 100° from the epicentre.

According to the theory here developed, we may obtain the times for the second phase of the preliminary tremors by increasing the times for the first phase in the ratio of 31·3 to 18, assuming for the purpose Benndorf's corrected values. The results are as follows:—

Times of Transit of Second Preliminary Tremors.

Arc.	Case I.	Case II.	Benndorff.
30°	13	9·9	11·9
60	20·7	16·5	17·5
90	25·2	22·6	23·8
120	28·5	27·1	27·5
150	30·6	30·6	30·5
180	31·3	31·1	31·3

The agreement here is not quite so good as in the case of the first phase. It could be improved by slightly increasing the depth of the shell through which there is variation of speed of propagation. Viewed as a problem in elasticity, this means that the modulus on which the speed of the second phase depends varies with depth according to a law not quite the same as that which holds for the modulus determining the propagation of the first phase,—a highly probable truth.

If the second phase be supposed to be due to the arrival of the first tremors of the distortional wave with displacements perpendicular to the direction of propagation, then the horizontal component of the displacement will be equal to the maximum displacement multiplied by the *sine* of the angle of emergence. Consequently, at great arcual distances the second phase, as recorded by a horizontal pendulum, should tend to become proportionately in greater evidence than the first phase. There are, indeed, cases in which the second phase has been mistaken for the first, the latter having had too small a horizontal component to produce a record.

It seems to me that the theory here sketched is amply sufficient to co-ordinate all the known phenomena. The first phase of the preliminary tremors is thus identified with the compressional waves passing through the body of the earth. No doubt, especially in the more heterogeneous crust, surfaces of discontinuity will start waves of distortional type along with the incident waves of compressional type.* But across the practically homogeneous nucleus the compressional waves will run ahead of their associates of other type, so that what emerges at the surface, although modified in detail, must be referable to these compressional waves. Similarly, when somewhat later the distortional waves flow in in quantity, there will be mingled with them waves of the compressional type. Nevertheless, the second phase will be largely composed of disturbances which have passed through the homogeneous nucleus as distortional waves, but have emerged modified in detail by refraction across discontinuous surfaces.

The distortional waves need not necessarily be more energetic than the compressional; but their generally greater amplitude, as measured on instruments recording horizontal motion, is at once explained in terms of the angle of emergence. Looking back to pages 222, 223, we see that for arcual distances greater than 60° the sine of the angle of emergence is greater than the cosine in ratios exceeding the value 2, rapidly increasing for greater arcual distances.

The longer periods which observation proves to be associated with the

* See my paper on "Reflexion and Refraction of Elastic Waves," *Phil. Mag.*, July 1899, and abstract of address on Earthquakes, *R.S.E. Proceedings*, vol. xxii., 1899.

second phase do not seem to find an immediate explanation along the lines of this theory. But may these not be explained as due to the intermingling of the quicker distortional vibrations with compressional vibrations of longer wave-length which, because of their longer period, have travelled slower than the compressional waves of shorter wave-length? True, the mathematical theory of elasticity does not recognise any relation between speed of propagation and length of wave; but this theory is only a first approximation to reality, and proves nothing, either one way or the other, as to what may occur in seismic vibrations. The fact that the first phase, when well developed, always begins with comparatively rapid oscillations, seems indeed to establish the truth that the shorter waves of a seismic disturbance do travel faster than the longer waves. If we take four seconds to be the shortest period, we find that the disturbance travelling with a speed of 12·23 kilometres per second will have a wave-length of nearly 49 kilometres.

It may be of some interest to compare the elastic constants of the material of the nucleus of the earth on the assumption that we are dealing with compressional and distortional waves. The ratio of the speeds of the two types is 31·3 to 16, or 1·74 to unity. The ratio of the wave-moduli will be as the square of this, or almost exactly 3 to 1. Hence in the notation of Thomson and Tait we have

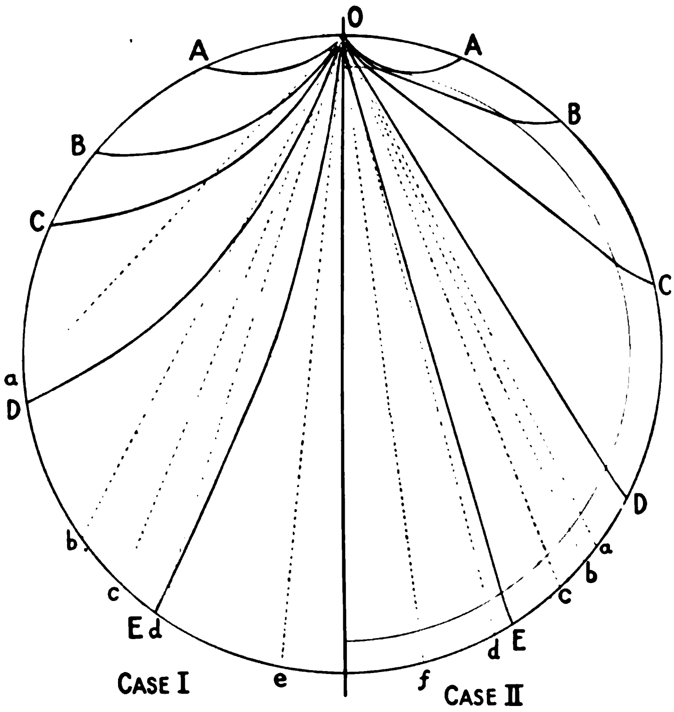
$$k + 4n/3 = 3n$$

where k is the incompressibility and n the rigidity. This gives

$$3k = 5n$$

a noteworthy result, showing that the inner parts of the earth almost accurately fulfil the conditions of isotropy possessed by the ideal elastic solid of Navier and Poisson. This conclusion seems to me to be an additional argument in favour of the view now being presented.

Here in the heart of the earth is a material at a high temperature and under great pressure, brought into a physical state suggesting homogeneity, though not necessarily implying it. As shown long ago by Tait, this globe is held together mainly by gravitational attraction. The cohesion between the molecules is, however, the force which is involved in the propagation of the elastic disturbances which radiate from a seismic centre. The view of the French elasticians was that true homogeneity required a definite relation between incompressibility and rigidity. This definite relation is not realised in the case of materials tested by ordinary combinations of stress and strain. This fact, however, was not admitted by de St Venant as disproving the uniconstant theory developed by Navier and Poisson; for as soon as an æolotropic stress is applied to our rods and wires, the material ceases to be truly isotropic. It would appear from the calculation just made that the interior of the earth is in a condition which at each point might be described as isotropic; and the relation required by the uniconstant theory is accurately satisfied by the constants calculated from the propagation of the two phases of the preliminary tremors when these are assumed to be respectively the compressional and distortional vibrations.



We shall now consider the significance of the last column of figures in the tables on pages 222 and 223, the columns headed "energy distribution." The meaning of the figures is best explained by consideration of particular cases in connection with the accompanying diagram.*

The diagram represents a section of the globe, and some of the particulars corresponding to each of the cases are figured on the one half. The full lines show the paths of the seismic disturbances as they radiate out from the origin O. Each ray corresponds to one of the particular set of values calculated and tabulated on pages 222, 223. The following short table

* Reproduced by permission of the Delegates of the Clarendon Press, the publishers of my book on *The Physics of Earthquake Phenomena*.

indicates the value of the parameter which corresponds to each ray, the rays being represented by the terminal letter on the diagram:—

Ray.	A.	B.	C.	D.	E.
Case I., parameter ϕ . .	1·5	1	0·8	0·5	0·2
Case II., parameter ϕ . .	1·8	1·7	1·5	1	0·5

To each full line OA, OB, OC, etc., there corresponds a dotted line Oa, Ob, Oc, etc., which starts tangential to the curved ray and is therefore the direction in which the disturbance begins to radiate outwards from the origin. Considerations of symmetry show at once that the angle which each dotted line makes with the surface at the origin is the same as the angle with which the ray emerges at the other end. In other words, this angle is equal to the emergence angle as tabulated above.

In the diagram the left-hand semicircle shows the rays for the first case, in which the variation of speed is assumed to take place throughout the whole globe; and the right-hand diagram the second case, in which the variation takes place only through the upper layer of thickness, equal to one-tenth of the radius. In the latter case the first ray OA lies wholly within the layer of varying speed of propagation, and is curved throughout its whole length. All the other rays represented pass partly through the interior of constant speed of propagation and are straight throughout a part of their course. Thus the rays OD, OE to distant points are very approximately coincident with chords, but for shorter rays such as OC and OB the chordal coincidence is not so close. We shall discuss this case in some detail.

The dotted line Oa in Case II. gives the direction in which a ray would have passed if the speed of propagation had been absolutely constant throughout the whole globe. This condition would have given rise to what we may call the spherical distribution of energy over the surface of the globe, half the energy being in fact distributed over the hemisphere of which the origin is the pole. But in the case represented in the right-hand semicircle the ray starting originally along the dotted line Oa becomes bent round by refraction so as to assume the position OA. The energy, of course, passes with it. Hence the energy, which in the simple case of spherical distribution would have been distributed over the part of the surface defined by the arc Oa with O as pole, is concentrated within the much smaller part of the surface defined by the arc OA. We are to imagine Oa to be one of a cone of rays which divides the spherical surface into two parts, defined respectively by the arcs into which Oa divides the semicircle. The semi-vertical angle of this cone is equal to $90° - e$, where e is the corresponding angle of emergence belonging to the ray OA; and the area on the spherical surface defined by the arc Oa is proportional to

$$1 + \cos(180° - 2e) = 1 - \cos 2e.$$

This number, divided by 2, the value when $e = 90°$, represents the fraction of the energy which is finally distributed over the surface defined by the arc OA. Thus in the particular case which has been the subject of discussion, 80·1 per cent. of the whole energy is found distributed over the comparatively small fraction of the surface whose boundary lies 21°·6 from the epicentre. In spherical distribution only 3·5 per cent. of the whole energy would have appeared over this surface. Glancing back to the table for Case II., we see that 50 per cent. of the whole energy is distributed over the small surface whose boundary lies about 7° from the epicentre, and that 75 per cent. is distributed over the surface which does not extend to 18° from the epicentre. In spherical distribution these surfaces would have received respectively only ⅓ and 2½ per cent. of the whole energy.

It is interesting to compare the two cases figured side by side on the diagram, and to notice how much more concentrated the energy is in the neighbourhood of the epicentre in Case II., which is characterised by a rapid variation of speed of propagation within the upper layers of the earth.

In these calculations I have, for simplicity, assumed the origin to be at the surface. This is never quite the case in large, world-shaking earthquakes. These originate at depths which may vary from 10 to 50 miles. Nevertheless, because of the curving of the seismic rays the energy will be distributed unequably in a manner similar to what is here indicated. The deeper the origin the less unequable will the final distribution be; but so long as the origin lies within the layer of changing velocity, there must be the curving round of the seismic rays, carrying their energy with them.

Let there be two earthquakes of equal intensity but with their origins at different depths. The one with the shallow focus will have its energy strongly concentrated towards the surface regions immediately in the vicinity of the epicentre; while the energy associated with the deeper focus will be less unequably distributed over the whole surface. The latter will be registered all the world over as a world-shaking earthquake, while the former may appear much more limited in its sphere of action, simply because of the small intensity of the tremors which pass to distant regions.

It would be possible, though somewhat laborious, to work out the surface distribution of energy for several depths of origin along the lines indicated above. If this were done, and if instruments could be constructed to give an accurate measure of the energy associated with seismic movements at the surface, we should be in possession of a method for determining the depths of origins—a problem which has hitherto baffled all endeavours.

(*Issued separately April* 9, 1908.)

XIII.—The Systematic Motions of the Stars. By Professor Dyson.

(MS. received February 3, 1908. Read February 17, 1908.)

MANY attempts have been made to interpret the systematic proper motions of the stars as the result of a movement of the solar system in space. In recent years several elaborate and extensive investigations have been made, but the increase of the available material in amount and accuracy has not led to a corresponding increase in the accuracy of the so-called "Solar Apex." Instead, determinations from different groups of stars have shown larger differences than can be attributed to accidental error, and in particular a determination by Kobold, using Bessel's method, has given results at variance with those obtained by Airy's method.

Professor Kapteyn was led to an examination of the fundamental hypothesis underlying all these investigations—viz., the solar motion apart, the proper motions of the stars show no preference for any special directions. Kapteyn found that the apparent proper motions of the stars (*i.e.* as seen by us, and therefore relative to the Sun) show drifts in two directions, and not in one only, as would be the case were the motion of the solar system the cause. Kapteyn's results, derived from an examination of 2500 stars observed by Bradley about 1755, and repeatedly re-observed in modern times, are given in a short paper in the British Association Report for 1905 (pp. 257–265). These results were confirmed by Mr Eddington (*Monthly Notices of the Royal Astronomical Society*, vol. lxvii. pp. 34–63) from the examination of the proper motions of over 4000 stars within 52° of the North Pole, which were observed by Groombridge about 1810, and re-observed at Greenwich about 1890.

The hypothesis that the stars are moving in two streams is of a revolutionary character, and calls for further investigation. In the following paper I have examined stars with large proper motions, the limits chosen being from 20″ to 80″ a century. The reason for choosing such stars is, that if the stars are moving in two streams, these streams would be indicated more strongly by excluding the stars of small proper motion. Again, limiting the stars in this manner, we eliminate the effect of errors of observation to a great extent, and may use stars whose proper motions are less accurately known than those of Bradley and Groombridge—for an error of 2″ or 3″ a century, though important in the case of small proper motion, is of little consequence when the proper motion exceeds 20″.

The stars whose proper motions are considered are well distributed over the whole sky: 157 are within 52° of the North Pole, the proper motions being taken from the Groombridge Catalogue; 165 are within 50° of the South Pole, the proper motions being taken from the Cape Astrographic Catalogue (only such stars being used as are found in two earlier catalogues), from Professor Porter's Catalogue of Stars with large proper motions (Cincinnati Observatory Publications, No. 12), and the Cape General Catalogue for 1900.

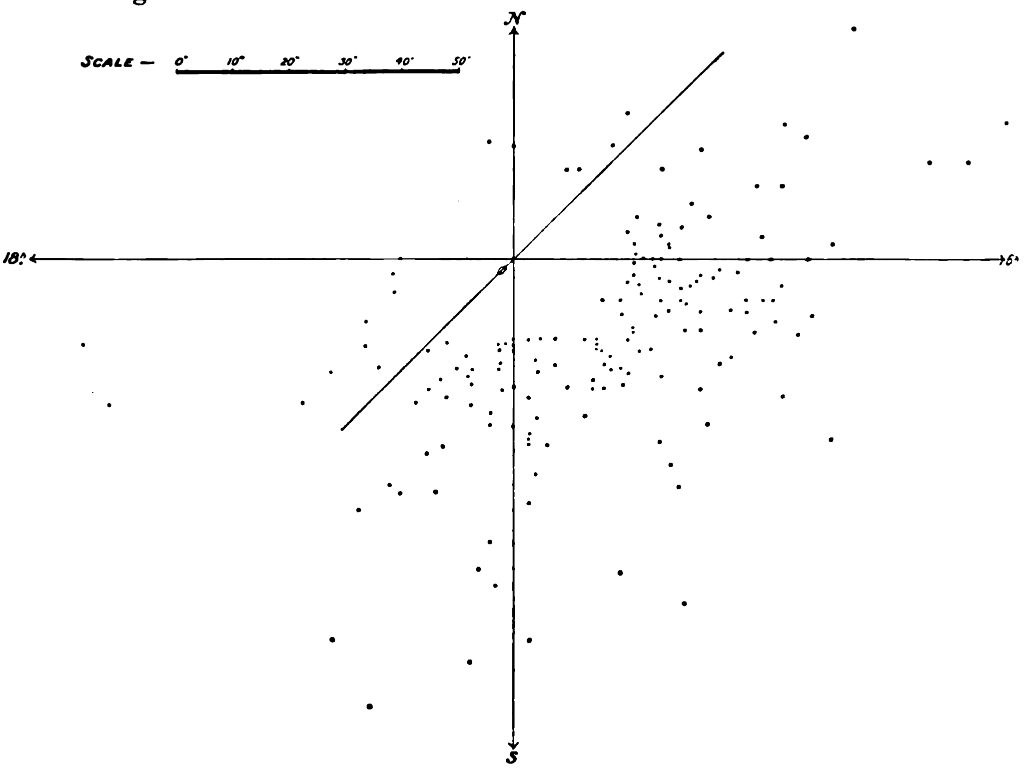


DIAGRAM 1.

The remainder are within a belt extending 30° N. and S. of the equator, and are taken from Professor Porter's Catalogue. They are distributed in R.A. as follows:—

R.A. 22^h–2^h	154 stars	R.A. 10^h–14^h	167 stars
R.A. 2 –6	126 stars	R.A. 14 –18	124 stars
R.A. 6 –10	92 stars	R.A. 18 - 22	121 stars

These 1100 stars have been treated as belonging to eight different areas, and their proper motions found as projected respectively on the tangent planes at the poles, and the tangents at the equator at the points 0^h, 4^h, 8^h, etc.

These proper motions were indicated graphically as in diagram 1, which refers to the area comprised between dec. −30° and +30° and R.A. 10^h to 14^h,

Centre of Area. Angle.	North Pole.	South Pole.	R.A. 0^h Dec. 0°.	R.A. 12^h Dec. 0°.	R.A. 4^h Dec. 0°.	R.A. 16^h Dec. 0°.	R.A. 8^h Dec. 0°.	R.A. 20^h Dec. 0°.
0° – 7½°	1	0	0	½	1	0	1	0
7½ – 15	3	0	0	0	0	0	0	1
15 – 22½	4	1	3	0	0	0	0	0
22½ – 30	0	0	2	1	2	1	0	0
30 – 37½	1	2	2	1	0	1	1	1
37½ – 45	2	3	0	2	2	1	1	0
45 – 52½	2	4	0	0	0	2	1	3
52½ – 60	4	2	2	3	0	0	1	2
60 – 67½	9	6	4	2	2	2	2	0
67½ – 75	9	7	3	5	1	1	1	3
75 – 82½	18	10	10	8	4	2	3	0
82½ – 90	8	15	9½	9½	3½	2½	1	3½
90 – 97½	11	12	13½	13½	6½	3½	3	1½
97½ – 105	10	10	10	16	5	8	5	1
105 – 112½	7	12	6	8	7	5	3	1
112½ – 120	9	11	11	6	3	5	2	0
120 – 127½	9	3	5	7	6	6	0	5
127½ – 135	10	8	2	4	7	3½	1	1
135 – 142½	1	3	4	9	6	3	1½	3
142½ – 150	2	8	6½	4	4	10	7	2½
150 – 157½	3	1	3½	3	7	5½	2½	6
157½ – 165	1	4	4	4	7	6½	5	3
165 – 172½	4	2	9	6	7	6½	7	4
172½ – 180	2	2	2	9½	9½	13½	3½	12
180 – 187½	0	2	3	11	5½	8½	2½	10½
187½ – 195	0	1	4½	3½	4	4½	6	9
195 – 202½	1	2	9	7	8	12½	12½	13
202½ – 210	0	4	6	7	3	3	2½	13½
210 – 217½	2	0	2½	4	1	2	7	4
217½ – 225	2	1	5	1	6½	1	4	6½
225 – 232½	0	2	7	1	2½	2	2	4
232½ – 240	1	0	2	2	1	1	1½	1½
240 – 247½	1	2	1	1	1	0	½	2½
247½ – 255	1	6	1	2	0	0	0	1
255 – 262½	2	2	0	2	0	0	0	1
262½ – 270	4	1	0	1½	0	1	0	0
270 – 277½	3	2	0	½	0	0	0	0
277½ – 285	1	2	0	0	0	0	0	0
285 – 292½	3	1	0	0	0	0	0	0
292½ – 300	1	2	0	0	0	0	0	0
300 – 307½	0	1	0	0	0	0	½	0
307½ – 315	1	2	0	0	0	0	½	1
315 – 322½	0	0	0	0	0	0	0	0
322½ – 330	0	1	0	0	1	0	0	0
330 – 337½	1	3	0	0	0	0	1	0
337½ – 345	1	1	0	0	0	0	0	0
345 – 352½	1	1	1	1	2	0	0	0
352½ – 360	1	0	0	½	0	0	0	0
Total	157	165	154	167	126	124	92	121

the displacement of each dot from the origin representing a century's proper motion projected on the tangent plane at R.A. 12^h, dec. 0°, the centre of the area. Inspection of the diagram shows that there is no pronounced maximum in the fourth quadrant opposite to the minimum in the second quadrant, but that there are two maxima approximately in the directions of the axes. Drawing lines at angles of 7½°, 15°, 22½°, etc., the number of dots in each angle were counted. The result is given in the foregoing table of the distribution in direction of the proper motions in the eight areas considered.

The directions in which the angles are measured are for the polar

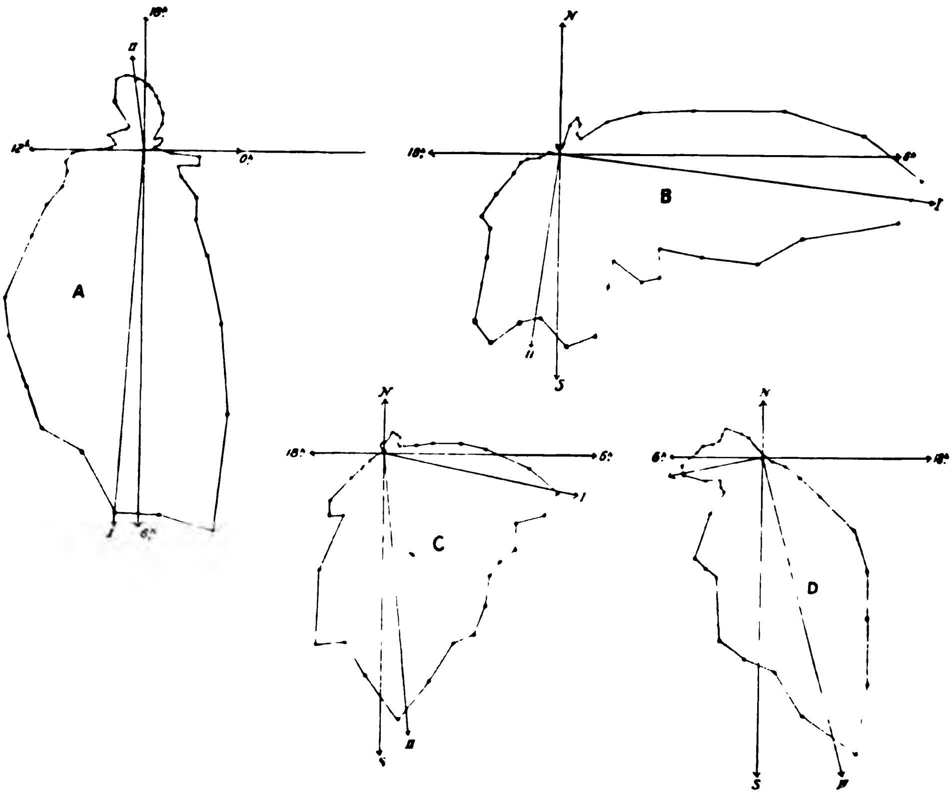


DIAGRAM 2.

regions from 0^h in a positive direction, and for the equatorial regions from the north round towards a point on the equator at R.A. 6^h. Thus the same directions in two parallel planes are placed side by side in the table.

The sums are taken for the corresponding directions in each pair of parallel planes, and the numbers smoothed by taking means and means again. The results are plotted in polar co-ordinates in diagram 2, the number of stars between 0° and 7½° being indicated by the radius vector in the direction 3¾°. If the proper motions were distributed at random in all directions, the figures obtained would be approximately circular. If a proper motion in one particular direction were added to a chance distribution of proper motions giving a *stream* of stars (such as would result, *e.g.*, from the motion of the solar system in space), the figures would be elongated but symmetrical about the direction of general drift. The figures of diagram 2 are not of this character, but inspection shows that they could be derived by the composition of two curves symmetrical about the directions marked I and II. These curves are in general similar to those calculated by Mr Eddington for the distribution of the direction of proper motion in a stream when proper motions of all magnitudes are included. I have endeavoured to effect the analysis graphically, mainly from the principle that the number of proper motions should be equal in two directions, equally inclined to the direction of the stream.

For the purpose of this analysis, the numbers were plotted in rectangular co-ordinates in diagram 3, the abscissæ corresponding to the direction and the ordinates to the number of stars moving within $3\frac{3}{4}°$ on either side of that direction. In A, whose centre may be taken as the North Pole, the abscissæ increase positively from 0^h to 24^h. In B, C, and D, 0° indicates a direction N. from the given centre (points on the equator at 0^h, 4^h, and 8^h), and the angles increase towards 6^h in each case.

The analysis of the resulting figures into two symmetrical curves is, to a fairly high degree of accuracy, simple, in consequence of the pronounced character of the maxima and their distance apart. Taking B as an example, the mode of analysis was as follows:—

(i) There is a pronounced maximum between 90° and 100°.

(ii) There is a maximum between 180° and 210°.

(iii) The second stream does not spread beyond 280°.

(iv) If 180° is the position of maximum, the curve of the second stream will, if symmetrical, reach the axis of x again at 80°; if the maximum is at 210°, at 140°.

Drawing provisional symmetrical curves on these two extreme hypotheses, and plotting the differences between the observations and the provisional curve II in each case on the left-hand side of the maximum, we have the material from which to construct curve I. If the maximum of II is at 180°, a symmetrical curve I with a maximum at $93\frac{3}{4}°$ (the highest dot in the diagram) will fit in excellently with the three or four observations on each side. If, however, the maximum of II is at 210°, curve II will only extend to 140°, and curve I near the maximum will be wholly independent of curve II. In this case the maximum of curve I will be near the position drawn at $97\frac{1}{2}°$.

(v) The maximum of curve I being thus fixed at between $93\frac{3}{4}°$ and $97\frac{1}{2}°$, the left-hand side of the curve is drawn from the observations and the right-hand side from symmetry. Curve II is then drawn again, and the curves are adjusted to fit in as well as possible with the observations. In

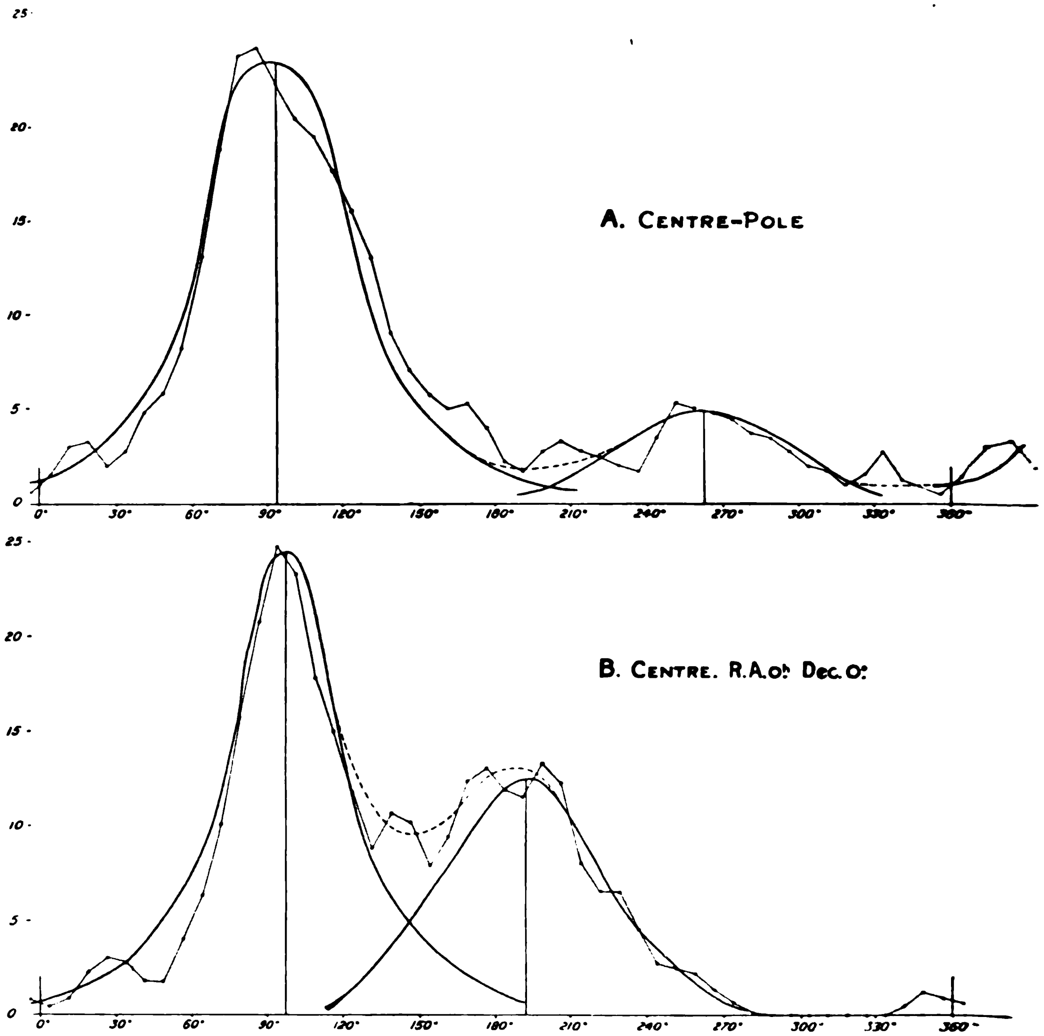


Diagram 3.

each case the form of the curves has been derived for curve I from the left-hand side and for curve II from the right-hand side.

The results are as follows:—

A. Poles. Stream I in direction $93\frac{3}{4}°$ or 6^h 15^m, with a maximum of $23\frac{1}{2}$; Stream II in direction $262\frac{1}{2}°$, or 17^h 30^m, with a maximum of 5.

B. Equator 0^h and 12^h. Stream I with a maximum of $24\frac{1}{2}$, in a direction $7\frac{1}{2}°$ S. (*i.e.* inclined $7\frac{1}{2}°$ to the equator); Stream II with a maximum $12\frac{1}{2}$, in direction 102° S.

C. Equator 4^h and 16^h. Stream I with a maximum of 11, in a direction 12° S.; and Stream II with a maximum of 16, in a direction 86° S.

D. Equator 8^h and 20^h. Stream I with a maximum of 5, in a direction 12° S.; Stream II with a maximum of $17\frac{1}{2}$, in a direction 105° S.

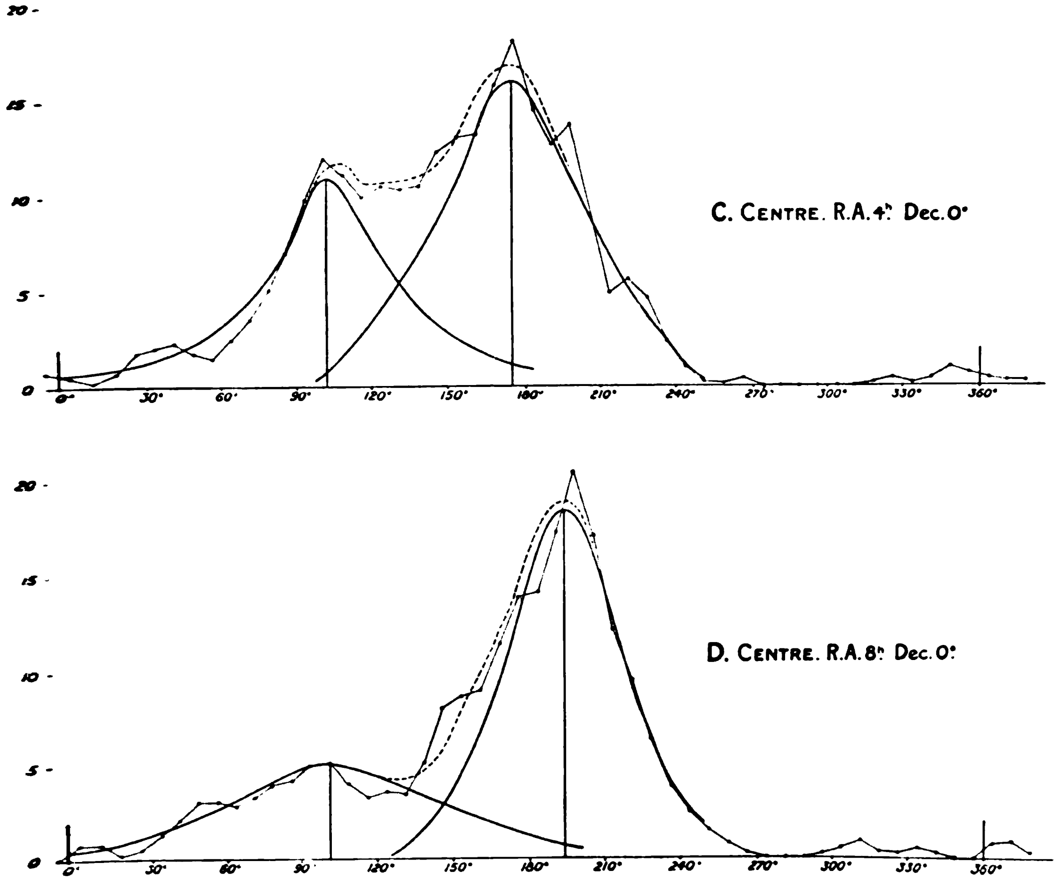


Diagram 3.

The four great circles through the pole, and the points 0^h, 4^h, 8^h on the equator belonging to Stream I, all pass within a degree of the point R.A. 6^h 15^m and dec. $-7°$.

The four belonging to Stream II converge to a point at R.A. 16^h dec. $-74°$,—the circles from A and C passing within 5° of this point, and those from B and D within 2°.

The high maxima of Stream I in regions A and B agree with the result that the apex of this stream is about 90° from the centres of the two regions. The diminution in C and D is in general agreement, except that the smallness of the case of D would indicate that the apex of the stream is somewhat nearer the centre of the region, say at R.A. 6^h 30^m, instead of R.A. 6^h 15^m.

The small maximum for Stream II in region A is in accordance with the position of the apex at a distance of 16° from it. For the other regions whose centres are 75°, 90°, and 75° from this point, the maxima should be approximately equal.

These results strongly support Professor Kapteyn's hypothesis that the stars are moving in two streams. The quick-moving stars here considered show the streams in a very pronounced manner. This is particularly the case with Stream II.

The positions found for the apices may be compared with those found by Kapteyn and Eddington :—

	Stream I.		Stream II.	
Kapteyn . . .	R.A. 85°	Dec. −11°	R.A. 260°	Dec. −48°.
Eddington . . .	„ 90°	„ −19°	„ 292°	„ −58°.
Present research . .	„ 94°	„ − 7°	„ 240°	„ −74°.

For Stream I the three determinations agree within 7° of the point R.A. 90°, dec. −12°, and for Stream II within 14° of the point R.A. 263°, dec. −60°.

(*Issued separately April* 9, 1908.)

XIV.—On a Sensitive State induced in Magnetic Materials by Thermal Treatment.

By **James G. Gray**, B.Sc., Lecturer on Physics in the University of Glasgow, and **Alexander D. Ross**, M.A., B.Sc., Assistant to the Professor of Natural Philosophy in the University of Glasgow. *Communicated by* Professor A. Gray, F.R.S.

(MS. received January 31, 1908. Read February 3, 1908.)

Part I.

Introductory.—Whilst engaged in carrying out a series of tests on the magnetic properties of annealed steel, the authors detected a peculiar variation in the susceptibility of the specimens. In the tests referred to, the magnetometric method was employed, the specimens being completely demagnetised prior to the annealing. It was noticed that the magnetisation curve obtained with a specimen in the freshly annealed condition differed materially from any subsequent magnetisation curve obtained with that specimen. Thus, a cylindrical rod was demagnetised, annealed at a high temperature, and a test gone through. A certain I-H curve was obtained. The rod was now demagnetised by reversals, the test repeated, and a new I-H curve constructed. The values of I obtained in the second test were in all cases lower than those obtained in the first; in other words, the magnetic quality of the material had been somewhat destroyed in the process of magnetisation.

As the presence of this effect is important in magnetic testing generally, the authors commenced a detailed examination of the phenomenon. They, however, subsequently found that the effect had been previously detected and partially investigated by Ewing.* His experiments have apparently been carried out with the object of obtaining merely a general idea of the nature of the sensitive state. The specimens, in the form of long wires, were annealed by passage through a bunsen flame, and thus only a rough estimate can be formed of the temperature to which the specimen was raised, and of the rate at which it cooled. As will be seen from the experiments to be described, these are important factors in the case. The authors' tests also show that the effect depends largely on the chemical constitution of the specimens and on their previous history.

* *Phil. Trans. Roy. Soc.*, 1885, p. 570, §§ 54–58. Similar tests have been carried out by Searle and Bedford (*Phil. Trans. Roy. Soc.*, 1902, p. 70), but, with the exception of some quantitative determinations of the magnitude of the effect, little further information is given.

Method of Investigation.—In the following experiments the magnetometric method was adopted throughout. The test specimen, in the form of a cylindrical rod, was placed in the "A" position of Gauss, within a magnetising solenoid. The field at the magnetometer needle, due to the current, was balanced in the usual way by a compensating coil of large radius, connected up in series with the solenoid. The adjustment was carried to such a degree of refinement that, on reversing in the circuit a current of considerably greater magnitude than the maximum current employed in the tests, the movement of the magnetometer needle was quite inappreciable. This balance was tested for various points on the

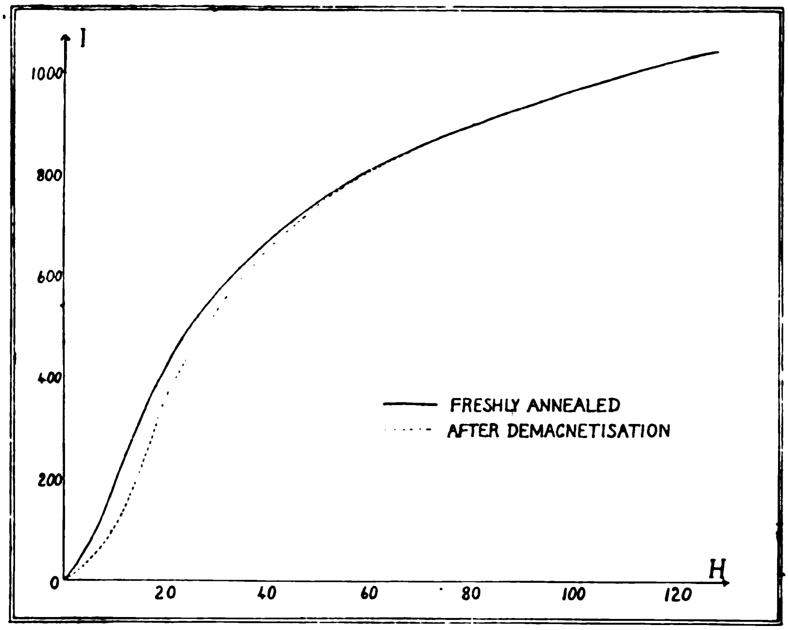


Fig. 1.—Steel specimen annealed at 900° C.

scale in the manner recommended by Erhard, and found to hold throughout. For the thermal treatment a Fletcher gas furnace was employed, the temperatures being measured by a platinum, platinum-iridium pyrometer.

Preliminary Tests.—In the first experiments several specimens were prepared from hard steel, completely demagnetised, and annealed at a temperature of 900° C. The specimens, when cold, were subjected to an I-H test, the fields extending from 0 to 125 C.G.S. units. In each case the test was repeated after the specimen had been demagnetised. The results obtained with the various specimens were in excellent agreement. Fig. 1 exhibits the I-H curves obtained with the specimens in the two states.

The continuous line shows the initial, or sensitive, condition of the specimen; the dotted line shows the condition after demagnetisation by reversals. It will be observed that the change is most pronounced at the steep part of the curve, whilst for fields greater than 60 C.G.S. units the susceptibility is practically unaltered. In subsequent work attention was therefore directed to field strengths below this value.

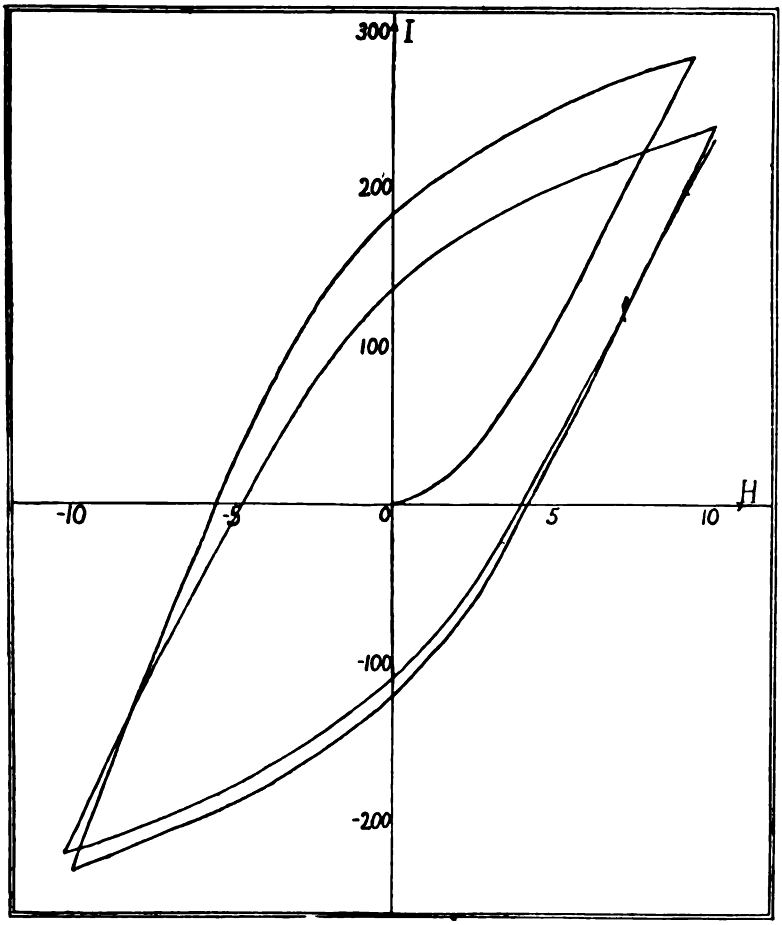


FIG. 2.

It was found that reannealing at 900° C. resulted in the specimen being restored to the magnetic condition defined by the continuous line.

Nature of the "Sensitive State."—As the enhanced magnetic quality of the steel induced by the thermal treatment was removed in the process of demagnetising by the method of reversals, it follows that if a specimen in the sensitive state is magnetised by a field H = H′, for which I = I′, and if H is now reduced to − H′ and then increased to + H′, the final value of I should be less than I′. That is to say, a cyclic alteration in H should not give a closed hysteresis loop. A specimen was accordingly tested in this manner and gave the curve shown in fig. 2.

In this test the magnetising field was increased to a maximum value of about 10 c.g.s. units, then reduced to −10 c.g.s. units, and this procedure was repeated several times. Each cyclic alteration of the values of H reduced

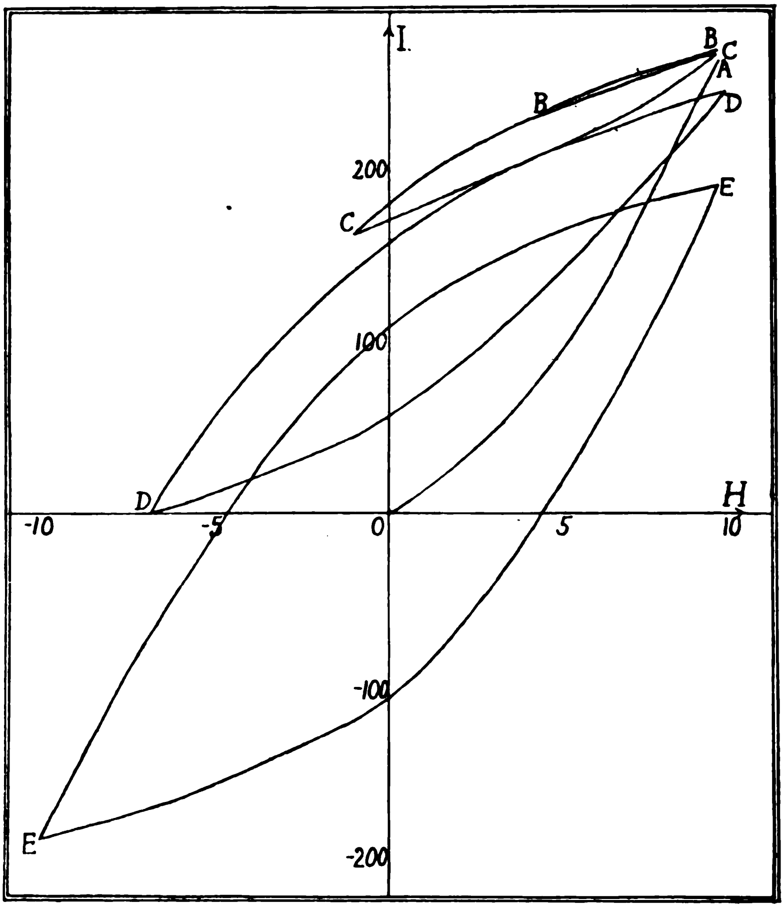


Fig. 3.

the corresponding values of I. The magnetic quality was very considerably reduced by the first cycle; it was further lessened, though to a diminishing extent, by the subsequent cycles. For H = 10 c.g.s. units, I was ultimately reduced by over 20 per cent. of its original value.

The form of the curves seemed to indicate that the deterioration in quality commenced on the first reversal of the field. This view was borne out by a test, the chief results of which are set out in fig. 3.

The line O A indicates the initial I-H curve obtained after annealing at 900° C., the point A corresponding to a field of 9·5 C.G.S. units. The magnetising field was now varied several times between the limits H = 9·5 and H = 4·5. The loop B B shows the cyclic condition finally arrived at. It will be noticed that the magnetic quality has not suffered diminution, but, on the contrary, has been slightly improved. The fact that cyclic treatment such as that described results in a slight augmentation of the value of I is, of course, well known. The loops were then repeated for a larger variation in H, viz. from H = 9·5 to H = −1. This treatment produced a diminution in the value of I, the final condition being that shown in the part C C of the curve. So long as the field does not fall below zero, the "sensitive state" persists, no matter how the magnetising field is varied. The feeble *reversed*, that is *negative*, field here employed has permanently diminished the susceptibility. Larger variations further destroyed the quality, as will be seen in the loops D D, E E, which represent the values obtained after the respective cyclic states had been established.

Several tests of this nature were carried out, and all agreed in showing that the deterioration in quality does not begin until the magnetising field has been reversed.

The gradual falling off in the magnetic quality gives initially a peculiar asymmetry to the curves (see fig. 2). The process of repeated cycles eventually results in symmetrical curves being obtained. This is well shown in the following table:—

H.	I.
+11	+334
−11	−270
+11	+277
−11	−261
+11	+262
−11	−255
+11	+255
−11	−252
+11	+249
−11	−249
+11	+249

Fig. 4 exhibits the results obtained with a specimen (annealed at 900° C.) which was first tested for field strengths up to 2 C.G.S. units, then demagnetised and tested for fields up to 5 C.G.S. units. The curves show that whilst the many variations of field between ± 2 C.G.S. units have resulted in a definite reduction in susceptibility, a variation in field strength between wider limits will still show an effect in reducing the susceptibility. The cycles carried out between H = ± 2 C.G.S. units have, however, lessened to a certain extent the susceptibility for higher fields. The dotted curve shows the magnetisation curve which would have been obtained had the specimen been freshly annealed and tested up to H = 5 C.G.S. units.

As is, of course, obvious, cyclic variation between wide limits of field strength destroys all trace of the "sensitive state" for fields lying between narrower limits.

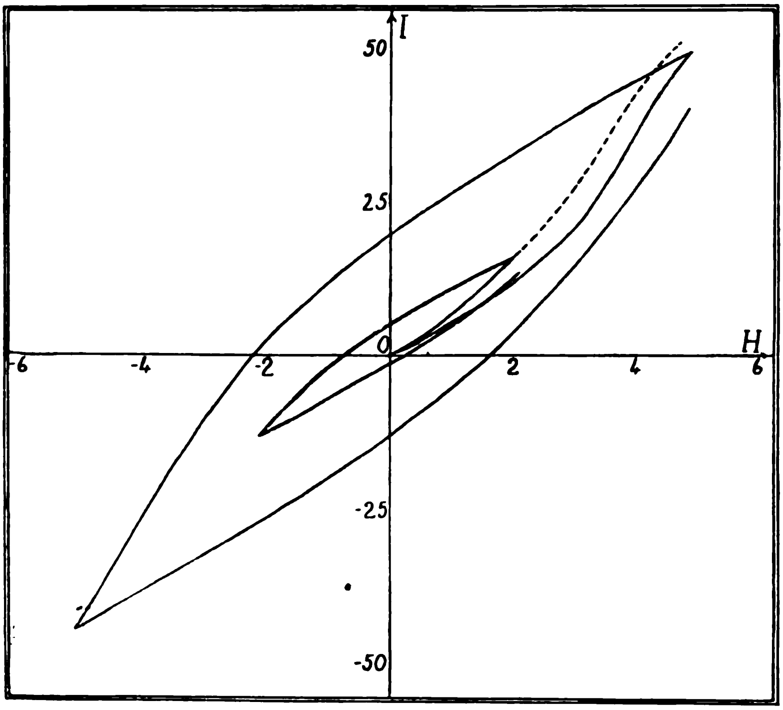


FIG. 4.

Scope of the Investigation.—So far, the tests were confined to a particular variety of steel, annealed at 900° C. It was now decided to investigate this so-called "sensitive state" in various kinds of steel after subjection to varying thermal treatment.

Rate of Cooling.—The experiments showed that in all varieties of steel the "sensitive state" was much less apparent in specimens quenched from high temperatures than in specimens slowly cooled. Further, it was shown that the magnitude of the effect did not depend to any large extent on the rate of cooling of the specimen, provided that the cooling was not very rapid.

Effect of Temperature.—Very moderate temperatures are sufficient to bring on the phenomenon in a slight degree. Several specimens heated in a steam coil to 100° C. showed decided indications of the "sensitive state." As the temperatures of heating are increased up to about 700° C. the effect becomes more and more pronounced; but little, if any, advantage is gained by carrying the temperature above 700° C.

Effect of Time.—A specimen was freshly annealed and allowed to rest for several days. It was then tested and gave a diagram which was identical with that which would have been obtained had the test followed immediately on the annealing. A second specimen was annealed, tested, demagnetised, and put aside for several days. On retesting, no trace of the "sensitive state" could be detected.

Effect of Vibration.—A number of similar specimens, the magnetic quality of which had been determined, were annealed at 900° C. These specimens were subjected to varying amounts of mechanical vibration. The first specimen was held vertically with its lower end at a distance of 1 metre from a stone slab and allowed to fall. This resulted in a considerable jarring of the specimen. The remaining specimens were given several falls, viz. 3, 5, 7, 10, 25, and 50 respectively.

The rods were afterwards tested, and the "sensitive state" was found to be less marked. In those specimens which received more than five falls the effect was reduced to less than one-half its previous value. A reduction of 40 per cent. was produced by a single fall. It appears to be practically impossible to completely remove the "sensitive state" by this treatment.

Varieties of Steel employed in the Experiments.—The following five varieties of steel were experimented upon: (1) very mild steel, (2) spindle steel, (3) thick steel wire, (4) magnet steel, and (5) a special hard steel. [The authors hope to furnish detailed chemical analyses of these steels in the second part of this paper.]

Mild Steel.—In this variety of steel the "sensitive state," though present, is but slight, there being an improvement in the susceptibility of about 3 per cent. for a field strength of 6 C.G.S. units. The effect is far from being persistent; one cycle is sufficient to render its existence almost indiscernible.

Spindle Steel.—This steel exhibits the phenomenon in a marked degree. After heating above the critical point, the intensity of magnetisation for a field of 10 C.G.S. units has acquired an increase of about 40 per cent. on its final value. The coercive force has also been considerably augmented by the treatment. One cycle removes nearly 70 per cent. of the total effect.

Steel Wire, Magnet Steel, and Hard Steel.—The results obtained for these varieties of steel are very similar to those obtained for spindle steel, and do not, therefore, call for special comment.

Cast Iron.—The authors now turned their attention to other magnetic substances. Cast iron was first tried. This material shows an effect smaller than that obtained with steel, though it is still considerable. In order to

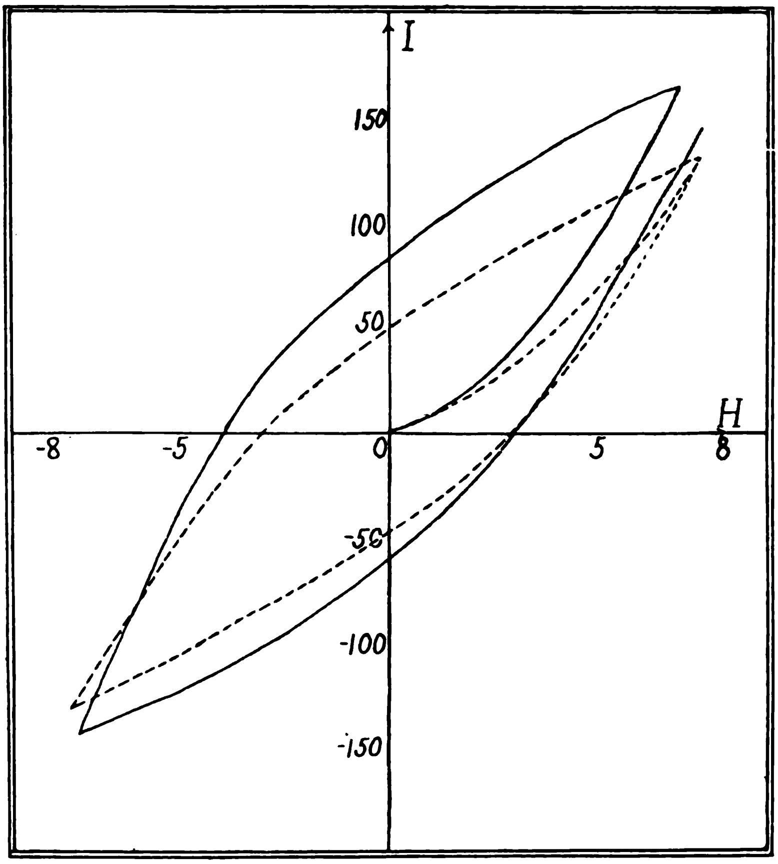


Fig. 5. Cast Iron Specimen.

——— after annealing at 900° C.

------ after demagnetising by cyclic reversals.

induce the state it is necessary to heat to a high temperature. Thus, for H = 10 C.G.S. units, the improvement in susceptibility is 24 per cent. for a temperature of 900° C. For a temperature of 500° C. the percentage is only 8, whilst heating to 200° C. produces no measurable effect. The first cycle removes 50 per cent. of the total improvement. The above diagram shows the initial and final curves obtained from a specimen cooled from 900° C.

Soft Iron.—An iron specimen cooled from 900° showed no indication of the "sensitive state."

Cobalt.—In a specimen of cobalt, cooled from 900° C., for $H=8$ an increase in susceptibility amounting to about 15 per cent. of the final value was observed.

In figs. 1 to 5 the values of I and H are given in C.G.S. units. In calculating the effective from the applied magnetising field, the factors investigated by Du Bois for cylindrical rods have in all cases been employed.

Tests are being made on specimens of nickel, and further investigation of the effects detailed above is in active progress. The authors hope to publish the results of these tests in a second part of this paper, to be brought before the Society at an early date.

The experiments described in the present paper were carried out in the Physical Laboratory of the University of Glasgow, and the authors desire to express their thanks to Professor Gray for the very helpful interest he has taken in the research.

Summary.

1. Specimens of steel cooled from high temperatures are, as regards their magnetic properties, in a "sensitive state."

2. After magnetisation, reversal of the field reduces the susceptibility.

3. Cyclic reversals of the field between definite limits bring the magnetic quality to a definite lower value for fields lying between these limits.

4. So long as the field is not changed in sign, the "sensitive state" persists.

5. Cyclic reversals of the field between narrow limits reduce the susceptibility for fields beyond these limits, but not to its lowest possible value.

6. The rate of cooling from the high temperature—if not very rapid—has little influence on the magnitude of the effect.

7. The "sensitive state" does not wear off with time.

8. Mechanical vibration of the specimens considerably reduces the phenomenon.

9. The "sensitive state" is most pronounced in hard steels.

10. The effect is induced to a slight degree in some varieties of steel by temperatures as low as 100° C. It increases with increasing temperatures to about 700° C. Further increase in temperature has little influence.

11. Similar effects are noticeable in cast iron and in cobalt. The "sensitive state" is absent, or is present to only a very slight extent, in soft iron.

12. The effect has a maximum value for fields which give large values of μ, and tends to zero as the saturation point is approached.

(*Issued separately April* 10, 1908.)

XV.—On a Test for Continuity. By W. H. Young, Sc.D.; F.R.S. *Communicated by* J. H. Maclagan Wedderburn.

(MS. received February 20, 1908. Read March 16, 1908.)

§ 1. The usual method of proving that a function defined as the limit of a sequence of continuous* functions is continuous is by proving that the convergence is uniform. This method may fail owing to the presence of points at which the convergence is non-uniform although the limiting function is continuous.† In such a case it would be necessary to apply a further test, *e.g.* that of Arzelà ("uniform convergence by segments").

In some cases the continuity may be proved directly by means of a totally different principle, without reference to modes of convergence at all. It is, in fact, a necessary and sufficient condition for the continuity of a function that it should be possible to express it at the same time as the limit of a monotone *ascending* and of a monotone *descending* sequence‡ of continuous functions.

This test may or may not, of course, be an easy one to apply in any particular problem, but in certain cases, both in theory and practice, the application is almost immediate.

The object of the present note is to call attention to the test, and show how it may be applied in a few simple cases.

§ 2. For instance, take the continued fraction

$$\frac{u_1}{v_1 +} \frac{u_2}{v_2 +} \cdots$$

where the u's and the v's are essentially positive continuous functions.

* Throughout this paper the word "continuous" will be used to mean "bounded and continuous" unless the contrary is stated.

† A point of non-uniform convergence where the limiting function is continuous is what I call an "invisible point of non-uniform convergence."

‡ $f_1, f_2, \ldots$ are said to form a monotone *ascending* sequence if

$$f_1 \leq f_2 \leq f_3 \leq \ldots ,$$

$f_1, f_2, \ldots$ being functions of any number of variables. The theorem on which the test depends is that "the limit of an *ascending* sequence of continuous functions is a *lower* semi-continuous function, and that the limit of a descending sequence of continuous functions is an upper semi-continuous function. A function which is the limit of both an ascending and a descending sequence is therefore both lower and upper semi-continuous, *i.e.* it is continuous." It may be added that there are no invisible points of non-uniform convergence, or divergence, in the case of monotone sequences of continuous functions.—W. H. Young, "On Monotone Sequences of Continuous Functions," *Camb. Phil. Soc. Proc.*, Lent Term, 1908.

The odd convergents form a monotone descending sequence having the continued fraction as limit, so that the latter is an upper semi-continuous function. The even convergents form a monotone ascending sequence having the continued fraction as limit, so that it is lower semi-continuous, and therefore, being also upper semi-continuous, is a continuous function, and this whatever the number of variables on which the u's and the v's depend.

§ 3. Again, the u's having the same meaning as before, the infinite series

$$u_1 - u_2 + u_3 - u_4 + \ldots ,$$

if the sum at each point is definite, necessarily represents a continuous function when the u's form a monotone descending sequence. In fact, it is the limit of the ascending sequence of functions constituted by the partial sums of the series

$$(u_1 - u_2) + (u_3 - u_4) + \ldots \qquad (1)$$

and also of the descending sequence constituted by the partial sums of the series

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots \qquad (2)$$

It should be noted that in the case of such a series "term-by-term integration" is allowable. In fact, each of the series (1) and (2) is so integrable, since it is monotone and represents a continuous function. This is therefore the case with the original series.

§ 4. If we omit the condition that the series

$$u_1 - u_2 + \ldots$$

should have a definite sum at each point, the monotone ascending sequence of continuous functions $f_1, f_2, \ldots$, where

$$f_n = (u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2n-1} - u_{2n}),$$

defines a limiting function f which is $\geq u_1 - u_2$ and is *lower* semi-continuous.

On the other hand, the monotone *descending* sequence of continuous functions $g_1, g_2 \ldots$, where

$$g_n = u_1 - (u_2 - u_3) - \ldots - (u_{2n-2} - u_{2n-1}),$$

defines a limiting function g which is $\leq u_1$ and is *upper* semi-continuous. Moreover, since

$$f_n \leq g_n,$$

$$u_1 - u_2 \leq f \leq g \leq u_1 \qquad (1)$$

Thus at each point P there are two possible values for the sum of the series, according as we proceed to the limit by taking an even or an odd number of terms in the partial summations; both these values are finite, and the first is not greater than the second.

If by any law we select one of these two values, or any intermediate value, at every point we get a function, say h, which may be said to be defined by the series, and we shall have at every point

$$f \leq h \leq g \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It now follows that, if at P the series have a definite sum, P is a point of continuity for each of the functions h defined by the series. For by (2) the least limit of $h(x)$ as x approaches P as limit $\geq$ the least limit of $f(x)$, $\geq f(\mathrm{P})$, since f is lower semi-continuous, $\geq h(\mathrm{P})$. Comparing, on the other hand, with g, the greatest limit of $h(x) \leq h(\mathrm{P})$, so that $h(x)$ has the limit $h(\mathrm{P})$, i.e. h is continuous at P.

§ 5. The argument used in the preceding article is of a perfectly general nature: it consists in the use of a special case of the following theorem, of which the proof * is now obvious:—

Theorem.—*The necessary and sufficient condition that a function* h(x) *should be continuous at* P *is that it should be possible to construct two monotone sequences of functions continuous at* P, *one ascending and one descending, having at* P *the same limit* h(P), *and such that, in the neighbourhood of* P, *the limits of the ascending sequence are not greater, and those of the descending sequence not less, than the corresponding values of* h(x).

This theorem constitutes a test for continuity at a point.

§ 6. A particular case of § 4 is when we have a series

$$a_0 - a_1x + a_2x^2 - \ldots \quad . \quad . \quad . \quad . \quad (1)$$

where all the a's are positive and

$$a_0 \geq a_1x \geq a_2x^2 \geq \ldots \quad . \quad . \quad . \quad . \quad (2)$$

for all positive values of x less than r, it being assumed that the series converges for all such values of x.

We can then assert that *the series represents a continuous function* f(x) *from* 0 *to* r, *the latter not included only when the series*

$$\mathrm{a}_0 - \mathrm{a}_1\mathrm{r} + \mathrm{a}_2\mathrm{r}^2 - \ldots \quad . \quad . \quad . \quad . \quad . \quad (3)$$

oscillates. In the latter case, moreover, any limit which f(x) *can approach as* x *approaches* r *lies between the limits of indeterminancy of* (3), *both included.*

* It involves the use of the theorem that "the limit of a descending (ascending) sequence of functions continuous at P is a function which is upper (lower) semi-continuous at P." This, like the previous theorem quoted on p. 249, of which it is a generalisation, will be found proved in the paper in the *Camb. Phil. Soc. Proc.* above referred to.

The argument of § 3 shows, in fact, that if the series (3) converges, the series (1) represents a continuous bounded function in the closed interval $(0, r)$.

If the series (3) diverge to the value $+\infty$, the series (1) still represents a continuous function $f(x)$ from 0 to r, the latter not included, by the preceding article. At each point $x < r$ we have

$$f(x) = (a_0 - a_1 x) + (a_2 x^2 - a_3 x^3) + \dots \qquad (4)$$

while the right-hand side of this equation, being the limit of a monotone increasing sequence of continuous functions, represents a lower semi-continuous function in the closed interval $(0, r)$. The value of this lower semi-continuous function at r being, however, $+\infty$, the function is continuous there, so that the series represents a continuous but unbounded function $f(x)$ throughout the closed interval $(0, r)$.

Similarly, if the series (3) diverge to the value $-\infty$, using the grouping (2) of the preceding article, and the fact that an upper semi-continuous function is continuous where it has the value $-\infty$, we prove the continuity of $f(x)$ in the closed interval $(0, r)$.

Finally, let the series (3) oscillate between the limits of indeterminancy L and U. Then, as in the case when the series (3) had the value $+\infty$, we get the equation (4), whose right-hand side represents a *lower* semi-continuous function in the closed interval $(0, r)$. The value of this function at r, being one of the values the series (3) is capable of assuming, is $\geq$ L. But this value is $\leq$ the limits of the values of the lower semi-continuous function in the neighbourhood, that is, is $\leq$ the limits of $f(x)$ as x approaches r. This shows that all such limits are $\geq$L. Similarly, using the grouping employed when the series (3) had the value $-\infty$, we show that all such limits are $\leq$U. Thus all such limits lie between L and U, both included.

§ 7. We now proceed to give two examples, the first constituting another example of the application of § 3; and the second, of the original test of § 1.

Example 1.—Consider the integral

$$U(y) = \int_0^\infty \frac{\sin x}{x+y} dx,$$

where y is any positive quantity. This may be written

$$\int_0^\pi + \int_\pi^{2\pi} + \dots = u_1(y) - u_2(y) + \dots$$

Here the u's are essentially positive and continuous functions of y and form a monotone descending sequence. Thus, by § 3, $U(y)$ is a continuous function of y, and its integral may be got by "term-by-term integration," and therefore by integration under the integral sign. Thus, for example,

$$\int_0^\infty \sin x \,.\, \log \frac{x+1}{x} dx = \int_0^1 U(y) dy.$$

Example 2.—*To prove that the infinite product*

$$\cos x \,.\, \cos \tfrac{1}{2}x \;\ldots\; \cos \frac{1}{n}x \,.\, \cos \frac{1}{n+1}x \;\ldots$$

is convergent and defines a bounded continuous function of x *for all finite values of* x.

Let

$$F_n(x) = \cos x \,.\, \cos \tfrac{1}{2}x \;\ldots\; \cos \frac{1}{n}x,$$

then $F_n(x)$, being the product of n continuous functions, is a continuous function of x for all finite values of x. Also, since the cosine of a quantity lying in the closed interval $\left(0, \frac{\pi}{2}\right) \geq 0$ and ≤ 1, $F_1, F_2 \ldots$ form a monotone *decreasing* sequence, whose limit is therefore an *upper* semi-continuous function of x in the closed interval $\left(0, \frac{\pi}{2}\right)$, lying between 0 and 1, both inclusive: this limit is, however, the infinite product in question.

Again, by the product form of $\sin x$, the infinite product

$$f(x) = \left(1 - \tfrac{1}{2}\frac{x^2}{4}\right)\left(1 - \tfrac{1}{2}\frac{x^2}{9}\right)\left(1 - \tfrac{1}{2}\frac{x^2}{16}\right) \ldots = \frac{\sqrt{2}}{\pi x} \sin \frac{\pi x}{\sqrt{2}}$$

is a continuous function of x throughout the closed interval $(0, 1)$, therefore the same is true of each of the functions

$$f_n(x) = \cos x \,.\, \cos \tfrac{1}{2}x \;\ldots\; \cos \frac{1}{n-1}x \,.\, \left(1 - \tfrac{1}{2}\frac{x^2}{n^2}\right)\left(1 - \tfrac{1}{2}\frac{x^2}{(n+1)^2}\right) \ldots$$

which form a monotone *increasing* sequence, since

$$\cos y \geq 1 - \tfrac{1}{2}y^2.$$

The limit of this latter sequence is therefore a *lower* semi-continuous function of x in the closed interval $\left(0, \frac{\pi}{2}\right)$; but this limit is the same infinite product as before, and is thus a continuous function of x in the closed interval $\left(0, \frac{\pi}{2}\right)$, lying between 0 and 1, *a fortiori* between -1 and $+1$. In the interval $\left(\frac{\pi}{2}, \pi\right)$ the same argument applies when we omit the first factor cos x, and it applies in each subsequent interval of the same length, omitting in succession 2, 3, ... factors of the infinite product. This shows that the product is a continuous function of x for all positive finite values and lies between -1 and $+1$. But the expression is unaltered by changing x into $-x$, so that the same is true for all negative finite values, which proves the required result.

§ 8. The use of the test in theoretical investigations is exemplified by the following proof of the continuity of an improper integral defined in the mode of De la Vallée-Poussin.

For brevity, it will be assumed that the function $f(x)$ to be integrated has a finite lower bound. The argument would require a repeated application if the lower as well as the upper bound were infinite.

Let $M_1, M_2, \ldots$ be any set of constantly increasing positive constants having infinity as limit, and let the function $f_n(x)$ differ only from $f(x)$ at the points where $f(x) > M_n$, and at these points $f_n(x) = M_n$.

Assuming the functions $f_n(x)$ to be integrable, and denoting $\int_a^x f_n(x)dx$ by $F_n(x)$, De la Vallée-Poussin defines the improper integral $F(x)$ of $f(x)$ to be the limit of $F_n(x)$, provided this latter limit exist and is finite and independent of the particular sequence $M_1, M_2, \ldots$.

Now we know that the proper integral $F_n(x)$ is a continuous function of x, and hence $F_1(x)$, $F_2(x)$, ... form a monotone increasing sequence of continuous functions, having $F(x)$ as limit, since the functions $f_1(x)$, $f_2(x)$, ... form a monotone *increasing* sequence having $f(x)$ as limit.

Denote by $F(b)$ the integral from a to b of $f(x)$, and let

$$G_n(x) = F(b) - \int_x^b f_n(x)dx.$$

Then it is plain that the functions $G_1(x)$, $G_2(x)$, ... form a monotone *decreasing* sequence, having $F(x)$ for limit. Hence $F(x)$ is continuous.

It will be noticed that the argument depends on $F(b)$ being finite. In fact, if we admit integrals with infinite values, the theorem of continuity no longer holds. We still have the first part of the argument holding good, viz. that $F_1(x)$, $F_2(x)$, ... form a monotone increasing sequence, whence it follows that $F(x)$ is, at a point of discontinuity, lower semi-continuous.

Suppose, for definiteness, that $f(x)$ is a positive function, so that $F(x)$ is a monotone increasing function of x. Then if $F(x)$ is first of all finite, and then becomes infinite, the only way in which this can happen is that up to *and including* some value $x=c$, $F(x)$ is finite, and then suddenly becomes infinite. Moreover, this is the only kind of discontinuity possible, since $F(x)$ cannot jump up to a finite value.

Example.—

Let $f(x) = (1-x)^{-\frac{1}{2}}$ in the closed interval $(0, 1)$,
and $ = (x-1)^{-2}$ when $x > 1$ but ≤ 2.

This function, being continuous but not bounded, and always positive, can be generated as the limit of such a sequence of functions as that used by De la Vallée-Poussin. We have, however,

$F(x) = 2 - 2(1-x)^{\frac{1}{2}}$ in the closed interval $(0, 1)$,
and $ = +\infty$, when $x > 1$ but ≤ 2.

It should be noticed, further, that the argument used in this article depends in no way on the form of the functions devised by De la Vallée-Poussin, but only on the fact that they form a monotone sequence.

§ 9. We now deduce two subsidiary tests which are sometimes useful. The first applies to series and the second to integrals. Both follow immediately from the principle enunciated in § 1. They are analogous to certain known tests for uniform convergence, but more general in form.

Theorem.—*If*

$$u_1 + u_2 + \dots$$

is a series of continuous functions of any number of independent variables which is convergent at all points of a certain region, then it represents a continuous function provided we can find a second series

$$U_1 + U_2 + \dots$$

where the U's *are positive continuous functions, whose sum is continuous, and are such that*

$$|u_i| \leq U_i$$

for all values of i *and each point of the region considered.*

For the function in question is the limit of the monotone ascending sequence $f_1, f_2, \dots$, where

$$f_n = u_1 + u_2 + \dots + u_{n-1} - U_n - U_{n+1} - \dots$$

and is evidently continuous.

It is also the limit of the monotone descending sequence $g_1, g_2, \dots$ where

$$g_n = u_1 + u_2 + \dots + u_{n-1} + U_n + U_{n+1} + \dots$$

and is evidently continuous.

Hence the result follows.

Theorem.—*If* $u(x, y)$ *is a continuous function of the ensemble* (x, y), *and* $\int_a^\infty u(x, y)dx$ *is convergent for every value of* y *considered, and if we* *can find a positive continuous function* U(x, y) *such that* $\int_a^\infty U(x, y)dx$ *is a continuous function of* y, *and also*

$$|u(x, y)| \leq U(x, y)$$

for all values of $x \geq a$ *and each value of* y, *then* $\int_a^\infty u(x, y)dx$ *is a continuous function of* y.

We have stated and shall prove this theorem for a single variable y; the argument is, however, quite general, and applies equally when there are n variables y. We may add, it also applies when there are m variables x, the integral being then the m-ple integral, and proper limitations are made with respect to the nature of the infinite region of integration.

The integral in question is, in fact, the limit of the monotone ascending sequence of continuous functions $f_1, f_2, \ldots.$ where

$$f_n(y) = \int_a^{x_n} u(x, y)dx - \int_{x_n}^\infty U(x, y)dx,$$

$a, x_1, x_2, \ldots.$ being a previously selected monotone ascending sequence of quantities having $+\infty$ as limit.

The integral is also the limit of the monotone descending sequence of continuous functions $g_1, g_2, \ldots.$ where

$$g_n(y) = \int_a^{x_n} u(x, y)dx + \int_{x_n}^\infty U(x, y)dx.$$

Hence the result follows.

§ 10. The theorem of § 5 gives us the following obvious modification of the tests of the preceding article for continuity at a point.

Theorem.—*If*

$$u_1 + u_2 + \ldots.$$

is a series of functions of any number of variables which are continuous at a point P *where the series is convergent, then it represents a (multi-valued) function which is continuous at* P, *provided we can find a second series*

$$U_1 + U_2 + \ldots.$$

where the U*'s are positive functions, which, as well as their sum, are continuous at* P, *and are such that, at all points of a certain neighbourhood of* P,

$$|u_i| \leq U_i$$

for all values of i.

Theorem.—*If* u(x, y) *is a continuous function of the ensemble* (x, y) *for the value* $y = y_0$, *and* $\int_a^\infty u(x, y)dx$ *is convergent for* $y = y_0$, *then, if we can find a positive function* U(x, y), *continuous for* $y = y_0$, *and such that* $\int_a^\infty U(x, y)$ *is continuous at* $y = y_0$, *while*

$$|u(x, y)| \leq U(x, y)$$

for all values of $\mathrm{x} \leq \mathrm{a}$ *and each value of* y *in a certain neighbourhood of* y, $\int_{\mathrm{a}}^{\infty} \mathrm{u}(\mathrm{x}, \mathrm{y}) d\mathrm{x}$ *is continuous at the point* $\mathrm{y}=\mathrm{y}_0$.

As an example of the application of the last test, we see that, $f(x, y)$ being a positive function continuous with respect to the ensemble (x, y), if $\int_0^{\infty} f(x, y) dx$ and $\int_0^{\infty} f(x, y) \sin x \, . \, dx$ are both convergent at $y=y_0$, and the former is a continuous function of y at y_0, so is the latter.

Addendum (received 9th March 1908).

§ 11. The theorem quoted in § 5 (footnote), on which the test given in that article depends, is a particular case of the following more general theorem:—*

"A monotone decreasing (increasing) sequence of functions $f_1, f_2, \ldots$. which are upper (lower) semi-continuous at P has for limit a function f which is also upper (lower) semi-continuous at P."

This latter theorem also may, of course, be used as a test for continuity at a definite point. The necessary and sufficient test for continuity at a point P will then be the possibility of expressing f both as the limit of a monotone decreasing sequence of functions which are upper semi-continuous (or in particular continuous) at P, and of a monotone increasing sequence of functions which are lower semi-continuous (or in particular continuous) at P.

§ 12. It should be noticed that the methods given above may be still further generalised. We have seen that to prove a function continuous throughout an interval, it is sufficient to show that it is the limit both of a monotone increasing and of a monotone decreasing sequence of continuous functions. The theorem quoted in the preceding article shows that *to prove that a function is pointwise discontinuous, it is sufficient to show that it is the limit both of a monotone increasing and of a monotone decreasing sequence of pointwise discontinuous functions.*

The theorem quoted, however, may be taken to apply not only to a fundamental interval, but to any fundamental perfect set; hence, if it is desired to show that a function is pointwise discontinuous with respect not only to an interval, but with respect to every perfect set in that interval, it is sufficient to show that it is the limit both of a monotone increasing and of a monotone decreasing sequence of functions with the same property.

It is a simplification in this case, however, that, in applying the test, we may use semi-continuous functions. To prove that a function is upper (lower) semi-continuous, we only need to show that it is the limit of a single

* *Loc. cit.*; for proof see *Mess. of Math.*, 1908.

monotone sequence of semi-continuous functions, viz. of a monotone decreasing (increasing) sequence of upper (lower) semi-continuous functions. These are particular cases of functions which are pointwise discontinuous with respect to every perfect set, and have, moreover, the advantage that they may be generated as the limits of monotone sequences of *continuous* functions. *To prove that a function is pointwise discontinuous with respect to every perfect set, it follows that it is sufficient to show that it is the limit of* (1) *a monotone decreasing sequence of lower semi-continuous functions, and* (2) *of a monotone increasing sequence of upper semi-continuous functions.*

It is interesting to remark that what I call *the upper function* of a sequence of continuous functions (that is, the function which at each point has the value of the highest possible limit approached by the values of the continuous functions there) satisfies the condition (1), while *the lower function* satisfies (2). This may be proved in the manner indicated in Hobson's *Functions of a Real Variable*, p. 552, line 22 *seq.*, where it is shown that the upper derivate, which is a particular case of an upper function, is the limit of a monotone decreasing sequence of functions $w_1, w_2, \ldots$ each of which is the limit of a monotone increasing sequence of continuous functions $v_1, v_2, \ldots$

It follows that *an upper (lower) function of a sequence of continuous* * *functions, and in particular an upper lower derivate, is upper (lower) semi-continuous except at the points of a set of the first category* (viz. the points at which one at least of the semi-continuous functions of the monotone sequence is discontinuous). This is true, moreover, not only with respect to the continuum, but with respect to every perfect set.

This, again, gives us a new proof of Baire's theorem, that (in the case when the upper and lower functions coincide) the limit of a sequence of continuous functions is pointwise discontinuous with respect to every perfect set. The converse, proved also by Baire, viz. that any function which is pointwise discontinuous with respect to every perfect set is the limit of a continuous function, shows that the conditions above given are not only sufficient but necessary for the function in question to have the required property.

* It may be added that this result can be still further generalised. The general result is that the upper (lower) function of a sequence of lower (upper) semi-continuous functions is upper (lower) semi-continuous except at the points of a set of the first category. Further, this is true with respect to the continuum or any perfect set.

(*Issued separately April* 10, 1908.)

XVI.—The Problem of a Spherical Gaseous Nebula.
By Lord Kelvin.

(MS. received March 9, 1908.)

This paper was begun about the close of 1906, in order to fulfil a promise given at the end of the paper "On the convective equilibrium of a gas under its own gravitation only," published in the *Philosophical Magazine*, 1887; and part of it was communicated by Lord Kelvin to the Royal Society of Edinburgh at its meeting on 21st January 1907. Since then, however, important additions have been made to it, and the subject has been dealt with more fully than was originally intended. Unfortunately the manuscript was left incomplete at Lord Kelvin's death. It ended with § 35.

However, from information which I received from Lord Kelvin while carrying out the earlier work connected with the paper, I have been able to write the sections from § 36 to the end. These complete all that Lord Kelvin desired to include in this communication; and they express, I believe, the views he held while writing the earlier sections.

The statement of mathematical solutions and numerical results separately, as an Appendix to the paper, under my own name, is in accordance with Lord Kelvin's wishes.

George Green,
Secretary.

§ 1.* If a fluid globe were given with any arbitrary distribution of temperature, subject only to the condition that it is uniform throughout every spherical surface concentric with the boundary, the cooling, and consequent augmentation of density of the fluid at its boundary, by radiation into space, would immediately give rise to an instability according to which some parts of the outermost portions of the globe would sink, and upward currents would consequently be developed in other portions. In any real fluid, whether gaseous or liquid, this kind of automatic stirring would tend to go on until a condition of approximate equilibrium is reached, in which any portion of the fluid descending or ascending would, by the thermodynamic action involved in change of pressure, always take the

* § 1 is extracted from "On Homer Lane's Problem of a Spherical Gaseous Nebula," *Nature*, Feb. 14, 1907.

temperature corresponding to its level, that is to say, its distance from the centre of the globe. The condition thus reached, when heat is continually being radiated away from the spherical boundary, is not perfect equilibrium. It is only an approximation to equilibrium, in which the temperature and density are each approximately uniform at any one distance from the centre, and vary slowly with time, the variable irregular convective currents being insufficient to cause any considerable deviation of the surfaces of equal density and temperature from sphericity.

§ 2. The problem of the convective equilibrium of temperature, pressure and density, in a wholly gaseous, spherical fluid mass, kept together by mutual gravitation of its parts, was first dealt with by the late Mr Homer Lane, who, as we are told by Mr T. J. J. See, was for many years connected with the U.S. Coast and Geodetic Survey at Washington. His work was published in the *American Journal of Science*, July 1870, under the title "On the theoretical Temperature of the Sun." *

In a letter to Joule, which was read before the Literary and Philosophical Society of Manchester, January 21, 1862, and published in the *Memoirs* of the Society under the title, "On the Convective Equilibrium of Temperature in the Atmosphere," † it was shown that natural up and down stirring of the earth's atmosphere, due to upward currents of somewhat warmer air, and return downward flow of somewhat cooler air, in different localities, causes the average temperature of the air to diminish from the earth's surface upwards to a definite limiting height, beyond which there is no air. It was also shown that, were it not for radiation of heat across the air, outwards from the earth's surface, and inwards from the sun, the temperature of the highly rarefied air close to the bounding surface would be

* The real subject of this paper is that stated in the text above. The application of the theory of gaseous convective equilibrium to sun heat and light is very largely vitiated by the greatness of the sun's mean density (1·4 times the standard density of water). Common air, oxygen, and carbonic acid gas show resistance to compression considerably in excess of the amount calculated according to Boyle's Law, when compressed to densities exceeding four, or five, or six, tenths of the standard density of water. There seems strong reason to believe that every fluid whose density exceeds a quarter of the standard density of water resists compression much more than according to Boyle's Law, whatever be the temperature of the fluid, however high, or however low. We may consider it indeed as quite certain that a large proportion of the sun's interior, if not indeed the whole of the sun's mass within the visible boundary, resists compression much more than according to Boyle's Law. It seems indeed most probable that the boundary, which we see when looking at the sun through an ordinary telescope, is in reality a surface of separation between a liquid and its vapour; and that all the fluid within this boundary resists compression so much more than according to Boyle's Law that it does not even approximately satisfy the conditions of Homer Lane's problem; and that in reality its density increases inwards to the centre vastly less than according to Homer Lane's solution (see § 55 below).

† Republished in Sir William Thomson's *Math. and Phys. Papers*, vol. iii. p. 255.

just over absolute zero; that is to say, temperature and density would come to zero at the same height as we ideally rise through the air to the boundary of the atmosphere. Homer Lane's problem gives us a corresponding law of zero density and zero temperature, at an absolutely defined spherical bounding surface (see § 27 below). In fact it is clear that if in Lane's problem we first deal only with a region adjoining the spherical boundary, and having all its dimensions very small in comparison with the radius, we have the same problem of convective equilibrium as that which was dealt with in my letter to Joule.

§ 3. According to the definition of "convective equilibrium" given in that letter, any fluid under the influence of gravity is said to be in convective equilibrium, if density and temperature are so distributed throughout the whole fluid mass that the surfaces of equal temperature and of equal pressure remain unchanged when currents are produced in it by any disturbing influence so gentle that changes of pressure due to inertia of the motions are negligible. The essence of convective equilibrium is that if a small spherical or cubic portion of the fluid in any position P is ideally enclosed in a sheath impermeable to heat, and expanded or contracted to the density of the fluid at any other place P′, its temperature will be altered by the expansion or contraction, from the temperature which it had at P, to the actual temperature of the fluid at P′. The formulas to express this condition were first given by Poisson. They are now generally known as the equations of adiabatic expansion or contraction, so named by Rankine. They may be written as follows, for the ideal case of a perfect gas:—

$$\frac{p}{p'}=\left(\frac{\rho}{\rho'}\right)^k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

$$\frac{\rho}{\rho'}=\left(\frac{t}{t'}\right)^{\frac{1}{k-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2);$$

$$\frac{p}{p'}=\left(\frac{t}{t'}\right)^{\frac{k}{k-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3);$$

where (t, ρ, p), (t', ρ', p') denote the temperatures, densities, and pressures, at any two places in the fluid (temperatures being reckoned from absolute zero); and k denotes the ratio of the thermal capacity of the gas when kept at constant pressure to its thermal capacity at constant volume, which, according to a common usage, is for brevity called "the ratio of specific heats." For dry air, at any temperature, and at any density, within the range of its approximate fulfilment of the gaseous laws, we have

$$k=1{\cdot}41\,; \quad \frac{k-1}{k}=\cdot 291\,; \quad \frac{k}{k-1}=3{\cdot}44 \quad . \quad . \quad . \quad (4).$$

For monatomic gases we have

$$k=\frac{5}{3}\,;\quad \frac{k-1}{k}=\frac{2}{5}\,;\quad \frac{k}{k-1}=\frac{5}{2} \quad . \quad . \quad . \quad . \quad . \; (5).$$

For real gases, we learn from the Kinetic Theory of Gases, and by observation, that k may have any value between 1 and $1\frac{2}{3}$, but that it cannot have any value greater than $1\frac{2}{3}$, or less than 1.

§ 4. To specify fully the quality of any gas, so far as concerns our present purpose, we need, besides k, the ratio of its specific heats, just one other numerical datum, the volume of a unit mass of it at unit temperature and unit pressure. This, which we shall denote by S, is commonly called the specific volume; and its reciprocal, 1/S, we shall call the specific density (D) of the gas. In terms of this notation, the Boyle and Charles gaseous laws are expressed by either of the equations

$$pv=\mathrm{S}t \;\; . \; . \; . \; (6), \quad \text{or} \quad p=\rho\mathrm{S}t \quad . \quad . \quad . \quad . \quad . \; (6');$$

where p, v, ρ, denote respectively the pressure, the volume of unit mass, and the density of the gas at temperature t, reckoned from absolute zero. Our unit of temperature throughout the present paper will be 273° C. Thus the Centigrade temperature corresponding to t in our notation is $273(t-1)$.

§ 5. In virtue of § 4, what is expressed by (1), (2), (3), equivalent as they are to two equations, may now, for working purposes, be expressed much more conveniently by the single formula (6), together with the following equation—

$$p=\mathrm{A}\rho^{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \; (7);$$

where A denotes what we may call the Adiabatic Constant, which is what the pressure would be, in adiabatic convective equilibrium, at unit density, if the fluid could be gaseous at so great a density as that.

§ 6. Looking to (6), remark that p being pressure per unit of area, the dimensions of pv are $\mathrm{L}^{-2}\times\mathrm{L}^{3}$ or L, if we express force in terms of an arbitrary unit, as in § 10 below; therefore S, though we call it specific volume, is a length. It is in fact, as we see by (9) below, equal to the height of the homogeneous atmosphere at unit temperature, in a place for which the heaviness of a unit mass is the force which we call unity in the reckoning of p.

§ 7. In the definition of what is commonly called the "height of the homogeneous atmosphere," and denoted by H, an idea very convenient for our present purpose is introduced. Let p be the pressure and ρ the density, at any point P within a fluid, liquid or gaseous, homogeneous or heterogeneous, in equilibrium under the influence of mutual gravitation between its parts; and let g be the gravitational attraction on a unit of mass at the position P. Let

$$g\rho H = p \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

This means that H is the height to which homogeneous liquid, of uniform density ρ, ideally under the influence of uniform gravity equal to g, must stand in a vertical tube to give pressure at its foot equal to p.

§ 8. The idea expressed by (8) is useful in connection with questions connected with internal pressure throughout a spherical liquid mass, such as the sun. It is also useful when we are considering pressure and temperature in gaseous fluids, such as our terrestrial atmosphere, or the outermost parts of the sun; which may be regarded as practically gaseous where the density is anything less than ·1.

§ 9. For a perfect gas, (8) divided by ρ, becomes

$$gH = St \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

By this we see, what is interesting to remark, that for the same temperature and same gaseous material, the "height of the homogeneous atmosphere" is the same for the air at the earth's surface and for the air at any height above the surface; and is the same for different barometric pressures. For different temperatures, it varies as the absolute temperature. For different gases at the same temperature, it is proportional to their specific volumes. For different forces of gravity, it is inversely proportional to them.

§ 10. Even for cosmical reckonings in respect to our present subject, and in many and varied terrestrial reckonings, it is convenient to take as unit of force the heaviness in mid-latitudes of the unit of mass. The unit of mass, for all nations and peoples of the earth, must for general convenience be founded on the existing French Metrical System. The unit may, according to the particular magnitude or character of substance of which the mass or quantity is to be specified, be conveniently taken as a milligram, or a gram, or a kilogram, or a metric ton (one thousand kilograms), or 10^9 tons.

§ 11. The choice of unit force, as mean terrestrial heaviness of unit mass, is very convenient for ordinary earthly purposes, but language in which it is adopted is, unless properly guarded and tacitly understood, always liable to ambiguity as to whether force or quantity of matter is meant. Thus if (using for a moment the moribund British Engineering reckonings in pounds, inches, etc.) we speak of 73 pounds of lead, there is no doubt that we mean quantity of a particular kind of matter; but if we speak of 73 pounds per square inch (which might be 73 pounds of lead, or of iron, or of stone) we mean a force. If we call the pressure on the boiler of a ship 73 pounds per square inch, we mean a somewhat greater pressure when the ship is in middle or northern latitudes than when she is on the equator; though the difference is, for pressures on safety-valves, practically negligible, being for example three-tenths per cent. between the equator and the latitude of Glasgow or Edinburgh.

§ 12. In the present paper we shall take as our unit of mass the mass of a cubic kilometre of water at standard density (which is 10^9 metric tons); and we shall take its heaviness in mid-latitudes as our unit of force. This means taking for g in (8) and (9), and in all future formulas, the ratio of gravity at the place under consideration, to terrestrial gravity in mid-latitudes. Hence (remembering that in § 4 we have chosen for our unit temperature reckoned from absolute zero the temperature of melting ice, being equal to 273° Centigrade above absolute zero) we see by (8) that S is simply the height in kilometres of the Homogeneous Atmosphere in mid-latitudes, at the freezing temperature. Thus, from known measurements of densities, we have the following table * of values of S for several different gases:—

Gas.	S.	
Air	7·988	kilometres.
Ammonia	13·414	"
Argon	5·767	"
Carbon dioxide	5·232	"
Carbon monoxide	8·370	"
Chlorine	3·297	"
Helium	58·354	"
Hydrogen	114·76	"
Nitrogen	8·256	"
Oxygen	7·233	"
Sulphur dioxide	3·709	"

§ 13. Consider now convective equilibrium in any part of a wholly gaseous globe, or in any part of a fluid globe so near the boundary as to have density small enough to let it fulfil the gaseous laws. Let z be depth measured inwards from any convenient point of reference. The differential equation of fluid equilibrium is

$$dp = g\rho dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

* If instead of taking 10^9 tons as our unit of mass we take a gram, the numbers in this table must each be multiplied by 10^5, and they will then be the values of S in centimetres instead of in kilometres.

Now, if the equilibrium is convective, we have by (3)

$$dp = \frac{k}{k-1}\,\frac{p'}{t'}\left(\frac{t}{t'}\right)^{\frac{1}{k-1}} dt \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Using this, and (2), in (10), and dividing both members by $\left(\frac{t}{t'}\right)^{\frac{1}{k-1}}$, we find

$$\frac{dt}{dz} = \frac{k-1}{k}\,\frac{g\rho' t'}{p'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Whence, by (6), we find

$$\frac{dt}{dz} = \frac{k-1}{k}\,\frac{g}{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13);$$

and, ((2) repeated)

$$\frac{\rho}{\rho'} = \left(\frac{t}{t'}\right)^{\frac{1}{k-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

§ 14. These are exceedingly important and interesting results. By (13) we see that in any part of a wholly gaseous spherical nebula, or in a gaseous atmosphere around a solid or liquid nucleus, in convective equilibrium, sufficiently stirred to have the same chemical constitution throughout, the temperature-gradient of increase inwards is in simple proportion to the force of gravity at different distances from the centre. We also see that in gaseous spherical nebulas of different chemical constitutions, or in gaseous atmospheres of different chemical constitutions, around solid or liquid nucleuses, the temperature-gradients at places of the same gravity are simply proportional to the values of $(k-1)/(kS)$ for the different gases or gaseous mixtures.

§ 15. For the terrestrial atmosphere we have by (4) $\frac{k}{k-1} = 3{\cdot}44$, and by the table in § 12, S = 7·988 kilometres. The temperature-gradient according to (13) is therefore, at the rate of our unit of temperature, or 273 degrees Centigrade, per 27·5 kilometres; or 1° C. in 100·6 metres. This is much greater than the temperature-gradient found by Welsh, in balloon ascents of about fifty years ago, which was only 1° C. in 161 metres.* Joule, with whom I had been in discussion on the subject in 1862, suggested to me that the discrepance might be accounted for by the condensation of vapour in upward currents of air. In endeavouring to test this suggestion, I made some calculations of which results are shown in the following table, extracted from a table given in my paper of 1862, referred to in § 2 above.

* Mr Shaw informs me that much investigation in later times gives a general average mean gradient of 1° C. per 164 metres. This is very nearly the same as it would be with no disturbance from radiation in air saturated with moisture, at 4° C.

Temperature Centigrade or $t-273{\cdot}7$.	Elevation from Earth's surface required to cool moist air by 1° C.
°	$\frac{dx}{-dt}$
0	Metres 152
5	168
10	186
15	207
20	229
25	252
30	274
35	284

§ 16. From this we see that an ascending current of moist air at 3° C. would sink in temperature at about the rate of 1° C. in 161 metres of ascent. This is exactly Welsh's gradient; "and we may conclude that at the times and places of his observations the lowering of temperature upwards was nearly the same as that which air saturated with moisture [at 3° C.] would experience in ascending."* But it is not to be supposed, indeed it cannot have been the case, that his observations were made in a single ascent through cloud. "It is to be remarked that except when the air is saturated, and when, therefore, an ascending current will always keep forming cloud, the effect of vapour of water, however near saturation, will be scarcely sensible on the cooling effect of expansion."†

§ 17. But, considering our terrestrial atmosphere as a whole, and the complicated circumstances of winds, and rain, and snow, and its heatings by radiations from the sun, and its coolings by radiation into starlit space, and its heatings and coolings by radiations to land and sea in different latitudes, we may feel sure that Joule's suggestion shows a cause contributing importantly to the general average temperature-gradient being less than it would be in dry air in convective equilibrium.

§ 18. For the solar atmosphere, we have approximately, $g=28$ (28 times middle latitude gravity at the earth's surface). By way of example, we may take S and k the same as for the terrestrial atmosphere, as we have not sufficient knowledge from spectrum analysis to allow us to guess other probable values of S and k for the mixture of gases constituting the upper parts of the sun's atmosphere, than those we know for the mixture of Oxygen, Nitrogen, Argon, and Carbonic Acid, which in the main constitutes our terrestrial atmosphere. Thus in the upper atmosphere of the sun,

* Quoted from the Manchester paper above referred to, *Math. and Phys. Papers*, vol. iii. p. 260.

† *Ibid.*

if in purely convective equilibrium, and undisturbed by radiations and other complications, the temperature would increase at the rate of 280 degrees Centigrade per kilometre downwards, and, looking forward to § 27 below, we see that the increase of temperature would start from absolute zero at the boundary, where density, pressure, and temperature, are all zero. It would require very robust faith in the suggestion of convective equilibrium for the gaseous atmosphere of the sun to believe in $+7°$ C., being the actual temperature of the sun's atmosphere at one kilometre below the boundary. I am afraid I cannot quite profess that faith. It seems to me that the enormous radiation from below would, if the upward and downward currents were moderately tranquil, overheat the air in the uppermost kilometre of the sun's atmosphere to far above the temperatures ranging from $-273°$ Centigrade to $+7°$ Centigrade, calculated as above from the adiabatic convective theory.

§ 19. Keeping, however, for the present by way of example to the calculated results of this theory, with the data for S and k chosen in § 15, we find that at ten and at fifty kilometres below the boundary, the temperatures, reckoned in Centigrade degrees above absolute zero, would be respectively 2800 and 14000. Calling these temperatures t' and t, and the densities at the same places ρ' and ρ, we find by (14)

$$\frac{\rho}{\rho'}=\left(\frac{14000}{2800}\right)^{\frac{5}{2}}=55{\cdot}9 \quad . \quad . \quad . \quad . \quad . \quad (15).$$

Suppose for example ρ' to be ·001 (1/1000 of the density of water), we should have $\rho = {\cdot}056$. This last is nearly but not quite too great a density for approximate fulfilment of the gaseous laws for the same gaseous mixture as our air. Thus, if not too much disturbed by radiation of heat from below, the uppermost fifty kilometres of the sun's atmosphere might be quite approximately in gaseous convective equilibrium; with density and temperature augmenting from zero at the boundary, to density ·056, and temperature 14000 Centigrade degrees above absolute zero, at the fifty kilometres depth. But, going down fifty kilometres deeper, we find that the temperature at one hundred kilometres depth would be 28000°, and the density would be ·316. This density is much too great to allow even an approximate fulfilment of the gaseous laws, by any substance known to us, even if its temperature were as high as 28000°. This single example is almost enough to demonstrate that the approximately gaseous outer shell of the sun cannot be as much as 100 kilometres thick,—a conclusion which may possibly be tested, demonstrated or contradicted, by sufficiently searching spectroscopic analysis. The character of the test would be to find the thickness of the outermost layer from which the bright spectrum lines proceed. If it were ·1″ as seen from the earth, it would be 73 kilometres thick.

§ 20. Considering the great force of gravity at the sun's surface (about 28 times terrestrial gravity), it is scarcely possible to conceive that any fluid, composed of the chemical elements known to us, could be gaseous in the sun's atmosphere at depths exceeding one hundred kilometres. I am forced to conclude that the uppermost luminous bright-line-emitting layer of our own sun's atmosphere, and of the atmosphere of any other sun of equal mass, and of not greater radius, cannot probably be as much as one hundred kilometres thick.

§ 21. There must have been a time, now very old, in the history of the sun, when the gravity at his boundary was much less than 28, and the thickness of his bright-line-emitting outermost layer very much greater than one hundred kilometres. Going far enough back through a sufficient number of million years, in all probability we find a time when the sun was wholly a gaseous spherical nebula from boundary to centre, and a splendid realisation of Homer Lane's problem. The mathematical solution of Homer Lane's problem will, for a spherical gaseous nebula of given mass, tell exactly what, under the condition of convective equilibrium, the density and temperature were at any point within the whole gaseous mass, when the central density was of any stated amount less than ·1; on the assumption that we know the specific volume, (S), and the ratio of specific heats, (k), for the actual mixture of gases constituting the nebula. It will also allow us to find, at the particular time when any stated quantity of heat has been radiated from the gaseous nebula into space, exactly what its radius was, what its central temperature and density were, and what were the temperature and density at any distance from the centre. Thus, on the assumption of S and k known, we have a complete history of the sun (or any other spherical star) for all the time before the central density had come to be as large as ·1.

§ 22. To pass from the case of convective equilibrium in a gaseous atmosphere so thin that the force of gravity is practically constant throughout its thickness, to the problem of convective equilibrium through any depth, considerable in comparison with the radius, or through the whole depth down to the centre, provided the fluid is gaseous so far, we have only to use (13) and (14), with the proper value of g, varying according to distance from the centre. Remembering that we are taking g in terms of terrestrial gravity, and that the mean density of the earth is 5·6 in terms of the standard density of water, which we are taking as our unit density, we have the following expression for g, in any spherical mass, m, having throughout equal densities, ρ, at equal distances, r, from the centre:—

$$g = \frac{m/r^2}{E/e^2} = \frac{3}{5{\cdot}6.e} \frac{\int_0^r dr\, r^2\rho}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (16),$$

where E denotes the earth's mass, and e the earth's radius. This expression we find by taking g as the force of gravity due to matter within the sphere of radius r, according to Newton's gravitational theorem, which tells us that a spherical shell of matter having equal density throughout each concentric spherical surface exerts no attraction on a point within it. Using this in (13) of § 13, with $dz = -dr$; multiplying both members by r^2, and introducing m to denote the mass of matter within the spherical surface of radius r, we find

$$-r^2\frac{dt}{dr} = \frac{3}{5{\cdot}6.e}\,\frac{k-1}{kS}\int_0^r dr\, r^2\rho = \frac{3}{5{\cdot}6.e}\,\frac{k-1}{kS}\,\frac{m}{4\pi} \quad . \quad . \quad (17).$$

Differentiating (17) with reference to r, we find

$$-\frac{d}{dr}\left[r^2\frac{dt}{dr}\right] = \frac{3}{5{\cdot}6.e}\,\frac{k-1}{kS}\,r^2\rho \quad . \quad . \quad . \quad . \quad . \quad (18).$$

§ 23. By (6), and (7), of §§ 4, 5, we find

$$\rho = \left(\frac{St}{A}\right)^{\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

where

$$\kappa = \frac{1}{k-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Eliminating ρ from (18) by (19), we find

$$-\frac{d}{dr}\left[r^2\frac{dt}{dr}\right] = \frac{r^2t^{\kappa}}{\sigma^2} \quad . \quad . \quad . \quad . \quad . \quad (21),$$

where

$$\sigma^2 = \frac{5{\cdot}6.e\,(\kappa+1)A^{\kappa}}{3S^{\kappa-1}} \quad . \quad . \quad . \quad . \quad . \quad (22).$$

§ 24. By putting

$$r = \frac{\sigma}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23),$$

we reduce (21) to the very simple form,

$$\frac{d^2t}{dx^2} = -\frac{t^{\kappa}}{x^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24);$$

the equation of the first and third members of (17), modified by (20) and (23), gives

$$\frac{m}{\mathrm{E}} = \frac{(\kappa+1)\mathrm{S}\sigma}{e^2}\frac{dt}{dx} \qquad \text{. . . . . . (25).}$$

§ 25. Let $t=\mathfrak{F}(x)$ be any particular solution of this equation; we find as a general solution with one disposable constant C,

$$t = \mathrm{C}\mathfrak{F}[x\mathrm{C}^{-\frac{1}{2}(\kappa-1)}] \qquad \text{. . . . . . (26),}$$

which we may immediately verify by substitution in (24). Here $\mathfrak{F}(x)$ may denote a solution for a gaseous atmosphere around a solid or liquid nucleus, or it may be the solution for a wholly gaseous globe, in which case $\mathfrak{F}(x)$ will be finite, and $\mathfrak{F}'(x)$ will be zero, when $x=\infty$. Each solution $\mathfrak{F}(x)$ must belong to one or other of two classes:—

Class A: that in which the density increases continuously from the spherical boundary to a finite maximum at the centre. In this class we have $d\rho/dr=0$ $(dt/dr=0)$, when $r=0$; or, which amounts to the same, $d\rho/dx=0$ $(dt/dx=0)$, when $x=\infty$.

Class B: that in which, in progress from the boundary inwards, we come to a place at which the density begins to diminish, or is infinite; or that in which the density increases continuously to an infinite value at the centre.

With units chosen to make $\mathfrak{F}(\infty)=1$, we shall denote the function $\mathfrak{F}$ of class A by Θ_κ, and call it Homer Lane's Function; because he first used it, and expressed in terms of it all the features of a wholly gaseous spherical nebula in convective equilibrium, and calculated it for the cases, $\kappa=1{\cdot}5$ and $\kappa=2{\cdot}5$ ($k=1\frac{2}{3}$ and $k=1{\cdot}4$). He did not give tables of numbers, but he represented his solutions by curves.* He did give some of his numbers for three points of each curve, and Mr Green, by very different methods of calculation, has found numbers for the case $\kappa=2{\cdot}5$, agreeing with them to within $\frac{1}{10}$th per cent.

§ 26. By improvements which Mr Green has made on previous methods of calculation of Homer Lane's Function, and which he describes in an Appendix to the present paper, he has calculated values of the function $\Theta_\kappa(x)$, and of its differential coefficient $\Theta'_\kappa(x)$, which are shown in five tables corresponding to the following five values of κ, 1·5, 2·5, 3, 4, ∞. For the four finite values of κ the practical range of each table is from $x=q$ to $x=\infty$, q denoting the value of x which makes $t=0$.

§ 27. There is such a value of x which is real in every case in which κ is positive and less than 5. This we see exemplified in the four diminishing

* *American Journal of Science*, July 1870, p. 69.

values of q found by Mr Green (·2737, ·1867, ·1450, ·0667)* for the four finite values of κ, 1·5, 2·5, 3, 4, and in the zero value of q for $\kappa=5$, the case described in § 29 below. In this case equation (24) has a solution in finite terms, which gives $t \doteq \sqrt{3}.x$ for infinitely small values of x, and therefore makes $q=0$, for $x=0$.

§ 28. Two interesting cases, $\kappa=1$, and $\kappa=5$, for each of which the differential equation (24) is soluble in finite terms, have been noticed, the former by Ritter,† the latter by Schuster.‡ Ritter's case yields in reality Laplace's celebrated law § of density for the earth's interior, ($\sin nr/r$), which Laplace suggested as a consequence of supposing the earth to be a liquid globe, having pressure increasing from the surface inwards in proportion to the augmentation of the square of the density. With Ritter, however, the value of n is taken equal to π/R, so as to make the density zero at the bounding surface ($r=R$). With Laplace, n is taken equal to $\frac{5}{6}\pi/R$ to fit terrestrial conditions, including a ratio of surface density to mean density which is approximately 1/2·5. The ratio of surface density to mean density given by Laplace's law, with $n=\frac{5}{6}\pi/R$, is in fact 1/2·4225, which is as near to 1/2·5 as our imperfect knowledge of the surface density of the earth requires.

§ 29. For the case $\kappa=5$, Schuster found a solution in finite terms, which with our present notation may be written as follows:—

$$\frac{A}{S}\rho^{\frac{1}{5}} = t = \Theta_5(x) = \frac{x\sqrt{3}}{\sqrt{(3x^2+1)}} \quad . \quad . \quad . \quad . \quad (27).$$

This makes $t=1$ at the centre ($\sigma/r=x=\infty$). At very great distances from the centre, ($x \doteq 0$), it makes

$$t \doteq x\sqrt{3} = \frac{\sqrt{3}\sigma}{r}, \quad \text{and} \quad \rho = \left(\frac{St}{A}\right)^5 \doteq \left(\frac{S\sqrt{3}}{A}\right)^5 x^5 = \left(\frac{S\sqrt{3}}{A}\right)^5 \frac{\sigma^5}{r^5} \quad . \quad . \quad (28).$$

Using (27) in (25), we find

$$\frac{m}{E} = \frac{(\kappa+1)S\sigma}{e^2}\frac{\sqrt{3}}{(3x^2+1)^{3/2}} \quad . \quad . \quad . \quad . \quad . \quad (29);$$

and if in this we put $x=0$, we find

$$\frac{M}{E} = \frac{(\kappa+1)S\sigma\sqrt{3}}{e^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

* See Appendix to the present paper, Tables I. ... IV.

† *Wiedemann's Annalen*, Bd. xi., 1880, p. 338.

‡ *Brit. Assoc. Report*, 1883, p. 428.

§ *Mécanique Céleste*, vol. v., livre xi., p. 49.

where M denotes the whole mass of the fluid. Thus we see that while the temperature and density both diminish to zero at infinite distance from the centre, the whole mass of the fluid is finite.

§ 30. It is both mathematically and physically very interesting to pursue our solutions beyond $\kappa=5$, to larger and larger values of κ up to $\kappa=\infty$: though we shall see in § 43 below, that, for all values of κ greater than 3 (or $k<1\frac{1}{3}$), insufficiency of gravitational energy causes us to lose the practical possibility of a natural realisation of the convective equilibrium on which we have been founding. But notwithstanding this large failure of the convective approximate equilibrium, we have a dynamical problem of true fluid equilibrium, continuous through the whole range of κ from -1 to $-\infty$, and from $+\infty$ to 0; that is to say, for all values of k from 0 to ∞. In fact, looking back to the hydrostatic equation (10), and the physical equations (1), or (7), and (16), we have the whole foundation of equations (17)....(26), in which we may regard t merely as a convenient mathematical symbol defined by (6′) in § 4. Any positive value of k is clearly admissible in (1), if we concern ourselves merely with a conceivable fluid having any law of relation between pressure and density which we please to give it, subject only to the condition that pressure is increased by increase of density. It is interesting to us now to remark, what is mathematically proved in § 44 below, that, unless $k>1\frac{1}{3}$, the repulsive quality in the fluid represented by k in equation (1) is not vigorous enough to give stable equilibrium to a very large globe of the fluid, in balancing the conglomerating effect of gravity.

§ 31. As to the range of cases in which κ has finite values greater than 5, we leave it for the present and pass on to $\kappa=\infty$, or $k=1$. In this case equation (1) becomes

$$\frac{\rho}{\rho'}=\frac{p}{p'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31);$$

which is simply Boyle's law of the "Spring of air," as he called it. It was on this law that Newton founded his calculation of the velocity of sound, and got a result that surprised him by being much too small. It was not till more than a hundred years later that the now well-known cause of the discrepance was discovered by Laplace, and a perfect agreement obtained between observation and dynamical theory. But at present we are only concerned with an ideal fluid which, irrespectively of temperature, exerts pressure in simple proportion to its density. This ideal fluid we shall call for brevity a Boylean gas.

§ 32. For this extreme case of $\kappa=\infty$, our differential equation (24) fails;

but we deal with the failure by expressing t in terms of ρ by (19), and then modifying the result by putting $\kappa=\infty$. We thus find

$$\frac{d^2 \log \rho}{dx^2} = -\frac{\rho}{x^4}: \text{ where } x=\frac{\sigma}{r} \quad . \quad . \quad . \quad . \quad (32);$$

σ denoting a linear constant given by (37) below. Equation (32) is the equation of equilibrium of any quantity of Boylean gas, when contained within a fixed spherical shell, under the influence of its own gravity, but uninfluenced by the gravitational attraction of any matter external to it. The value of σ might, but not without considerable difficulty, be found from (22) by putting $\kappa=\infty$. But it is easier and more clear to work out afresh, as in § 33 below, the equation of equilibrium of a Boylean gas, unencumbered by the exuviae of the adiabatic principle from which our present problem emerges.

§ 33. Let

$$p = \mathrm{B}\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

where B denotes what we may call the Boylean constant for the particular gas considered; being its pressure at unit density. According to our units, as explained in §§ 10, 11, 12, B is a linear quantity. The analytical expression of the hydrostatic equilibrium is

$$dp = -g\rho dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34),$$

where [(16) repeated]

$$g=\frac{m/r^2}{\mathrm{E}/e^2}=\frac{3}{5\cdot 6.e}\frac{\int_0^r dr\, r^2\rho}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (35).$$

Eliminating p from (34) by (33), and multiplying both members by r^2, we find

$$-r^2\frac{d\log\rho}{dr}=\frac{3}{5\cdot 6.e.\mathrm{B}}\int_0^r dr\, r^2\rho=\frac{e^2}{\mathrm{B}}\frac{m}{\mathrm{E}} \quad . \quad . \quad . \quad (36).$$

Differentiating this with reference to r, and then transforming from r to x as in equations (21)...(24) above, we find (32), with the following expression for σ:—

$$\sigma^2=\frac{5\cdot 6}{3}e\mathrm{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).$$

The equation of the first and third members of (36) gives

$$\frac{m}{\mathrm{E}}=\frac{\mathrm{B}\sigma}{e^2}\frac{d\log\rho}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$$

§ 34. Let now $\rho=\mathrm{F}(x)$ be any particular solution of (32); we find as a general solution with one disposable constant C,

$$\rho=\mathrm{CF}\left(\frac{x}{\sqrt{\mathrm{C}}}\right) \quad . \quad . \quad . \quad . \quad . \quad (39),$$

which we may immediately verify by substitution in (32) (compare § 25 above). The particular solution F must belong to one or other of the two classes, class A and class B, defined in § 25 above.

§ 35. We shall denote by $\Psi(x)$ what $F(x)$ of § 34 becomes, when the particular solution of (32), denoted by F, is of class A, with units so adjusted as to make $\Psi(\infty)=1$; that is to say, central density unity. Mr Green in his Appendix to the present paper has calculated $\Psi(x)$ and $\Psi'(x)/\Psi(x)$, through the range from $x=\infty$ to $x=\cdot 1$. His results are shown in Table V. of the Appendix. Thus we may consider $\Psi(x)$ and its differential coefficient $\Psi'(x)$ as known for all values of x through that range.

---

§ 36. Using this solution, $\Psi(x)$, instead of F in (39) above, we find that the solution of class A, which makes the central density C, is

$$\rho = C\Psi\left(\frac{x}{\sqrt{C}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (40);$$

and when we insert this expression for ρ in equation (38) we obtain

$$\frac{m}{E} = \frac{B\sigma}{e^2}\frac{1}{\sqrt{C}}\frac{\Psi'\left(\frac{x}{\sqrt{C}}\right)}{\Psi\left(\frac{x}{\sqrt{C}}\right)} \quad . \quad . \quad . \quad . \quad . \quad (41).$$

§ 37. From equations (40) and (41), with values of $\Psi\left(\frac{x}{\sqrt{C}}\right)$ and $\Psi'\left(\frac{x}{\sqrt{C}}\right)\Big/\Psi\left(\frac{x}{\sqrt{C}}\right)$ obtained from the curves of $\Psi(x)$ and $\Psi'(x)/\Psi(x)$ in the range from $x=\infty$ to $x=\cdot 1$, and with the relation $r=\frac{\sigma}{x}$ where σ is given by (37) above, we can tell exactly the density at any point of a spherical mass of an ideal Boylean gas, and the mass of gas within each spherical surface of radius r, when the gas is in equilibrium under its own gravitation only, and has a density at its centre of any stated amount C. It is interesting to examine by means of these solutions the changes in ρ and m at any given distance from the centre when the central density C increases by any small amount dC; and to find also the changes in the radius of the spherical shell enclosing a given mass m, required to allow the mass to continue in equilibrium when the central density is increasing or diminishing continuously. The following table shows the values of ρ or $C\Psi\left(\frac{x}{\sqrt{C}}\right)$, and $e^2m/EB\sigma$ or $\Psi'\left(\frac{x}{\sqrt{C}}\right)/\sqrt{C}\Psi\left(\frac{x}{\sqrt{C}}\right)$, for several of the larger values of r, corresponding to the central densities 1 and 1·21 respectively.

$\frac{\sigma}{r}$	ρ	$\frac{e^2}{\mathrm{EB}\sigma}m$	ρ	$\frac{e^2}{\mathrm{EB}\sigma}m$
∞	1	0	1·21	0
·275	·2491	6·697	·2511	7·19
·250	·2076	7·905	·2069	8·39
·225	·1673	9·38	·1647	9·86
·200	·1295	11·20	·1260	11·64
·195	·1223	11·61	·1189	12·03
·190	·1153	12·04	·1118	12·46
·185	·1084	12·50	·1048	12·89
·180	·1017	12·97	·0982	13·35
·175	·0952	13·47	·0918	13·83
·170	·0889	13·99	·0855	14·34
·165	·0828	14·53	·0795	14·86
·160	·0769	15·10	·0738	15·40
·155	·0712	15·71	·0681	16·10
·150	·0657	16·34	·0628	16·59
·145	·0605	17·01	·0577	17·22
·140	·0554	17·71	·0529	18·04
·135	·0506	18·45	·0483	18·61
·130	·0461	19·23	·0439	19·35
·125	·0418	20·06	·0398	20·14
·120	·0377	20·95	·0359	20·98
·115	·0339	21·89	·0322	21·88
·110	·0303	22·88	·0288	22·82
·105	·0269	23·95	·0257	23·83
·100	·0238	25·10	·0227	24·94

§ 38. From this table we see that it is possible to have the same mass of an ideal Boylean gas ($e^2m/\mathrm{EB}\sigma \doteqdot 21·9$) distributed in two different equilibrium conditions within a given sphere ($\sigma/r \doteqdot ·115$). We see also that in all smaller spheres the mass has increased, and in greater spheres it has decreased through the alteration of density at the centre from 1 to 1·21. Indeed, when we trace the changes in the condition of any stated mass of a Boylean gas as its central density ideally increases from very small to very great values, we find that its radius diminishes till a certain central density has been reached, after which it increases till it becomes infinite.

§ 39. By taking any two values of C in equation (26) above, and comparing the two solutions thus obtained as in § 37, it may be verified that results similar to those found in the case of a finite mass of an ideal Boylean gas, are found also in the case of a finite mass of any gas for which $\kappa > 3$, or $k < 1\frac{1}{3}$; while for any finite mass of a gas for which $\kappa < 3$, an increase in the density at the centre is always accompanied by a decrease in the radius of the shell enclosing the mass in equilibrium. These differences in the behaviour of the Boylean gas from that of gases for which $\kappa < 3$, and the resemblances of the Boylean gas and of gases for which $\kappa > 3$ (of which it may be regarded as the limiting case, $\kappa = \infty$), become of interest when we come to the question of the possibility of equilibrium of a mass of gas which is gradually losing energy by radiation into space. The result found above that there are two equilibrium conditions of a mass of any gas for which $\kappa>3$, and one equilibrium condition of a mass of any gas for which $\kappa<3$, within a given sphere, makes it desirable to investigate the nature of the equilibrium in each case, and leads us to the consideration of the energy required to maintain a mass of gas in equilibrium, within a sphere of radius R, in balancing the condensing influence o gravity.

§ 40. Let K_v denote the thermal capacity at constant volume of the particular gas considered. The energy within unit volume of the gas at temperature t is $K_v\rho t$; and the total energy I, within a sphere of radius R is given by

$$I=4\pi K_v\int_0^R dr\, r^2\rho t=K_v\int_0^R dm\, t \quad . \quad . \quad . \quad . \quad (42).$$

By using equation (6), and then integrating by parts, we obtain

$$I=\frac{4\pi K_v}{S}\int_0^R dr\, r^2 p=\frac{4\pi K_v}{S}\left[\left(\frac{1}{3}r^3 p\right)_0^R-\frac{1}{3}\int_0^R dr\, r^3\frac{dp}{dr}\right] \quad . \quad . \quad (43);$$

and since $p=0$ at the outer boundary of the sphere and $r=0$ at the centre, we have

$$I=-\frac{4\pi K_v}{3S}\int_0^R dr\, r^3\frac{dp}{dr} \quad . \quad . \quad . \quad . \quad (44).$$

Substituting now the expression given for $-\frac{dp}{dr}$ in the equation of hydrostatic equilibrium (34), we obtain finally

$$I=\frac{4\pi K_v}{3S}\int_0^R dr\, r^3 g\rho \quad . \quad . \quad . \quad . \quad . \quad (45).$$

§ 41. The work which is done by the gravitational attraction of the matter within any layer of gas $4\pi r^2\rho dr$ in bringing that layer from an infinite distance to its final position in the sphere is given by

$$dw=4\pi r^2\rho\, dr.\, gr \quad . \quad . \quad . \quad . \quad . \quad . \quad (46);$$

and the work done by gravity in collecting the whole sphere of radius R is therefore

$$W=4\pi\int_0^R dr\, r^3 g\rho=\frac{e^2}{E}\int_0^R dm\frac{m}{r} \quad . \quad . \quad . \quad . \quad (47).$$

§ 42. From equations (45) and (47) we obtain, as the ratio of the intrinsic energy within the sphere of gas to the work done by gravity in collecting the whole mass from an infinite distance,

$$\frac{I}{W} = \frac{K_v}{3S} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48).$$

If K_p be the specific heat of the gas at constant pressure, we have $S = K_p - K_v$, and equation (48) may now be written in the form

$$\frac{I}{W} = \frac{K_v}{3(K_p - K_v)} = \frac{1}{3(k-1)} = \frac{\kappa}{3} \quad . \quad . \quad . \quad . \quad (49).$$

§ 43. According to this theorem, it is convenient to divide gases into two species: species P, gases for which the ratio (k) of thermal capacity pressure constant to thermal capacity volume constant is greater than $1\frac{1}{3}$; species Q, gases for which k is less than $1\frac{1}{3}$. And the theorem expressed mathematically in equations (48) and (49) may be stated thus:—"A spherical globe of gas, given in equilibrium with any arbitrary distribution of temperature having isothermal surfaces spherical, has less heat if the gas is of species P, and more heat if of species Q, than the thermal equivalent of the work which would be done by the mutual gravitational attraction between all its parts, in ideal shrinkage from an infinitely rare distribution of the whole mass to the given condition of density." *

§ 44. It is easy to show from the theorem of §§ 42, 43 that the equilibrium of a globe of Q gas is essentially unstable. Let us first suppose for a moment that by a slight disturbance of the equilibrium condition the ratio I/W for the globe of Q gas becomes greater than that required for equilibrium by equation (49). Unless the excess of internal energy were quickly radiated away, the repulsive force which the globe of gas possesses by virtue of its internal energy would more than balance the condensing influence of gravity, and the globe would tend to expand. Since the internal energy lost in expansion is exactly equivalent to the work done against gravity, we see that the ratio I/W would continue to increase and the globe would become farther from an equilibrium condition than before. The expansion of the globe would therefore go on at an ever increasing speed till the density of the gas becomes infinitely small throughout.

If, on the other hand, through a slight disturbance of the equilibrium condition, the ratio I/W becomes less than that required for equilibrium, the globe of gas would in this case tend to contract. The increase in the internal energy due to any slight condensation would be exactly equal to the thermal equivalent of the work done by gravitation; and the ratio

* Quoted from "On Homer Lane's Problem of a Spherical Gaseous Nebula," *Nature*, Feb. 14, 1907.

I/W would therefore go on diminishing instead of increasing, as it would require to do if the gas is to be restored to a condition of equilibrium.

§ 45. "From this we see that if a globe of gas Q is given in a state of equilibrium, with the requisite heat given to it no matter how, and left to itself in waveless quiescent ether, it would, through gradual loss of heat, immediately cease to be in equilibrium, and would begin to fall inwards towards its centre, until in the central regions it becomes so dense that it ceases to obey Boyle's Law; that is to say, ceases to be a gas. Then, notwithstanding the above theorem, it can come to approximate convective equilibrium as a cooling liquid globe surrounded by an atmosphere of its own vapour." *

§ 46. But if, after being given in convective equilibrium as in § 45, heat be properly and sufficiently supplied to the globe of Q gas at its centre, the whole gaseous mass can be kept in the condition of convective equilibrium.

§ 47. The theorem of §§ 42, 43 is given by Professor Perry on page 252 of *Nature* for July 13, 1899; and in the short article "On Homer Lane's Problem of a Spherical Gaseous Nebula," published in *Nature*, February 14, 1907, I have referred to it as Perry's theorem. Since this was written, however, I have found the same theorem given by A. Ritter on pp. 160–162 of *Wiedemann's Annalen*, Bd. viii., 1879, with the same conclusion from it as that stated in § 44 above, namely, that when $k < 1\frac{1}{3}$ the equilibrium of the gaseous spherical mass is unstable.

§ 48. In the theorem of Ritter and of Perry, given in section 42, convective equilibrium is not assumed. For the purposes of our problem indicated in § 21, it is desirable to obtain expressions for the energy and the gravitational work of a mass M in equilibrium with a stated density at its centre, in terms of the notation of §§ 23 . . . 25 above. Thus, taking as our solution with central temperature C (equation 26),

$$t = C\Theta(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

where

$$z = xC^{-\frac{1}{2}(\kappa-1)}; \; r = \sigma C^{-\frac{1}{2}(\kappa-1)}/z;$$

and where σ is given in terms of the Adiabatic Constant, A, by (22); we have from equations (25) and (50)

$$\frac{m}{E} = \frac{(\kappa+1)S\sigma C^{-\frac{1}{2}(\kappa-3)}}{e^2}\Theta'(z) \quad . \quad . \quad . \quad . \quad (51),$$

and by differentiating this we obtain—

$$\frac{dm}{E} = \frac{(\kappa+1)S\sigma C^{-\frac{1}{2}(\kappa-3)}}{e^2}\Theta''(z)dz \quad . \quad . \quad . \quad . \quad (52).$$

* Quoted from "On Homer Lane's Problem of a Spherical Gaseous Nebula," *Nature*, Feb. 14, 1907.

§ 49. With these values of t and dm substituted in the third member of equation (42), the expression for the internal energy, i, of the gas within a sphere of radius r becomes

$$i = \mathrm{K}_v \int_0^r dm\, t = -\frac{\mathrm{K}_v \mathrm{E}(\kappa+1)\mathrm{S}\sigma\mathrm{C}^{-\frac{1}{2}(\kappa-5)}}{e^2} \int_z^\infty dz \Theta''(z)\Theta(z) \quad . \quad . \quad . \quad (53).$$

By putting $\Theta''(z) = -[\Theta(z)]^\kappa / z^4$ in this, and then integrating by parts as in § 40, equation (43), we may write i in the form—

$$i = \frac{\mathrm{K}_v \mathrm{E}(\kappa+1)\mathrm{S}\sigma\mathrm{C}^{-\frac{1}{2}(\kappa-5)}}{e^2} \left[\frac{[\Theta(z)]^{\kappa+1}}{3z^3} + \frac{\kappa+1}{3} \int_z^\infty dz \frac{[\Theta(z)]^\kappa}{z^3} \Theta'(z) \right] \quad . \quad . \quad (54).$$

Similarly, from the third member of (47), with the values of m and dm given in (51) and (52) above, we obtain the following expression for the gravitational work, w, done in collecting the gas within a sphere of radius r from infinite space—

$$w = \frac{\mathrm{E}(\kappa+1)^2\mathrm{S}^2\sigma\mathrm{C}^{-\frac{1}{2}(\kappa-5)}}{e^2} \int_z^\infty dz \frac{[\Theta(z)]^\kappa}{z^3} \Theta'(z) \quad . \quad . \quad . \quad . \quad (55).$$

It is easy to verify from these equations for i and w that with $\mathrm{S} = \mathrm{K}_p - \mathrm{K}_v$, as in § 42,

$$i = \frac{\mathrm{K}_v \mathrm{E}(\kappa+1)\mathrm{S}\sigma\mathrm{C}^{-\frac{1}{2}(\kappa-5)}}{e^2} \frac{[\Theta(z)]^{\kappa+1}}{3z^3} + \frac{\kappa}{3} w \quad . \quad . \quad . \quad . \quad (56).$$

§ 50. For the complete mass of gas, M, which can be in convective equilibrium under the influence of its own gravitation only, with central temperature C, we have the following results:—

$$\frac{\mathrm{M}}{\mathrm{E}} = \frac{(\kappa+1)\mathrm{S}\sigma\mathrm{C}^{-\frac{1}{2}(\kappa-3)}}{e^2} \Theta'_\kappa(q) \quad . \quad . \quad . \quad . \quad . \quad . \quad (57);$$

$$\mathrm{R} = \frac{\sigma\mathrm{C}^{-\frac{1}{2}(\kappa-1)}}{q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58);$$

$$\mathrm{I} = \frac{\mathrm{K}_v \mathrm{E}(\kappa+1)^2\mathrm{S}\sigma\mathrm{C}^{-\frac{1}{2}(\kappa-5)}}{3\,e^2} \int_q^\infty dz \frac{[\Theta(z)]^\kappa}{z^3} \Theta'(z) \quad . \quad . \quad . \quad (59);$$

$$\mathrm{W} = \frac{\mathrm{E}(\kappa+1)^2\mathrm{S}^2\sigma\mathrm{C}^{-\frac{1}{2}(\kappa-5)}}{e^2} \int_q^\infty dz \frac{[\Theta(z)]^\kappa}{z^3} \Theta'(z) \quad . \quad . \quad . \quad (60);$$

with

$$\sigma^2 = \frac{5{\cdot}6.e(\kappa+1)\mathrm{A}^\kappa}{3.\mathrm{S}^{\kappa-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad [(22)\text{ repeated}].$$

The two equations (59) and (60) give as before

$$\frac{I}{W}=\frac{\kappa}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

§ 51. The equations of §§ 48 .. 50, with equation (19), give the solution of Homer Lane's problem for all values of κ for which the function $\Theta_\kappa(z)$ and its derivative $\Theta'_\kappa(z)$ have been completely determined, namely for $\kappa=1$ and $\kappa=5$, referred to in §§ 28, 29 above, and for the values 1·5, 2·5, 3, 4, for which the Homer Lane functions and their derivatives are given in the Appendix to the present paper (Tables I. ... IV.). It is important to remark that these equations indicate clearly the critical case $\kappa=3$, and that they also reveal some interesting peculiarities of the case $\kappa=5$; which we have found to be the smallest value of κ for which a finite mass of gas is unable to arrange itself in equilibrium within a finite boundary (see §§ 27, 29).

Equation (57) shows that in spherical nebulas, for whose gaseous stuff $\kappa=3$, the total mass of any gas which can exist in the equilibrium condition corresponding to a definite central temperature, when so distributed throughout its whole volume that the temperature and density at every point are related to each other in accordance with a chosen value of the adiabatic constant A, can also be brought into the equilibrium condition corresponding to any smaller central temperature, through gradual loss of energy, without disturbing the relation of temperature and density at any point of the mass.

Equations (59) and (60) show that in spherical gaseous nebulas, for whose gaseous stuff $\kappa=5$, the total internal energy, and the gravitational work, corresponding to each equilibrium distribution of gas, has the same value, whatever be the central temperature or total mass, provided temperature and density at each point within the mass are related to each other in accordance with the same value of the adiabatic constant in each case.

§ 52. We may now apply the above equations to obtain the complete solution of our problem of § 21:—to determine for any spherical gaseous nebula of given mass, initially in convective equilibrium, exactly what its radius was, what its central temperature and density were, and what were the temperature and density at any distance from the centre, at the time when a stated quantity of heat has been radiated into space. Looking to equation (57), we see that throughout all approximate equilibrium conditions of a constant total mass the relation

$$\sigma C^{-\frac{1}{2}(\kappa-3)}=\mathcal{M} \text{ (a constant)} \quad . \quad . \quad . \quad . \quad (62)$$

holds: and, with this condition, equation (51) shows that, during the gradual loss of heat from the nebula, the value of z for each stated mass m, concentric with the boundary, is constant. We have accordingly for the mass m

$$r = \frac{\sigma C^{-\frac{1}{2}(\kappa-1)}}{z} = \frac{\mathcal{M}}{z} \cdot \frac{1}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

where C varies slowly as time goes on. If we suppose C_1 to be the initial central temperature of the nebula, and C_2 its central temperature after a quantity of heat H has been lost by radiation, by applying (62) in the equations given above we easily find (with suffixes 1 and 2, referring to the initial and final conditions respectively) the following results:—

$$\left.\begin{array}{lcl} t_2 = \dfrac{C_2}{C_1} t_1 & ; & \rho_2 = \left(\dfrac{C_2}{C_1}\right)^3 \rho_1 \\ R_2 = \dfrac{C_1}{C_2} R_1 & ; & r_2 = \dfrac{C_1}{C_2} r_1 \\ I_2 = \dfrac{C_2}{C_1} I_1 & ; & W_2 = \dfrac{C_2}{C_1} W_1 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (64);$$

in which $t_2, t_1, \rho_2, \rho_1, r_2, r_1$, all refer to points on the spherical surface enclosing a stated mass m. The total heat lost by radiation may now be written—

$$H = (W_2 - W_1) - (I_2 - I_1) = \frac{C_2 - C_1}{C_1}(W_1 - I_1) \quad . \quad . \quad . \quad (65);$$

and for an infinitesimal change in the condition of the whole mass at any time this becomes

$$\delta H = \frac{\delta C}{C}(W - I) \quad . \quad . \quad . \quad . \quad . \quad (66).$$

§ 53. These are interesting results. Remembering that $I_1 = \kappa/3.W_1$, we see by (65) and (66) that the central temperature of a globe of gas P in equilibrium increases through gradual loss of heat by radiation into space. We then see also by (64) that the internal energy of a globe of gas P, continuing in a condition of approximate equilibrium while heat is being radiated away across its boundary, would go on increasing, and the work done by the mutual gravitation of its parts would go on increasing, till the gas in the central regions becomes too dense to obey Boyle's Law. At the same time, the radius of the globe would diminish. In other words, the repulsive power which the globe of gas P possesses by virtue of its internal energy, while in approximate equilibrium, is, owing to gradual loss of energy by radiation, at each instant just insufficient to exactly balance the attractive force due to the mutual gravitation of its parts. The globe is therefore compelled to contract; and, as the heat due to the contraction is not radiated away so quickly as it is produced, the shrinkage of the globe is accompanied by augmentation of its internal energy.

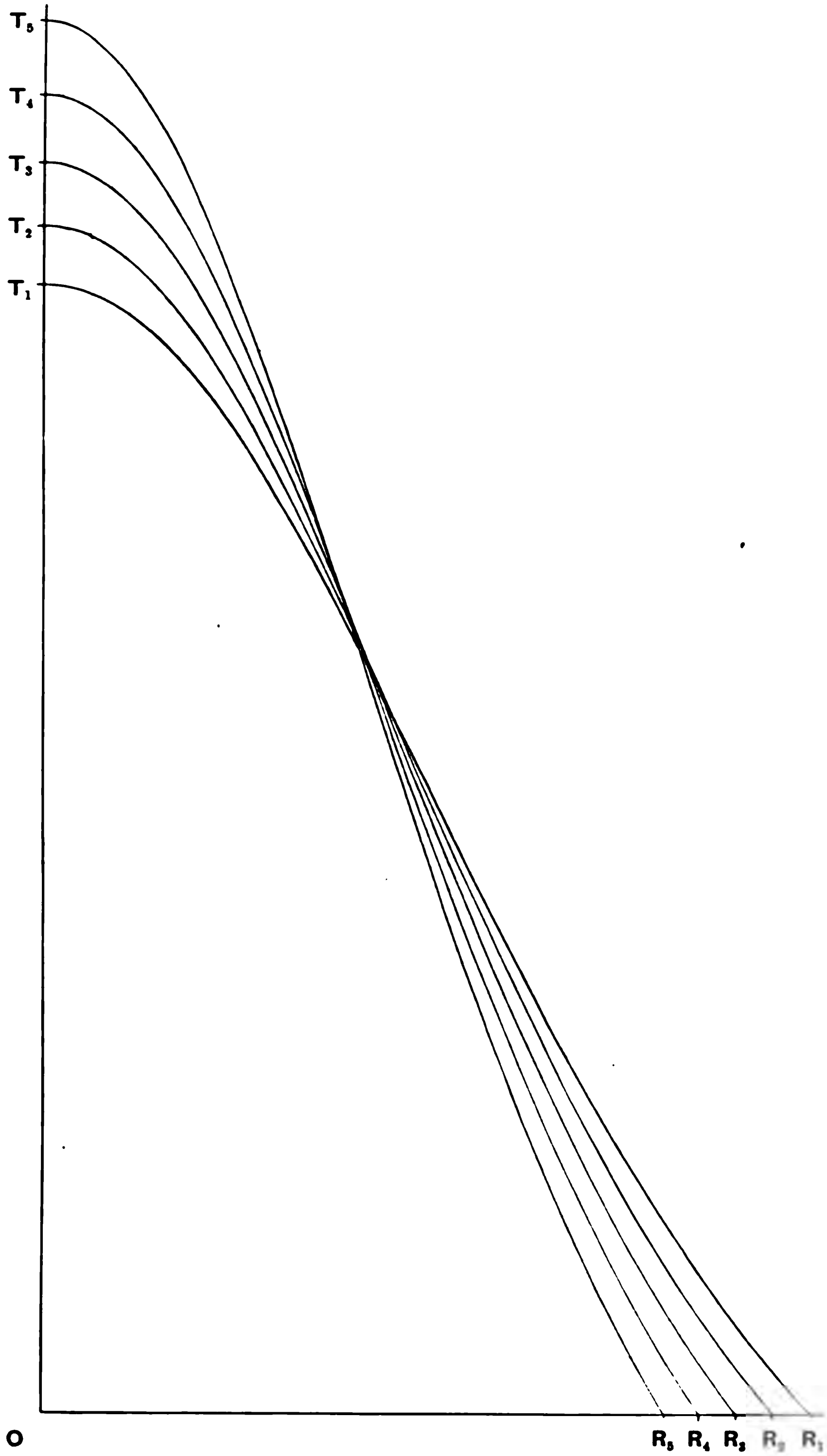
In figures 1 and 2 curves are shown illustrating five successive stages, numbered 1, 2, 3, 4, 5 respectively, in the history of a constant mass of any monatomic gas ($\kappa = 1·5$; $k = 1\frac{2}{3}$) in approximate convective equilibrium while heat is being radiated from it into space. The abscissas represent distance from the centre. The ordinates in figure 1 represent temperature, reckoned from absolute zero; OT_1, . . . OT_5, being proportional to 1, 1·052, 1·108, 1·169; 1·235: figure 2 gives the corresponding density curves.

§ 54. The remarkable result we have arrived at for P gases (for which alone, as we have seen, convective equilibrium can be realised), that the internal energy of a given mass in approximate convective equilibrium increases through gradual loss of heat by radiation into space, was first suggested as a possibility by Homer Lane; the suggestion being given in his paper referred to in § 2. To understand it more fully, go back to equation (62), and observe that in the case of P gases σ is continually diminishing, while the globe is shrinking through loss of heat. The adiabatic constant A, which determines the relation between temperature and density throughout the fluid at any instant, must therefore also continually diminish as time goes on [see (22) above]. Thus, we find from equation (19) that, although the density and temperature of the gas near the centre of the sphere are increasing, as we see from figures 1 and 2, and the total energy is increasing, in reality the temperatures at places of the same density are continually diminishing. And this diminution of temperature at places of the same density causes a diminution of the elastic resistance of the gas to compression which allows the gravitational forces to effect a contraction of the gaseous mass.

§ 55. It seems certain that, as the condensation illustrated in figures 1 and 2 continues with increasing total energy, a time must come when the resistance to compression of the matter in the central regions must become much more than in accordance with the laws of perfect gases; and after that occurs, the cooling at the surface, with continual mixing of cooled fluid throughout the interior mass, must ultimately check the process of becoming hotter in the central regions, and bring about a gradual cooling of the whole mass.

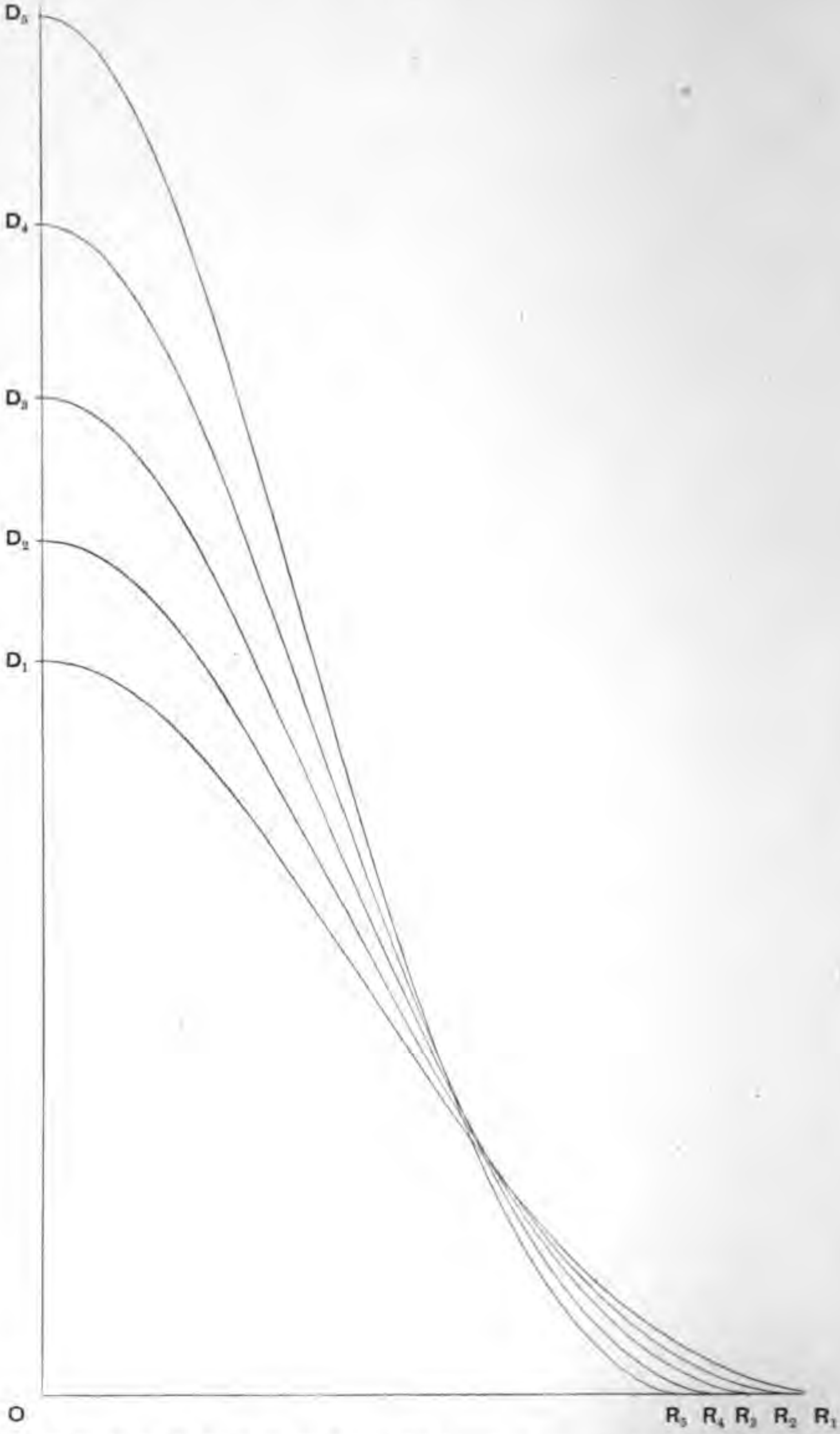
§ 56. The application of the above theory of approximate convective equilibrium to the sun, regarded as a mass of matter in the monatomic state, requires that the law of increase of density from the surface inwards should be such that the density at the centre is about six times the mean density (see Appendix, § 16). The mean density of the sun is about 1·4, the density of water being taken as unity. From this fact itself it seems certain that the sun is not gaseous as a whole. Disregarding, therefore, the high velocities which we know to exist in portions of the sun's atmo-

Fig. 1.



$T_1 R_1$, $T_2 R_2$, $T_3 R_3$, $T_4 R_4$, $T_5 R_5$, are temperature curves for a constant mass of monatomic gas in equilibrium, at five stages of its history, numbered 1, 2, 3, 4, 5, in order of time, while it is losing heat by radiation into space.

FIG. 2.



$D_1 R_1$, $D_2 R_2$, $D_3 R_3$, $D_4 R_4$, $D_5 R_5$, are density curves, corresponding respectively to the temperature curves $T_1 R_1$, $T_2 R_2$, $T_3 R_3$, $T_4 R_4$, $T_5 R_5$, of figure 1.

sphere, and which are, according to the definition given in § 3, inconsistent with a condition of convective equilibrium, we are still forced to conclude that Homer Lane's exquisite mathematical theory gives no approximation to the present condition of the sun, because of his great average density. "This was emphasised by Professor Perry in the seventh paragraph, headed 'Gaseous Stars,' of his letter to Sir Norman Lockyer on 'The Life of a Star' (*Nature*, July 13, 1899), which contains the following sentence:—

'It seems to me that speculation on this basis of perfectly gaseous stuff ought to cease when the density of the gas at the centre of the star approaches 0·1, or one-tenth of the density of ordinary water in the laboratory.'" *

§ 57. According to a promise in the 1887 paper to the *Philosophical Magazine* "On the equilibrium of a gas under its own gravitation only," I now give examples of the application of this theory of convective equilibrium to spherical masses of argon and of nitrogen; choosing, for illustration, amounts of matter equal to the masses of the sun, earth, and moon, with density at the centre 0·1 in each case.

Assuming

$$t = C\Theta[xC^{-\frac{1}{2}(\kappa-1)}] \quad . \quad . \quad . \quad . \quad . \quad (67)$$

as the solution of equation (24), which gives central density 0·1, we find from equation (19)

$$0{\cdot}1 = \left(\frac{SC}{A}\right)^{\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad (68);$$

and, as in this case we suppose the total mass M of the nebula to be known, we can determine C by applying equation (25) above. Thus

$$\frac{M}{E} = \frac{(\kappa+1)S\sigma C^{-\frac{1}{2}(\kappa-3)}}{e^2}\Theta'_{\kappa}(q) \quad . \quad . \quad . \quad . \quad (69),$$

where q denotes the value of x for which $\Theta_{\kappa}(q) = 0$. Eliminating A and σ by means of equations (22) and (68), we obtain

$$C = \left(\frac{e}{S}\right) \cdot 3770 . \left[\Theta'_{\kappa}(q)\right]^{-\frac{2}{3}} \frac{1}{\kappa+1} \left(\frac{M}{E}\right)^{\frac{2}{3}} \quad . \quad . \quad . \quad (70).$$

From equations (68) and (22) we can determine the following expressions for A and σ:—

$$\frac{A}{e} = {\cdot}8122\left[\Theta'_{\kappa}(q)\right]^{-\frac{2}{3}} \frac{{\cdot}1^{\frac{\kappa-3}{3\kappa}}}{\kappa+1} \left(\frac{M}{E}\right)^{\frac{2}{3}} \quad . \quad . \quad . \quad (71)$$

$$\frac{\sigma}{\sqrt{(eS)}} = \left(\frac{e}{S}\right)^{\frac{\kappa}{2}} {\cdot}1366\left[{\cdot}1366 . \Theta'_{\kappa}(q)\right]^{-\frac{\kappa}{3}} \frac{{\cdot}1^{\frac{\kappa-3}{6}}}{(\kappa+1)^{\frac{1}{2}(\kappa-1)}} \left(\frac{M}{E}\right)^{\frac{\kappa}{3}} \quad . \quad . \quad (72).$$

* Quoted from "On Homer Lane's Problem of a Spherical Gaseous Nebula," *Nature*, Feb. 14, 1907.

The radius of the outer boundary of the nebula is given by

$$R=\frac{\sigma}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73),$$

where

$$xC^{-\frac{1}{2}(\kappa-1)}=q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).$$

We have therefore $R=\sigma q^{-1}C^{-\frac{1}{2}(\kappa-1)}$, which, by means of equations (53) and (55), may be written in the form

$$\frac{R}{e}=2{\cdot}6527\Big[\Theta'_{\kappa}(q)\Big]^{-\frac{1}{2}}q^{-1}\left(\frac{M}{E}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (75).$$

For argon we have $k=1\frac{2}{3}$, or $\kappa=1{\cdot}5$; and $S=5{\cdot}767$ kilometres; and for nitrogen we have $k=1{\cdot}4$, or $\kappa=2{\cdot}5$; and $S=8{\cdot}256$ kilometres. With these values of S and κ, inserted in the above formulas, we obtain the results shown in the following table:—

Matter in nebula.	Total mass, that of	Central density.	Central temperature in Centigrade degrees above absolute zero.	Central pressure in metric tons per sq. kilometre.	Radius of boundary in kilometres.	Adiabatic Constant in kilometres.
Argon	Sun	·1	$1{\cdot}105\times10^{8}$	$2{\cdot}33\times10^{14}$	$3{\cdot}04\times10^{6}$	$1{\cdot}08\times10^{7}$
”	Earth	·1	$2{\cdot}342\times10^{4}$	$4{\cdot}95\times10^{10}$	$4{\cdot}42\times10^{4}$	$2{\cdot}30\times10^{4}$
”	Moon	·1	$1{\cdot}243\times10^{3}$	$2{\cdot}63\times10^{9}$	$1{\cdot}02\times10^{4}$	$1{\cdot}22\times10^{2}$
Nitrogen	Sun	·1	$6{\cdot}383\times10^{7}$	$1{\cdot}92\times10^{14}$	$4{\cdot}79\times10^{6}$	$4{\cdot}82\times10^{6}$
”	Earth	·1	$1{\cdot}353\times10^{4}$	$4{\cdot}07\times10^{10}$	$6{\cdot}97\times10^{4}$	$1{\cdot}02\times10^{3}$
”	Moon	·1	$7{\cdot}185\times10^{2}$	$2{\cdot}16\times10^{9}$	$1{\cdot}61\times10^{4}$	54·3

§ 58. The curves of figures 3 and 4 represent temperature and density at different distances from the centres of nebulas for which κ has the values, 1·5, 2·5, 3, 4. The temperature curves are drawn from the numbers given in the third columns of Tables I.... IV. of the Appendix: the density curves, from the numbers given in the fourth columns. With properly chosen scales of ordinates and abscissas, the curves shown may represent the condition of any gaseous mass, corresponding to any of the solutions (26) above. Thus, with scales so chosen that $OR_{\kappa}=R=\sigma q^{-1}C^{-\frac{1}{2}(\kappa-1)}$, and $OT=C$, each curve, TR_{κ}, represents the temperature reckoned from absolute zero; and with $OD_{\kappa}=(SC/A)^{\kappa}$, each curve, $D_{\kappa}R_{\kappa}$, represents the density; in a nebula composed of gas for which κ has one of the values given above, when the central temperature is C.

Each curve shown meets the axis of R at a finite angle; this angle being so small for the density curves that they appear to meet OR tangentially.

FIG. 3.

T $R_{1\cdot5}$ is the curve of temperature, and $D_{1\cdot5}$ $R_{1\cdot5}$ is the curve of density, for a monatomic gas ($k=1\frac{2}{3}$). T $R_{2\cdot5}$, and $D_{2\cdot5}$ $R_{2\cdot5}$ are corresponding curves for a diatomic gas ($k=1\cdot4$).

Fig. 4.

T R_3 is the curve of temperature, and D_3 R_3 is the curve of density, for a gas for which $k=1\frac{1}{3}$. T R_4 and D_4 R_4 are corresponding curves for a gas for which $k=1\frac{1}{4}$.

APPENDIX.

By George Green.

§ 1. In order to determine the conditions of temperature, pressure, and density, at any distance from the centre of a spherical mass of gas in convective equilibrium, held together by the mutual gravitation of its parts, it is necessary to find a solution of the equation,

$$\frac{d^2t}{dx^2} = -\frac{t^\kappa}{x^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

In this equation,

x is inversely proportional to r, the distance of any point from the centre of the sphere:

$$\left[r = \frac{\sigma}{x}\right]$$

t is the temperature of the gas at any point of a spherical surface of radius r:

t^κ is proportional to the density where t is the temperature:

$$\left[\rho = \left(\frac{\mathrm{S}t}{\mathrm{A}}\right)^\kappa\right]$$

$\frac{dt}{dx}$ is proportional to the mass of gas within the surface of radius r:

$$\left[\frac{dt}{dx} = \frac{e^2}{(\kappa+1)\mathrm{S}\sigma}\frac{m}{\mathrm{E}}\right]$$

and κ is equal to $1/(k-1)$, where k is the ratio of specific heats of the gas considered (see §§ 22 ... 24 of the above paper).

§ 2. Solutions of this equation can be found which correspond to a mass of gas around a solid or liquid nucleus. These may become of interest later. Solutions can also be found which correspond to an infinite sphere of gas, with an infinite density at the centre. But the solutions which it is now desirable to obtain are those which can be applied to the case of a spherical mass of gas which has a finite density at the centre. This is expressed mathematically by saying that at $x = \infty$, we have $t = \mathrm{C}$, $\frac{dt}{dx} = 0$.

Lord Kelvin has shown, in his paper "On the equilibrium of a gas under its own gravitation only," *Phil. Mag.*, March 1887, and in § 25 of the preceding paper, that when any complete solution $\mathfrak{Z}(x)$ has been found, it is possible to derive from it a general solution with one disposable constant C, $C\mathfrak{Z}\left[xC^{-\frac{1}{2}(\kappa-1)}\right]$. Accordingly, it is convenient to deal only with the particular solution for which $t=1$; $\frac{dt}{dx}=0$; at $x=\infty$, denoted by $\Theta_\kappa(x)$, and called the Homer Lane Function.

§ 3. Homer Lane, in his paper "On the theoretical temperature of the sun," gives analytical solutions of equation (1) for the cases $k=1\frac{2}{3}$ and $k=1{\cdot}4$, which correspond to a monatomic and to a diatomic gas respectively. His method of obtaining these solutions is equivalent to the following. Assume

$$t=\Theta_\kappa(x)=1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\frac{a_3}{x^6}+\ \ldots\ \text{etc.} \qquad (2)$$

to be the required solution of (1), where a_1, a_2, a_3, etc. are to be determined. Then

$$\frac{d^2\Theta_\kappa}{dx^2}=\frac{2.3.a_1}{x^4}+\frac{4.5.a_2}{x^6}+\frac{6.7.a_3}{x^8}+\text{etc.} \qquad (3).$$

And the coefficients a_1, a_2, a_3, etc. can be determined from the equation—

$$\left(\frac{2.3.a_1}{x^4}+\frac{4.5.a_2}{x^6}+\frac{6.7.a_3}{x^8}+\text{etc.}\right)=-\frac{\left(1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\frac{a_3}{x^6}+\text{etc.}\right)^\kappa}{x^4} \qquad (4)$$

The solutions given by Homer Lane are, in the present notation,

$$\Theta_{1\cdot5}(x)=1-\frac{1}{6x^2}+\frac{1}{80x^4}-\frac{1}{1440x^6}+\frac{1}{31104x^8}-\ \ldots\ \text{etc.} \qquad (5);$$

$$\Theta_{2\cdot5}(x)=1-\frac{1}{6x^2}+\frac{1}{48x^4}-\frac{5}{2016x^6}+\frac{125}{435456x^8}-\ \ldots\ \text{etc.} \qquad (6).$$

These terms are sufficient to give a satisfactory determination of t for all values of x equal to or greater than unity; that is, at all points in the gas within distance σ from the centre of the sphere, where the radius of the boundary is σ/q, and q is the value of x for which $\Theta_\kappa(x)=0$.

The labour of calculating additional terms of these series being very great, and no great precision being necessary, Homer Lane merely employed a step-by-step process, involving the use of numerical differences, to obtain approximately the value of t at points whose distance from the centre of the sphere was greater than that corresponding to x unity in equations (5) and (6). When a fairly small value of t had been reached by this method, he was able to complete the calculation as far as $t=0$ $(x=q)$ by means of approximate formulas which can be derived in a manner similar to that described above.

§ 4. With a view to obtaining greater accuracy in the results for monatomic gases, Mr T. J. J. See has extended Homer Lane's series (5) as far as the term containing x^{20}; and with the aid of additional terms, obtained by means of logarithmic differences of preceding terms, he has calculated the values of t, ρ, m, etc., at a place very close to the boundary of the gas. From this he has been able to find with great accuracy the radius of the spherical boundary, and the total mass of gas, corresponding to the Homer Lane Function $\Theta_{1\cdot5}(x)$ (see § 17). These results are published in a paper entitled "Researches on the physical constitution of the heavenly bodies" (*Astr. Nachr.* No. 4053, Bd. 169, Nov. 1905). They were found after Table I., on page 299 of the present paper, had been completed by the entirely different method given below, and they are a confirmation of its usefulness.

§ 5. Eight years after the publication of Homer Lane's famous paper, the problem of the convective equilibrium of a spherical mass of gas under its own gravitation only was dealt with very fully by A. Ritter, in a series of papers entitled "Untersuchungen über die höhe der Atmosphäre und die Constitution gasförmiger Weltkörper," published in *Wiedemann's Annalen*, 1878–1882. Numerical solutions of equation (1) are given for the following values of κ, 1·5, 2, 2·44, 3, 4, 5; these solutions being obtained wholly by a graphical process, similar to the process described in § 7 below.

§ 6. Professor Schuster, in a short paper to the British Association at Southport in 1883, pointed out that it was possible to obtain solutions of equation (1) in finite terms in the two cases $\kappa=1$, and $\kappa=5$ ($k=2$ and $k=1\cdot2$). For $\kappa=1$, the solution, in the present notation, is—

$$\Theta_1(x)=x\sin\frac{1}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7);$$

a result which was first given by Ritter. For $\kappa=5$, the solution is

$$\Theta_5(x)=\frac{x\sqrt{3}}{\sqrt{(3x^2+1)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

§ 7. The method of obtaining numerical solutions of equation (1) which has been adopted throughout the present paper, is derived from that indicated by Lord Kelvin on page 291 of his paper to the *Philosophical Magazine*, 1887, referred to in § 2 above. An arbitrary trial curve, t_0, fulfilling the initial conditions $t=A$; $\frac{dt}{dx}=A'$; at $x=a$; is taken for t. From this curve, t_0, a curve representing $-\frac{t_0^\kappa}{x^4}$, or $\frac{d^2t_0}{dx^2}$, is obtained by direct calculation. One integration performed on this calculated curve gives the means of drawing $\frac{dt_1}{dx}$; and the curve representing t_1, which is obtained by integrating $\frac{dt_1}{dx}$, is then found to be a closer approximation to the true curve for t than the curve t_0 chosen arbitrarily. This process may be repeated to obtain t_2, using the curve t_1 as a new trial curve for t: and so on. When t_i differs very little from t_{i-1}, t_i may be regarded as a very close approximation to the true curve for t.

§ 8. It was found that the best way to carry out this plan of obtaining a numerical solution by means of two successive integrations was to choose the interval of integration, from any point at which t and $\frac{dt}{dx}$ were known, sufficiently small that the trial value chosen for t, for the point at the end of the interval, could be determined with any degree of accuracy required, by means of numerical differences of the values of t already determined for the end points of preceding intervals. The arbitrary trial curve is in this case a curve which coincides with the true curve in each interval preceding the one considered, and in this interval it closely approximates to the true curve. It has been found that one application of the process of double integration to this trial curve gives t and $\frac{dt}{dx}$ at the end of the interval with any accuracy desired; the accuracy depending on the smallness of the interval chosen. By taking the curves for $-\frac{t^5}{x^4}$ and $\frac{dt}{dx}$ as straight lines within each successive interval treated, and by making a roughly estimated allowance at each step for the error thus introduced, the process can be carried out very quickly and quite satisfactorily, as a process of step-by-step calculation, without the assistance of carefully drawn curves.

§ 9. The accuracy of the above process can easily be proved analytically for the case of any very short interval; but when such a process is applied to a succession of intervals, there is certain to be a cumulative error, which may, or may not, increase without limit as the work proceeds. So far as it is possible to judge, however, the process of § 8 seems to be practically applicable to obtain numerical solutions of differential equations of the form

$$\frac{d^2t}{dx^2}=f(x, t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

provided f has the opposite sign from t. It has been applied with very satisfactory results to the two equations—

$$\frac{d^2t}{dx^2} = -t \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

and

$$\frac{d^2t}{dx^2} = -\frac{t^5}{x^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

for each of which the direct verification of the solutions obtained is possible; the solution of the latter equation obtained was $\Theta_5(x)$, which is given in § 6 above.

In these cases, and in the case of any Homer Lane Function, if t throughout a number of intervals becomes greater than its true value, the absolute value of $\frac{d^2t}{dx^2}$ also becomes greater than its true value, and $\frac{dt}{dx}$ is therefore tending to become less than it would otherwise be. Hence the successive additions to t in each interval in which it is greater than its true value are, or tend to become, less than they would be with a correct t. Thus, when t has become too great in any interval, throughout the succeeding intervals it tends to return to its true value rather than to go without limit away from it. From similar considerations it may be judged that when t has become less than its true value, in the succeeding intervals it tends to return towards its true value.

§ 10. The numerical values of the Homer Lane Function $\Theta(x)$ with its differential coefficient $\Theta'(x)$, in the interval from $x=\infty$ to $x=q$, given in Tables I.... IV., and the values of the Boylean Function $\Psi(x)$, and the function $\Psi'(x)/\Psi(x)$, in the interval from $x=\cdot25$, to $x=\cdot1$, given in Table V., have all been obtained by the method of § 8. In each of the Homer Lane Functions, a beginning of the calculation was made from the following approximate equation:—

$$\Theta_\kappa(x) \doteq 1 - \frac{1}{6x^2} + \frac{\kappa}{120x^4} - \quad . \quad . \quad . \quad . \quad . \quad (12),$$

easily derived from equations (2) and (4) above. From $x=\infty$ to $x=\cdot25$, the Boylean Function was calculated by a method described below.

After Tables I.... IV. had been completed, as it was still desirable to be able to verify the results obtained by the step-by-step process at some point close to the final point, q, and as the labour of calculating successive terms of the series (2) soon becomes very great, while the number of the terms required to give a sufficiently good result also becomes greater and greater as x diminishes, the form of expansion given in § 11 below was tried, and found to be useful.

§ 11. Assume as a solution of equation (1),

$$\Theta_\kappa(x) = \frac{1}{\left(1 + \frac{a_1}{x^2} + \frac{a_2}{x^4} + \frac{a_3}{x^6} + \text{etc.}\right)^n} \quad . \quad . \quad . \quad . \quad . \quad (13).$$

We can write $\Theta''_\kappa(x)$ in either of the following forms:—

$$\frac{-n\left(\frac{2.3.a_1}{x^4}+\frac{4.5.a_2}{x^6}+\frac{6.7.a_3}{x^8}\text{ etc.}+\right)\left(1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\text{ etc.}+\right)+n(n+1)\left(\frac{2a_1}{x^3}+\frac{4a_2}{x^5}+\frac{6a_3}{x^7}+\text{etc.}\right)^2}{\left(1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\frac{a_3}{x^6}+\text{etc.}\right)^{n+2}};$$

$$\frac{-\frac{2.3.a_1n}{x^4}}{\left(1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\text{etc.}\right)^{n+1}}-\frac{n\left(\frac{4.5.a_2}{x^6}+\frac{6.7.a_3}{x^8}+\text{etc.}\right)\left(1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\text{etc.}\right)-n(n+1)\left(\frac{2a_1}{x^3}+\frac{4a_2}{x^5}+\text{etc.}\right)^2}{\left(1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\frac{a_3}{x^6}+\text{etc.}\right)^{n+2}}.$$

If we now choose n in the first form, so that $(n+2)=\kappa n$, and equate the numerator to $-\frac{1}{x^4}$, we obtain $\Theta''_\kappa(x)=-[\Theta_\kappa(x)]^\kappa/x^4$. And the coefficients a_1, a_2, a_3, etc. can be determined from the following equations:—

$$\left.\begin{aligned}
&2.3.a_1=\frac{1}{n};\ 4.5.a_2=(\beta.1^2-2.3)a_1^2;\ 6.7.a_3=(2\beta.1.2.-2.3.-4.5.)a_1a_2;\\
&2i(2i+1)a_i=\sum_{r=1}^{r=\frac{i-2}{2}}\{2\beta.r(i-r)-2r(2r+1)-(2i-2r)(2i-2r+1)\}a_ra_{i-r}\\
&\qquad\qquad+\left\{\beta\left(\frac{i}{2}\right)^2-i(i+1)\right\}a_{\frac{i}{2}}^2\ (i\text{ even});\\
&2i(2i+1)a_i=\sum_{r=i}^{r=\frac{i-1}{2}}\{2\beta.r(i-r)-2r(2r+1)-(2i-2r)(2i-2r+1)\}a_ra_{i-r}(i\text{ odd});\\
&\text{where}\\
&\qquad\beta=4(n+1)=\frac{4(\kappa+1)}{(\kappa-1)}.
\end{aligned}\right\}\quad\ldots(14)$$

Similarly, if we choose n so that $(n+1)=\kappa n$, and then choose a_1 so that $2.3.a_1.n=1$, the first term of the second form of $\Theta''_\kappa(x)$ becomes $-[\Theta_\kappa(x)]^\kappa/x^4$; and the second term, equated to zero, gives us the following equations to determine $a_2, a_3, \ldots\ldots$, etc.:—

$$\left.\begin{aligned}
&4.5.a_2=\beta a_1^2;\ 6.7.a_3=(2\beta.1.2-4.5)a_1a_2;\ 8.9.a_4=(2\beta.1.3.-6.7)a_1a_3+(\beta.2^2-4.5)a_2^2;\\
&2i(2i+1)a_i=\{2\beta.1.(i-1)-(2i-2)(2i-1)\}a_1a_{i-1}+\sum_{r=2}^{r=\frac{i-2}{2}}\{2\beta.r(i-r)-2r(2r+1)\\
&\qquad-(2i-2r)(2i-2r+1)\}a_ra_{i-r}+\left\{\beta\left(\frac{i}{2}\right)^2-i(i+1)\right\}a_{\frac{i}{2}}^2\ (i\text{ even});\\
&2i(2i+1)a_i=\{2\beta.1.(i-1)-(2i-2)(2i-1)\}a_1a_{i-1}+\sum_{r=2}^{r=\frac{i-1}{2}}\{2\beta.r(i-r)-2r(2r+1)\\
&\qquad-(2i-2r)(2i-2r+1)\}a_ra_{i-r}(i\text{ odd});\ \text{where }\beta=4(n+1)=\frac{4\kappa}{(\kappa-1)}.
\end{aligned}\right\}\quad\ldots(15)$$

§ 12. In the particular case of equation (1) for which $\kappa=\infty$ ($k=1$), and which corresponds to an ideal gas which obeys Boyle's Law for all pressures, the differential equation becomes

$$\frac{d^2 \log \rho}{dx^2} = -\frac{\rho}{x^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Assume as the solution of this equation which gives $\rho=1$, when $x=\infty$,

$$\rho = \Psi(x) = \frac{1}{\left(1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\frac{a_3}{x^6}+\text{etc.}\right)^n} \quad . \quad . \quad . \quad . \quad (17).$$

We may, in this case also, write $\dfrac{d^2 \log \Psi(x)}{dx^2}$ in the two forms:—

$$\frac{-n\left(\frac{2.3.a_1}{x^4}+\frac{4.5.a_2}{x^6}+\text{etc.}\right)\left(1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\text{etc.}\right)+n\left(\frac{2a_1}{x^3}+\frac{4a_2}{x^5}+\text{etc.}\right)^2}{\left(1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\frac{a_3}{x^6}+\text{etc.}\right)^2};$$

$$\frac{-\frac{2.3.a_1 n}{x^4}}{\left(1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\frac{a_3}{x^6}+\text{etc.}\right)} - \frac{n\left(\frac{4.5.a_2}{x^6}+\frac{6.7.a_3}{x^8}+\text{etc.}\right)\left(1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\text{etc.}\right)-n\left(\frac{2a_1}{x^3}+\frac{4a_2}{x^5}+\text{etc.}\right)^2}{\left(1+\frac{a_1}{x^2}+\frac{a_2}{x^4}+\frac{a_3}{x^6}+\text{etc.}\right)^2}.$$

By taking $n=2$ in the first of these forms, and $n=1$ in the second, and proceeding exactly as in § 11, we obtain two expressions for $\Psi(x)$ in the form (17), the coefficients a_1, a_2, a_3, etc. being again determined from the series of equations (14), when $n=2$, and from equations (15), when $n=1$.

§ 13. At present only the numerical values given by the above solutions are of importance; and for this purpose, the expression (13) with $n=\dfrac{2}{\kappa-1}$ has been used to calculate the Homer Lane Functions, and the expression (17) with $n=2$ to calculate the Boylean Function. The following table gives the logarithms of several of the coefficients a_1, a_2, etc. for the functions given in Tables I. … V.; the letter n indicating that the term is negative:—

κ	1·5	2·5	3	4	∞
log a_1	$\bar{2}$·61979	$\bar{1}$·09691	$\bar{1}$·22185	$\bar{1}$·39794	$\bar{2}$·92082
log a_2	$\bar{3}$·08468	$\bar{3}$·41567	$\bar{3}$·44370	$\bar{3}$·31876	$\bar{4}$·84164 *n*
log a_3	$\bar{5}$·81361	$\bar{5}$·94369	$\bar{5}$·82045	$\bar{6}$·91736	$\bar{5}$·13921
log a_4	$\bar{6}$·59587	$\bar{6}$·45524	$\bar{6}$·10924	$\bar{7}$·23612	$\bar{7}$·61229 *n*
log a_5	$\bar{7}$·39883	$\bar{8}$·96928	$\bar{8}$·46991	$\bar{9}$ 5540 *n*	$\bar{8}$·19003
log a_6	$\bar{8}$·21416	$\bar{9}$·49350	$\bar{10}$·72486	$\bar{11}$·8517	$\bar{10}$·88708 *n*
log a_7	$\bar{9}$·03834	$\bar{10}$·01706	$\bar{11}$·19035	*n*	$\bar{11}$·5332
log a_8	$\bar{11}$·86917	$\bar{12}$·54628	$\bar{14}$·84984		*n*
log a_9	$\bar{12}$·70515	$\bar{13}$·07868	$\bar{14}$·25996		
log a_{10}	$\bar{13}$·54520	$\bar{15}$·60715	$\bar{16}$·9440 *n*		
log a_{11}	$\bar{14}$·38854	$\bar{16}$·13498	$\bar{17}$·9204		
log a_{12}	$\bar{15}$·23460	$\bar{18}$·67438	*n*		
log a_{13}		$\bar{19}$·1967			
log a_{14}		$\bar{21}$·7511			
log a_{15}		$\bar{22}$·2473			
log a_{16}		$\bar{24}$·8548			
log a_{17}		$\bar{25}$·4061			
log a_{18}		$\bar{27}$·6267			

§ 14. In each of the functions (13) and (17) the coefficients become positive and negative alternately; and for small values of x, close enough to q, the series ultimately diverge. If x is not too small, several terms at the beginning of the series converge fairly rapidly, and give a close approximation to the result. The following values of the Homer Lane Function calculated by means of the numbers given above are correct to the last figure shown:—

$$\Theta_{1\cdot5}(\cdot4) = \cdot3159 \;;\; \Theta'_{1\cdot5}(\cdot4) = 2\cdot133 \;;\; \Theta_{1\cdot5}(\cdot35) = \cdot2007 \;;\; \Theta'_{1\cdot5}(\cdot35) = 2\cdot464;$$
$$\Theta_{2\cdot5}(\cdot3) = \cdot24186;\; \Theta'_{2\cdot5}(\cdot3) = 2\cdot0078;\; \Theta_{2\cdot5}(\cdot25) = \cdot13768;\; \Theta'_{2\cdot5}(\cdot25) = 2\cdot1448;$$
$$\Theta_3\,(\cdot25) = \cdot2093 \;;\; \Theta'_3\,(\cdot25) = 1\cdot922 \;;\; \Theta_3\,(\cdot2) = \cdot111 \;;\; \Theta'_3\,(\cdot2) = 2\cdot00;$$
$$\Theta_4\,(\cdot25) = \cdot3180 \;;\; \Theta'_4\,(\cdot25) = 1\cdot582 \;;\; \Theta_4\,(\cdot2) = \cdot2359 \;;\; \Theta'_4\,(\cdot2\,) = 1\cdot695.$$

By comparing these results with the corresponding values of the Homer Lane Function given in Tables I. ... IV., we see that the numbers obtained by the step-by-step process of § 8 are sufficiently accurate for our purpose. It is possible to calculate q and $\Theta'_\kappa(q)$ from the values of $\Theta_\kappa(x)$ and $\Theta'_\kappa(x)$ at a point at which they are given with sufficient accuracy by (13) above, by means of the approximate formulas used by Homer Lane and referred to in § 3 above. It will be seen later, however, that Homer Lane's calculated results agree with those given in the tables.

§ 15. We can now estimate from the numbers given in § 14 the degree of probable error in the values of q and $\Theta'(q)$ found by the step-by-step process. Similar reasoning to that of § 9, by which we show that errors arising in the working out of this process do not tend to increase as the work proceeds, leads to the following results:—

The value of $q(\cdot2737)$ in Table I. is correct to the nearest figure, while

$\Theta'_{1\cdot5}(q)$ (2·707) is less than it ought to be by about 1 in 600; the value of q(·18676) in Table II. is too high, the error being less than 1 in 3000, while $\Theta'_{2\cdot5}(q)$ (2·188) is probably too high, the true value lying close to 2·188. In Tables III. and IV. q and $\Theta'(q)$ may be taken as correct to about 1 in 200. In Table V., the final values of the Boylean Function are believed to be correct to one per cent.

Greater accuracy has been aimed at in the case $\kappa = 2\cdot5$ than in any of the other cases, owing to its theoretical importance, being applicable to stars composed of gases such as our terrestrial atmosphere. This was the case dealt with by Lord Kelvin in his 1887 paper referred to in § 2 above. The numerical results given in that paper were obtained by an approximate process using radii of curvature calculated for successive small arcs of the curve; and they are in satisfactory agreement with the results given in Table II.

§ 16. One important numerical result which can be derived from the values of q_κ and $\Theta'_\kappa(q)$ is the value of the ratio $\frac{\text{central density}}{\text{mean density}}$ in a nebula composed of any gas for which κ is 1·5, 2·5, 3, or 4. This ratio is given by the expression

$$\rho = \frac{\text{central density}}{\text{mean density}} = \frac{1}{3.\Theta'_\kappa(q).q^3_\kappa}.$$

For $\kappa = 1\cdot5$ (monatomic gases)

$$\rho = \frac{1}{3.2\cdot707.\cdot2737^3} = 6\cdot006.$$

For $\kappa = 2\cdot5$ (diatomic gases)

$$\rho = \frac{1}{3.2\cdot188.\cdot18676^3} = 23\cdot39,$$

or, accurate to nearest figure,

$$\rho = \frac{1}{3.2\cdot188.\cdot18673^3} = 23\cdot40.$$

(For $\kappa = 2\cdot44$ or $k = 1\cdot41$, Ritter gives the value of ρ as 23.)

For $\kappa = 3$, and $\kappa = 4$, the values of the ratio are 54·2 and 625 respectively. For $\kappa = 5$, the ratio is infinite.

§ 17. The following table of results is given for comparison:—

	$q_{1\cdot5}$	$\Theta'_{1\cdot5}(q)$	ρ	$q_{2\cdot5}$	$\Theta'_{2\cdot5}(q)$	ρ
Homer Lane,	·2735	2·741	5·943	·18674	2·188	23·40
See,	·27368	2·7097	6·0014			
Ritter,	·274	2·70	6			

§ 18. In each of Tables I....V. below, the second column gives the value of $1/x$, which is also the value of r when σ is unity in equations (23) and (32) of Lord Kelvin's paper. In Tables I....IV., column 3 enables us to find the curve of temperature, and column 4 the curve of density, for any spherical nebula for which κ has the value, 1·5, 2·5, 3, or 4 (see equations (26) and (19) of the paper): column 5 gives the mass of gas within each sphere of radius r [see equation (51)]. In Table V., column 3 enables us to determine the density at any point in a spherical mass of ideal Boylean Gas, and column 4 the mass of gas within radius r (see equations (40) and (41) of § 36).

TABLE I.

x	r	$\Theta_{1\cdot5}$	$\Theta_{1\cdot5}^{1\cdot5}$	$\Theta'_{1\cdot5}$
∞	0	1·0000	1·0000	0
40	·0250	·9999	·9998	$5·20 \times 10^{-6}$
20	·0500	·9996	·9994	$4·16 \times 10^{-5}$
10	·0100	·9983	·9975	$3·33 \times 10^{-4}$
8	·1250	·9974	·9961	$6·50 \times 10^{-4}$
6	·1667	·9954	·9931	·00154
5	·2000	·9934	·9901	·00265
4	·2500	·9896	·9844	·00511
3	·3333	·9816	·9725	·01235
2	·5000	·9590	·9391	·0402
1	1·0000	·8447	·7764	·2884
·95	1·0526	·8292	·7551	·3309
·90	1·1111	·8114	·7309	·3817
·85	1·1765	·7909	·7034	·4430
·80	1·2500	·7669	·6716	·5174
·75	1·3333	·7388	·6350	·6083
·70	1·4286	·7057	·5928	·7198
·65	1·5385	·6664	·5440	·8573
·60	1·6667	·6194	·4875	1·027
·55	1·8182	·5640	·4236	1·236
·50	2·0000	·4949	·3482	1·491
·49	2·0408	·4797	·3322	1·547
·48	2·0833	·4640	·3161	1·606
·47	2·1277	·4477	·2996	1·666
·46	2·1739	·4307	·2827	1·728
·45	2·2222	·4130	·2654	1·792
·44	2·2727	·3948	·2481	1·858
·43	2·3256	·3759	·2305	1·925
·42	2·3810	·3563	·2127	1·993
·41	2·4390	·3360	·1948	2·061
·40	2·5000	·3151	·1769	2·130
·39	2·5641	·2934	·1589	2·199
·38	2·6316	·2711	·1412	2·267
·37	2·7027	·2481	·1236	2·334
·36	2·7778	·2244	·1063	2·399
·35	2·8571	·2002	·0896	2·460
·34	2·9412	·1753	·0734	2·517
·33	3·0303	·1499	·0580	2·569
·32	3·1250	·1239	·0436	2·614
·31	3·2258	·0976	·0305	2·652
·30	3·3333	·0709	·0189	2·680
·29	3·4483	·0441	·00926	2·698
·28	3·5714	·0171	·00223	2·706
q = ·2737	3·6536	·0000	·00000	2·707

TABLE II.

x	r	$\theta_{2\cdot5}$	$\theta_{2\cdot5}^{2\cdot5}$	$\theta'_{2\cdot5}$
∞	0	1·00000	1·00000	0
100	·0100	·99998	·99994	$3{\cdot}33 \times 10^{-7}$
50	·0200	·99993	·99983	$2{\cdot}67 \times 10^{-6}$
25	·0400	·99973	·99933	$2{\cdot}13 \times 10^{-4}$
10	·1000	·99834	·99584	·000332
8	·1250	·99740	·99351	·000659
6	·1667	·99539	·98851	·00153
5	·2000	·99337	·98350	·00264
4	·2500	·98966	·97436	·00513
3	·3333	·98174	·95496	·01201
2	·5000	·95960	·90203	·03917
1	1·0000	·85194	·66992	·26287
·98	1·0204	·84655	·65937	·27671
·96	1·0417	·84087	·64837	·29148
·94	1·0638	·83488	·63689	·30726
·92	1·0870	·82857	·62493	·32414
·90	1·1111	·82192	·61245	·34218
·88	1·1364	·81488	·59943	·36150
·86	1·1628	·80745	·58585	·38219
·84	1·1905	·79959	·57184	·40437
·82	1·2195	·79127	·55695	·42816
·80	1·2346	·78246	·54157	·45368
·78	1·2821	·77312	·52555	·48109
·76	1·3158	·76321	·50887	·51052
·74	1·3514	·75269	·49153	·54214
·72	1·3889	·74152	·47348	·57613
·70	1·4286	·72963	·45474	·61266
·68	1·4706	·71700	·43531	·65193
·66	1·5152	·70355	·41518	·69415
·64	1·5625	·68922	·39436	·73951
·62	1·6129	·67395	·37289	·78822
·60	1·6667	·65767	·35077	·84050
·58	1·7241	·64032	·32808	·89654
·56	1·7857	·62179	·30487	·95652
·54	1·8519	·60203	·28122	1·02058
·52	1·9231	·58094	·25724	1·08883
·50	2·0000	·55845	·23306	1·16130
·48	2·0833	·53447	·20883	1·23805
·46	2·1739	·50890	·18475	1·31868
·44	2·2727	·48170	·16104	1·4029
·42	2·3810	·45277	·13794	1·4903
·40	2·5000	·42206	·11573	1·5801
·38	2·6316	·38956	·09472	1·6709
·36	2·7778	·35524	·07521	1·7613
·34	2·9412	·31912	·05753	1·8493
·32	3·1250	·28129	·04197	1·9326
·30	3·3333	·24187	·02877	2·0084
·29	3·4483	·22161	·02312	2·0425
·28	3·5714	·20103	·01812	2·0736
·27	3·7037	·18015	·01377	2·1013
·26	3·8462	·15901	·01008	2·1254
·25	4·0000	·13766	·007031	2·1454
·24	4·1667	·11612	·004595	2·1613
·23	4·3478	·09444	·002741	2·1731
·22	4·5455	·07267	·001424	2·1813
·21	4·7619	·05083	·000583	2·1858
·20	5·0000	·02896	·000143	2·1876
·19	5·2632	·00708	·0000042	2·1879
η=·18676	5·3545	·00000	·00000000	2·1880

TABLE III.

x	r	θ_3	θ_3^3	θ'_3
∞	0	1·0000	1·0000	0
100	·010	1·0000	1 0000	$3·33 \times 10^{-7}$
50	·020	·9999	·9998	$2·66 \times 10^{-6}$
25	·040	·9997	·9992	$2·13 \times 10^{-5}$
10	·100	·9983	·9949	$3·32 \times 10^{-4}$
5	·200	·9934	·9803	$2·63 \times 10^{-3}$
4	·250	·9897	·9694	$5·11 \times 10^{-3}$
3	·333	·9818	·9464	·0129
2	·500	·9598	·8842	·0386
1	1·000	·855	·625	·252
·875	1·143	·818	·547	·348
·750	1·333	·766	·450	·493
·625	1·600	·692	·331	·714
·500	2·000	·583	·198	1·046
·375	2·667	·425	·0768	1·497
·250	4·000	·209	·00913	1·922
·240	4·167	·190	·00685	1·944
·230	4·348	·170	·00495	1·963
·220	4·546	·151	·00342	1·979
·210	4·762	·131	·00224	1·993
·200	5·000	·111	·00136	2·002
·190	5·263	·0908	·000748	2·010
·180	5·556	·0707	·000353	2·014
·170	5·882	·0505	·000129	2·016
·160	6·250	·0303	·0000279	2·017
·150	6·667	·0102	·00000166	2·0175
q = ·145	6·897	·0000	·00000000	2·0175

TABLE IV.

x	r	θ_4	θ_4^4	θ'_4
∞	0	1·0000	1·0000	0
100	·010	1·0000	1·0000	$3·33 \times 10^{-7}$
50	·020	·9999	·9997	$2·66 \times 10^{-6}$
25	·040	·9997	·9989	$2·13 \times 10^{-5}$
10	·100	·9983	·9934	$3·32 \times 20^{-4}$
5	·200	·9934	·9737	$2·625 \times 10^{-3}$
4	·250	·9897	·9595	$5·070 \times 10^{-3}$
3	·333	·9819	·9295	$1·180 \times 10^{-2}$
2	·500	·9603	·8505	$3·780 \times 10^{-2}$
1	1·000	·8604	·548	·2327
·875	1·143	·8265	·467	·3157
·750	1·333	·780	·370	·437
·625	1·600	·715	·261	·615
·500	2·000	·623	·151	·872
·375	2·667	·493	·0591	1·214
·250	4·000	·318	·0102	1·582
·200	5·000	·236	·00310	1·697
·190	5·263	·219	·00230	1·715
·180	5·556	·202	·00165	1·731
·170	5·882	·184	·00115	1·746
·160	6·250	·167	·000773	1·759
·150	6·667	·149	·000494	1·770
·140	7·143	·131	·000298	1·778
·130	7·692	·114	·000166	1·785
·120	8·333	·0957	·0000837	1·790
·110	9·09	·0777	·0000358	1·793
·100	10·00	·0598	·0000128	1·795
·090	11·11	·0418	·00000306	1·796
·080	12·50	·0239	·000000325	1·797
·070	14·29	·0069	·000000002	1·797
q = ·0667	14·99	·0000	·000000000	1·797

TABLE V.

x	r	$\Psi(x)$	$\Psi'(x)/\Psi(x)$
∞	0	1·0000	0
100	·010	·99998	$3·33 \times 10^{-7}$
50	·020	·99993	$2·67 \times 10^{-6}$
25	·040	·9997	$2·13 \times 10^{-5}$
10	·100	·9984	·000333
5	·200	·9934	·002656
2	·500	·9597	·04065
1	1·000	·8531	·3029
·8	1·250	·7851	·5625
·6	1·667	·6649	1·204
·5	2·000	·5713	1·895
·4	2·500	·4462	3·27
·3	3·333	·2907	5·71
·25	4·000	·2076	7·91
·245	4·082	·1994	8·18
·240	4·167	·1913	8·46
·235	4·255	·1833	8·75
·230	4·348	·1753	9·06
·225	4·444	·1673	9·38
·220	4·545	·1595	9·71
·215	4·651	·1518	10·06
·210	4·762	·1443	10·42
·205	4·878	·1368	10·80
·200	5·000	·1295	11·20
·195	5·128	·1223	11·61
·190	5·263	·1153	12·04
·185	5·405	·1085	12·50
·180	5·556	·1017	12·97
·175	5·714	·0952	13·47
·170	5·882	·0889	13·99
·165	6·061	·0828	14·53
·160	6·250	·0769	15·10
·155	6·452	·0712	15·71
·150	6·667	·0657	16·34
·145	6·897	·0605	17·01
·140	7·143	·0554	17·71
·135	7·407	·0506	18·45
·130	7·692	·0461	19·23
·125	8·000	·0418	20·06
·120	8·333	·0377	20·95
·115	8·696	·0339	21·89
·110	9·091	·0303	22·88
·105	9·524	·0269	23·95
·100	10·000	·0238	25·10

(*Issued separately May* 6, 1908).

XVII.—The Theory of Skew Determinants in the Historical Order of Development up to 1865. By Thomas Muir, LL.D.

(MS. received November 18, 1907. Read December 2, 1907.)

My last communication in reference to the history of skew determinants dealt with the period 1827–1857 (*Proc. Roy. Soc. Edin.*, xxiii. pp. 181–217). The present paper continues the history up to the year 1865, but in addition contains an account of two writings belonging to the previous period, namely, by Brioschi (1855) and Bellavitis (1857).

Brioschi (1855, March).

[Sur l'analogie entre une classe de déterminants d'ordre pair et les déterminants binaires. *Crelle's Journal*, lii. pp. 133–141. See also *Annali di sci. mat. e fis.*, vi. pp. 430–432.]

After explaining that his purpose is to generalise a result of Hermite's (*Comptes rendus . . . Acad. des Sci.*, Paris, xl. pp. 249–254) regarding determinants of the fourth order, Brioschi sets out by establishing a necessary lemma regarding determinants of any even order whatever. It is this lemma which is of importance to us in the present connection. Taking the determinant

$$\sum(\pm a_{11}\, a_{22} \,.\,.\,.\,.\, a_{2m,2m}), \quad \text{or A say,}$$

he multiplies it by the equivalent determinant

$$\begin{vmatrix} a_{12} & -a_{11} & a_{14} & -a_{13} & \cdots\cdots & a_{1,2m} & -a_{1,2m-1} \\ a_{22} & -a_{21} & a_{24} & -a_{23} & \cdots\cdots & a_{2,2m} & -a_{2,2m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot \\ a_{2m,2} & -a_{2m,1} & a_{2m,4} & -a_{2m,3} & \cdots\cdots & a_{2m,2m} & -a_{2m,2m-1} \end{vmatrix}$$

obtaining the result

$$A^2 = \begin{vmatrix} \cdot & l_{12} & l_{13} & \cdots\cdots & l_{1,2m} \\ l_{21} & \cdot & l_{23} & \cdots\cdots & l_{2,2m} \\ l_{31} & l_{32} & \cdot & \cdots\cdots & l_{3,2m} \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot \\ l_{2m,1} & l_{2m,} & l_{2m,3} & \cdots\cdots & \cdot \end{vmatrix}$$

where

$$l_{rs} \equiv a_{r1}a_{s2} - a_{r2}a_{s1} + a_{r3}a_{s4} - a_{r4}a_{s3} + \cdots\cdots$$

and where therefore

$$l_{rs} = -l_{sr}.$$

Using then Cayley's theorem regarding a determinant which is "gauche symétrique," he concludes that A is expressible as a rational function of the l's. This result he might have put in the form *Any even-ordered determinant is expressible as a Pfaffian*: and at a later date it would have been written

$$\sum(\pm a_{11}a_{22}\dots a_{2m,2m}) = \left| \begin{array}{ccccc} l_{12} & l_{13} & l_{14} & \dots & l_{1,2m} \\ & l_{23} & l_{24} & \dots & l_{2,2m} \\ & & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ & & & & l_{2m-1,2m} \end{array} \right|.$$

The rest of the paper is occupied with the consideration of the special case where

$$l_{12} = l_{34} = l_{56} = \cdots\cdots = l_{2m-1,2m}$$

and all the other l's vanish.

BELLAVITIS (1857).

[Sposizione elementare della teorica dei determinanti. *Memorie* . . . *Istituto Veneto* . . . vii. pp. 67–144.]

For the determinant which Cayley named "gauche," Bellavitis introduces the term *pseudosimmetrico*, and for "gauche symétrique" he introduces *emisimmetrico* (§ 41). In the matter of notation also he suggests a change, denoting (§ 54) the Pfaffian which is the square root of

$$\left| \begin{array}{cccc} 0 & b_a & c_a & d_a \\ a_b & 0 & c_b & d_b \\ a_c & b_c & 0 & d_c \\ a_d & b_d & c_d & 0 \end{array} \right|$$

by

$$\text{Pf.}(a, b, c, d).$$

Nothing else is worth noting save the carefulness of the exposition (§§ 51–54, 59). Part of Cayley's theorem regarding a bordered skew symmetric determinant appears as a theorem regarding a non-coaxial primary minor of a skew symmetric determinant (§ 59).

SCHEIBNER, W. (1859, July).

[Ueber Halbdeterminanten. *Berichte . . . Gess. d. Wiss. zu Leipzig: Math.-phys. Cl.*, xi. pp. 151–159.]

This paper is not put forward by its author as containing new matter, being in fact such an exposition of the theory of Pfaffians as would suitably have formed a chapter, and a good one, of a text-book like Brioschi's or Baltzer's.

From the vanishing of a zero-axial skew determinant of odd order Scheibner reaches the already known fact that the product of two of its coaxial primary minors is equal to the square of a non-coaxial primary minor. In a quite fresh manner it is then shown that the square root of this is a rational and integral expression (Pfaffian), whose law of formation is thereafter established. Naturally following on this comes the proof (p. 156) that each of the non-coaxial primary minors is the product of two Pfaffians, the result being written in the form

$$A_{pq} = (p+1, \ldots, 2_m, 0, \ldots, p-1)(q+1, \ldots, 2_m, 0, \ldots, q-1),$$

where the suffixes of the elements of the original determinant are 0, 1, . . . , $2m$. On a later page (p. 158) it is shown that a similar proposition holds when the original determinant is of even order, namely,

$$A_{pq} = (-1)^p(0, 1, 2, \ldots, 2_{m+1})(q+1, \ldots, p-1, p+1, \ldots, q-1).$$

Cayley's theorem regarding a "bordered" skew symmetric determinant thus appears broken up into two parts.

The paper concludes with the suggestions that a skew symmetric determinant should be called a *Wechseldeterminante*, that its square root should be called a *Halbdeterminante*, and that the latter should be denoted by

$$\left| \begin{matrix} a_{01} & a_{02} & a_{03} & \cdots & a_{0p} \\ & a_{12} & a_{13} & \cdots & a_{1p} \\ & & a_{23} & \cdots & a_{2p} \\ & & & \cdots\cdots & \\ & & & & a_{p-1,p} \end{matrix} \right|,$$

an expression which would thus be an alternative for $(0, 1, 2, \ldots, p)$, and which would vanish when p was even.

SOUILLART, C. (1860, Sept.).

[Note sur la question 405 et sur une composition de carrés. *Nouv. Annales de Math.*, xix. pp. 320–322.]

Souillart's subject is the skew determinant

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix},$$

and his observations are (1) that it is equal to

$$(a^2+b^2+c^2+d^2)^2,$$

and (2) that if it be multiplied by the similar determinant which is equal to

$$(p^2+q^2+r^2+s^2)^2$$

the result is a determinant of the same form, whether the multiplication be row-by-row or column-by-column. The object, of course, is to prove Euler's theorem* that the product of two sums of four squares is a sum of four squares.

CAYLEY, A. (1860, Dec.).

[Note on the theory of determinants. *Philos. Magazine*, xxi. pp. 180–185: or *Collected Math. Papers*, v. pp. 45–49.]

After expounding his, or rather Cauchy's last, mode of partitioning the ordinary expansion of a determinant, and giving his own diagrammatic representation of the partition, Cayley applies it to the expansion of a zero-axial skew determinant, showing, of course, that when of odd order it vanishes, and that when of even order it is expressible as a rational integral function of the elements.

TRUDI, N. (1862).

[TEORIA DE' DETERMINANTI, E LORO APPLICAZIONI, di Nicola Trudi: xii.+268 pp., Napoli.]

To "determinanti gobbi" Trudi devotes sixteen pages (pp. 78–94) of his text-book, the exposition, which is not a little influenced by Brioschi and Baltzer, being full and simple. There are only one or two points in it

* *Novi Commentarii Acad. Petropolitanae*, xv. (1770), pp. 75–106. For the conclusion reached see *Nouv. Annales de Math.*, xv. pp. 403–407.

worth noting. In the first place, there is his opening proposition that *in any zero-axial skew determinant, conjugate minors, if of even order, are equal, and if of odd order differ only in sign*: this is a slight generalisation of a previously known result. In the second place (p. 80), Jacobi's general theorem

$$\begin{vmatrix} A_{rr} & A_{rs} \\ A_{sr} & A_{ss} \end{vmatrix} = \Delta \times \text{compl. minor of} \begin{vmatrix} a_{rr} & a_{rs} \\ a_{sr} & a_{ss} \end{vmatrix}$$

is applied to the case where Δ is a zero-axial skew determinant of *even* order, Δ_{2m} say, and where, therefore, $A_{rr} = 0 = A_{ss}$ and $A_{sr} = -A_{rs}$, and the said complementary minor is a determinant of the same kind as Δ_{2m} but of the order $2m-2$: and it is thus seen that if Δ_{2m-2} be a square, so also must Δ_{2m}. The use to which this is put is evident.

JANNI, G. (1863).

[Teorica di determinanti simmetrici gobbi. *Giornale di Mat.*, i. pp. 275–278.]

Janni's final result is a troublesome rule for finding the expression whose square is a skew determinant of even order, the line of thought, so far as it goes, being similar to Scheibner's (1859).

CREMONA, L. (1864).

D'OVIDIO, TORELLI, MAGNI (1865).

[Quistione 32. *Giornale di Mat.*, ii. p. 62; iii. pp. 5–7, 7–10, 10–14.]

Cremona's theorem is that *if* Δ *be a skew determinant having its diagonal elements* $a_{11}, a_{22}, \ldots, a_{nn}$ *each equal to* z, *then the product of any two rows or any two columns of the adjugate determinant contains* Δ *as a factor, and the determinant of the* n^2 *cofactors equals* Δ^{n-2}. Proofs are given by E. D'Ovidio, G. Torelli, and A. Magni; but the second alone need be attended to here, as the two others are less direct, being connected with the subject of orthogonal substitution.

Starting with the known result

$$a_{r1}A_{s1} + \cdots + a_{rs}A_{ss} + \cdots + a_{rn}A_{sn} \left.\begin{aligned} &= \Delta \quad \text{when } s = r \\ &= 0 \quad \text{when } s \neq r \end{aligned}\right\}$$

Torelli by subtraction of $2a_{rs}A_{ss}$ and change of signs obtains

$$a_{r1}A_{1s} + \cdots + a_{rs}A_{ss} + \cdots + a_{rn}A_{ns} \left.\begin{aligned} &= -\Delta + 2zA_{rs} \quad \text{when } s = r \\ &= 2zA_{rs} \qquad\quad \text{when } s \neq r \end{aligned}\right\}$$

and thence by addition, whatever s may be,

$$a_{r1}(A_{s1}+A_{1s})+ \cdots + a_{rn}(A_{sn}+A_{ns}) = 2zA_{rs}.$$

Writing ω_{sr} for $(A_{sr}+A_{rs})\div 2z$ he thus has the set of n equations in ω_{s1}, $\omega_{s2}, \ldots, \omega_{sn}$,

$$\left.\begin{array}{l} a_{11}\omega_{s1}+ \cdots + a_{1n}\omega_{sn} = A_{1s}, \\ a_{21}\omega_{s1}+ \cdots + a_{2n}\omega_{sn} = A_{2s}, \\ \cdots\cdots\cdots\cdots \\ a_{n1}\omega_{s1}+ \cdots + a_{nn}\omega_{sn} = A_{ns}. \end{array}\right\}$$

the peculiarity of which is that the right-hand members are cofactors of a column of elements of the determinant formed from the coefficients on the left. The solution is thus

$$\omega_{sr} = \frac{A_{1s}A_{1r}+A_{2s}A_{2r}+ \cdots + A_{ns}A_{nr}}{\Delta},$$

whence

$$\left(A_{1s}, A_{2s}, \ldots, A_{ns}\right)\!\!\left(A_{1r}, A_{2r}, \ldots, A_{nr}\right) = \frac{A_{sr}+A_{rs}}{2z}\Delta,$$

as desired. Using this n^2 times, he, of course, obtains for the square of the adjugate the expression

$$\Delta^n . \begin{vmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1n} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \omega_{n1} & \omega_{n2} & \cdots & \omega_{nn} \end{vmatrix},$$

and, it being known otherwise that the square of the adjugate is Δ_2^{n-2}, it follows that

$$| \omega_{11} \quad \omega_{22} \quad \cdots \quad \omega_{nn} | = \Delta^{n-2},$$

which is the other result wanted.

In regard to the elements ω one fact is noted, and is worth noting. Since A_{sr} may be expressed as an aggregate of terms in $z^0, z^1, z^2, \ldots,$ namely, say

$$A_{sr} = \Theta_0 + \Theta_1 z + \Theta_2 z^2 + \cdots$$

and since A_{rs} is got from A_{sr} by altering the signs of all the $(n-1)^2$ elements and then changing $-z$ into z, there results when n is even

$$A_{sr}+A_{rs} = 2\Theta_1 z + 2\Theta_3 z^3 + \cdots;$$

in other words, $A_{sr}+A_{rs}$ is then divisible by $2z$.

Two "observations" are added, the first in regard to the case where $z=0$, and the second in regard to an alternative proof of the first part of the foregoing. The latter is interesting in that the expression for $(A_{sr}+A_{rs})\,\Delta$ is not found at once as a whole, but is viewed as consisting of two parts corresponding to $A_{sr}\Delta$ and $A_{rs}\Delta$, the reason being the known existence* of a general theorem of determinants to the effect that if the product of $| a_{11} \ldots a_{nn} |$ and $| b_{11} \ldots b_{nn} |$, obtained in row-by-row fashion, be $| c_{11} \ldots c_{nn} |$, then

$$A_{rs} \cdot | b_{11} \ldots\ldots b_{nn} | = b_{1s}C_{r1} + \cdots + b_{ns}C_{rn}.$$

This is seen to be immediately applicable on making $| a_{11} \ldots a_{nn} |$ identical with Δ above and the b's identical with the a's, and it, of course, implies that if the product obtained in column-by-column fashion be $| c'_{11} \ldots c'_{nn} |$, then

$$A_{sr} \cdot | b_{11} \ldots\ldots b_{nn} | = b_{s1}C'_{1r} + \cdots + b_{sn}C'_{nr}.$$

Making the said necessary specialisations and noting that the two differently formed axisymmetric products are then identical (in other words, that $C_{rs} = C_{sr} = C'_{rs} = C'_{sr}$), Torelli obtains

$$A_{rs} \cdot \Delta = a_{1s} \cdot \sum_{i=1}^{i=n} A_{ri} A_{1i} + \cdots + a_{ss} \sum_{i=1}^{i=n} A_{ri} A_{si} + \cdots + a_{ns} \sum_{i=1}^{i=n} A_{ri} A_{ni},$$

$$A_{sr} \cdot \Delta = a_{s1} \cdot \sum_{i=1}^{i=n} A_{ri} A_{1i} + \cdots + a_{ss} \sum_{i=1}^{i=n} A_{ri} A_{si} + \cdots + a_{sn} \sum_{i=1}^{i=n} A_{ri} A_{ni};$$

whence by addition

$$(A_{rs} + A_{sr})\Delta = 2a_{ss} \sum_{i=1}^{i=n} A_{ri} A_{si}.$$

CAYLEY, A. (1865, Oct.).

[A supplementary memoir on the theory of matrices. *Philos. Trans. Roy. Soc.* (London), clvi. pp. 25–35: or *Collected Math. Papers*, v. pp. 438–448.]

The expression of an even-ordered determinant, Δ_{2m}, as a Pfaffian being necessary for the second of the two investigations contained in his paper, Cayley effects the transformation (§§ 15–17) in substantially the same way as that devised by Brioschi ten years previously, the one point of difference being that the form of Δ_{2m} which is employed as a multiplier is got from Δ_{2m} by reversing the order of the columns and then changing the signs of the elements in the last m columns. Thus, Δ_4 being

$$\begin{matrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{matrix},$$

* Rubini's *Elementi d'Algebra* (1866), p. 277, is referred to.

the square found for it by row-by-row multiplication is, in Cayley's notation,

	$(d, c, -b, -a)$	$(h, g, -f, -e)$	$(l, k, -j, -i)$	$(p, o, -m, -n)$
(a, b, c, d)	"	"	"	"
(e, f, g, h)	"	"	"	"
(i, j, k, l)	"	"	"	"
(m, n, o, p)	"	"	"	"

which is readily seen to be zero-axial skew.

Another expression is, of course, got by treating the conjugate of Δ_4 in the same manner.

Brioschi's paper of 1855 is not referred to.

Horner, Jos. (1865, Oct.).

[Notes on determinants. *Quart. Journ. of Math.*, viii. pp. 157–162.]

The second of Horner's three notes consists of a fresh proof that a zero-axial skew determinant of even order, Δ_{2m} say, is the square of a rational function of the elements.

Δ_{2m} multiplied by the square of the product of the non-zero elements of the first row is evidently equal to a zero-axial skew determinant of the same order, Δ'_{2m} say, having 0, 1, 1, . . . , 1 for its first row. But by performing in order the operations which we may conveniently specify by

$$\mathrm{row}_{2m} - \mathrm{row}_{2m-1}, \quad \mathrm{row}_{2m-1} - \mathrm{row}_{2m-2}, \quad \ldots, \quad \mathrm{row}_3 - \mathrm{row}_2,$$
$$\mathrm{col}_{2m} - \mathrm{col}_{2m-1}, \quad \mathrm{col}_{2m-1} - \mathrm{col}_{2m-2}, \quad \ldots, \quad \mathrm{col}_3 - \mathrm{col}_2,$$

it is seen that for Δ'_{2m} we may substitute a zero-axial skew determinant of the next lower even order, Δ_{2m-2} say. The factor thus shown to connect Δ_{2m} and Δ_{2m-2} being a square, the little that needs to be added is evident.

LIST OF AUTHORS

whose writings are herein dealt with.

(*Issued separately May* 13, 1908.)

XVIII.—Sunset and Twilight Curves, and Related Phenomena. By D. M. Y. Sommerville. *Communicated by* Professor Dyson. (With Three Plates.)

(MS. received February 22, 1908. Read May 4, 1908.)

§ 1. The objects of the present paper are, first, to describe certain curves which approximate to the graphs of the time of sunset (or sunrise) and end of twilight (or daybreak) all the year round for various latitudes; second, to tabulate the yearly phenomena of light and darkness for different latitudes under various conditions.

The time of sunset depends upon the latitude and the sun's declination,

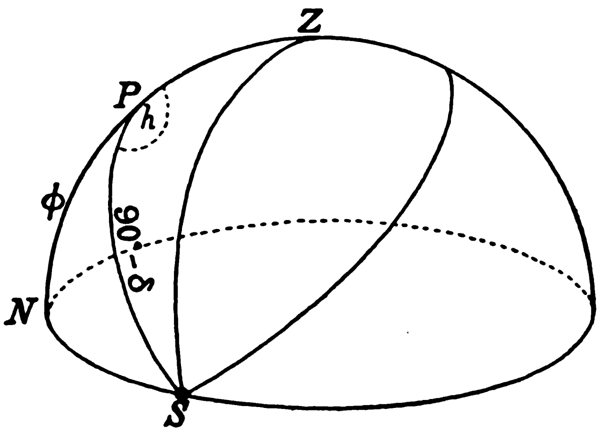


Fig. 1.

and the extent of its variations depends upon the obliquity of the ecliptic. The extent of the variations of twilight depends also upon a certain angle a, which is the maximum depression of the sun's centre below the horizon for which the light reflected by the upper strata of the atmosphere is sensible. The investigations will be conducted, in the first place, on the assumption of the present value of the obliquity $\epsilon = 23° 27'$, roughly, and the observed value of $a = 18°$. We shall then extend the investigations by assuming both ϵ and a as arbitrary within the range 0° to 90°.

I. The Sunset-Curves.

§ 2. Let Z be the zenith, P the North Pole, S the sun on the horizon, and N the north point. Then NP is the latitude = ϕ, PS is the sun's north polar distance = $90° - \delta$ (δ being the declination, north), and ZPS = h is the sun's hour-angle.

Then, in the right-angled triangle SNP,

$$\cos h = -\tan\delta\tan\phi .$$

h, expressed in hours, is the local apparent time of sunset, *i.e.* the number of hours that have elapsed when the sun's centre is on the horizon since its passage across the meridian. If E is the equation of time, *i.e.* the mean time of meridian transit, $h+E$ is the local mean time of sunset. Finally, if λ is the west longitude of the place, expressed in hours, $h+E+\lambda$ is the Greenwich mean time of sunset. A correction is usually made for refraction, r. At the equator r is about 2 minutes, and at latitude 60° it varies from 4 minutes at the equinoxes to 7 minutes at the solstices. The G.M.T. of apparent sunset is then $h+E+\lambda+r$, and the G.M.T. of

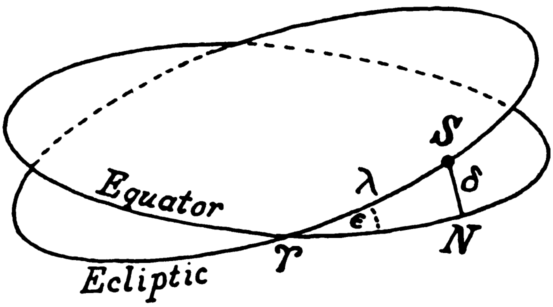


FIG. 2.

apparent sunrise is $12-h+E+\lambda-r$. The time of sunset corrected for refraction may also be calculated by the equation for twilight putting $a=34'$, the amount of mean horizontal refraction. Parallax may be neglected (though it would require to be taken into account in the case of the moon).

§ 3. In order to discuss the curves which represent the graph of the time of sunset for any place all the year round, it is desirable to get an approximate equation. For this we require an expression for the sun's declination at any time.

Let S (fig. 2) be the sun's position at any time, and project S upon the equator. Then NS = δ, ♈S is the sun's longitude = λ, and the angle N♈S = ϵ.

We have then

$$\sin\delta = \sin\epsilon\sin\lambda .$$

If we neglect the eccentricity of the earth's orbit the sun's motion in longitude will be uniform and λ will be proportional to the "equinoctial time."

Putting, therefore, $\lambda = x$, we have the two equations

$$\cos y = -\tan\phi\tan\delta\,,$$
$$\sin\delta = \sin\epsilon\sin x\,,$$

which, when δ is eliminated, give the equation of a curve of approximate sunset (apparent time) referred to rectangular axes.*

A rougher approximation, which leads to simpler results, can be obtained by supposing N to move uniformly. Then, putting ♈N $= x$, we get

$$\tan\delta = \tan\epsilon\sin x\,,$$

and the equation of the curve assumes the simple form

$$\cos y = -\tan\phi\tan\epsilon\sin x\,.$$

§ 4. It will be sufficient, in the first place, to take the latter approximation. We have then the equation

$$\cos y = k\sin x\,.$$

This is the equation of a repeating curve, and the portion contained between $x = -\frac{\pi}{2}$ and $\frac{3\pi}{2}$ and $y = 0$ and π represents approximately the graph of the time of sunset for one year from one winter solstice to another.

The form of the curve depends upon the value of k. $k = \pm 1$ divides the family of curves into two classes.

For $k = \pm 1$

$$\cos y = \pm\sin x = \cos\left(\frac{\pi}{2} \mp x\right);$$

$$\text{hence} \qquad \frac{\pi}{2} \mp x = 2n\pi \pm y\,,$$

and we have two systems of straight lines

$$x \pm y = n\pi + \frac{\pi}{2}\,.$$

Let $|k| < 1$.

$$\text{When} \qquad x = n\pi\,, \qquad y = m\pi + \frac{\pi}{2}\,,$$

$$x = 2n\pi + \frac{\pi}{2}\,,\ y = 2m\pi \pm \cos^{-1}k\,,$$

* The effect of the equation of time, reducing apparent time to mean time, will be most marked in low latitudes, and it will produce a skewness in the curve, displacing the maximum towards the autumnal equinox and, if the latitude is sufficiently small, producing another maximum a little before the vernal equinox. The minimum will be intensified and displaced also towards the autumnal equinox. In what follows we shall consider only apparent time.

$$x = 2n\pi - \frac{\pi}{2},\ y = (2m+1)\pi \pm \cos^{-1} k\,;$$

y cannot lie between $m\pi \pm \cos^{-1}|k|$.

The curve therefore consists of an infinite number of repetitions of sinuous curves lying between the pairs of parallels $y = m\pi - \cos^{-1}|k|$ and $y = (m-1)\pi + \cos^{-1}|k|$, and the concavities and convexities of consecutive branches are opposed.

Let $|k| > 1$.

When $x = n\pi,\ y = m\pi + \frac{\pi}{2}$,

$$y = 2m\pi,\ x = n\pi + (-)^n \sin^{-1}\frac{1}{k},$$

$$y = (2m+1)\pi,\ x = n\pi - (-)^n \sin^{-1}\frac{1}{k}\,;$$

x cannot lie between $n\pi + \sin^{-1}\frac{1}{|k|}$ and $(n+1)\pi - \sin^{-1}\frac{1}{|k|}$.

The curve therefore consists of an infinite number of repetitions of sinuous curves lying between the pairs of parallels $x = n\pi \pm \sin^{-1}\frac{1}{k}$, with the concavities and convexities of consecutive branches opposed.

Another special case occurs when $\phi = 0$; then

$$\cos y = 0 \quad \text{or} \quad y = n\pi + \frac{\pi}{2}.$$

Also if $\phi = 90°$,

$$\sin x = 0 \quad \text{or} \quad x = n\pi.$$

§ 5. With the other equation the only essential difference is in the form of the curve in the limiting case $|k| = 1$.

We have

$$\frac{dy}{dx} = \frac{\tan\phi \sec^2\delta \cos x}{\operatorname{cosec}\epsilon \cos\delta \sin y}.$$

Putting $\phi = 90° - \epsilon$, $\delta = \epsilon$, $x = \frac{\pi}{2}$, $y = \pi$, and evaluating the limit, we get $\frac{dy}{dx} = \sec\epsilon = \tan 47°\ 28'$. The same value is obtained when $\delta = -\epsilon$, $x = -\frac{\pi}{2}$, $y = 0$; while when $\delta = 0$, $x = 0$, $y = \frac{\pi}{2}$, we get $\frac{dy}{dx} = \cos\epsilon = \tan 42°\ 32'$. So that the curve does not differ very greatly from a straight line.

§ 6. Corresponding to these complete curves, we have the curves of sunset. The limiting case for north latitudes is $k = -1$, or $\tan\phi \tan\epsilon = 1$, which gives $\phi = 90° - \epsilon$,—*i.e.* the Arctic circle. For the equator $\phi = 0$, giving a straight line, $y=6$ hours; for the North Pole we have two vertical straight lines, $x=0$ and $x=\frac{1}{2}$ year, corresponding to the equinoxes. Outside the Arctic circle $\phi<90°-\epsilon$ and we have a horizontal sinuous curve with maximum at the summer solstice and minimum at the winter solstice. Within the Arctic circle the sinuous curve is vertical, cutting $y=0$ and $y=12$ hours at right angles, leaving open gaps at the solstices. These gaps correspond to a period of perpetual day in summer which continues so long as $\delta>90°-\phi$, and a period of perpetual night in winter which continues so long as $\delta<\phi-90°$. In every case the ordinate at the equinoxes is six hours.

For southern latitudes the phenomena are, of course, similar but reversed.

II. The Twilight-Curves.

§ 7. The twilight-curves are somewhat more complicated. It is found by observation that twilight lasts so long as the sun is not more than 18° below the horizon.

Here

$$ZS=108°,\quad PS=90°-\delta,\quad ZP=90°-\phi,\quad \text{and}\quad \angle ZPS=y.$$

Hence

$$\cos 108° = \sin\delta\sin\phi+\cos\delta\cos\phi\cos y,$$

$$\text{or}\qquad \cos y = -(\sin\delta\sin\phi+\sin 18°)\sec\delta\sec\phi,$$

where δ has to be expressed in terms of x by an equation such as

$$\sin\delta=\sin\epsilon\sin x.$$

We shall assume throughout that ϕ is positive. For southern latitudes the seasons are simply reversed, for the equation is unaltered if the signs of both ϕ and δ be reversed. We shall also write the general angle a instead of 18°, only putting in its value as it is required.

§ 8. We shall now determine some critical points in the curves, and in this way we shall obtain the critical values of ϕ which mark a change in the character of the curve. Some of these values of ϕ will be impossible for the actual values of a and ϵ. As they will afterwards be required in tabulating the general yearly phenomena of light and darkness under various conditions, it is convenient to notice them all here as they come out. The possible critical values of ϕ for $a = 18°$ and $\epsilon = 23°\ 27'$ will be given in numbers.

When $x=n\pi+\dfrac{\pi}{2}$, $\delta=(-)^n\epsilon$, and

$$\cos y = -\{(-)^n\sin\epsilon\sin\phi+\sin a\}\sec\epsilon\sec\phi.$$

y is imaginary if

$$|(-)^n \sin\epsilon \sin\phi + \sin a| > \cos\epsilon \cos\phi .$$

At summer solstice n is even, and y is imaginary if $\cos(\epsilon+\phi) < \sin a$,

$$\text{i.e. if } \phi > 90^\circ - \epsilon - a, \text{ i.e. } > 48^\circ\ 33'.$$

At winter solstice n is odd and there are two cases:

1. $\sin a > \sin\epsilon \sin\phi$.

   y is imaginary if $\cos(\phi-\epsilon) < \sin a$,

$$\text{i.e. if either } \phi > 90^\circ + \epsilon - a \quad \text{or} \quad \phi < \epsilon + a - 90^\circ.$$

2. $\sin a < \sin\epsilon \sin\phi$.

   y is imaginary if $\cos(\phi+\epsilon) < -\sin a$,

$$\text{i.e. if } \phi > 90^\circ - \epsilon + a, \text{ i.e. } \geq 84^\circ\ 33'.$$

When $y = 2n\pi$, $\cos y = 1$, $\cos(\phi-\delta) = \cos(90^\circ + a)$; therefore $\delta = \phi - 90^\circ - a$, which is impossible if $\phi < 90^\circ + a - \epsilon$, i.e. $< 84^\circ\ 33'$.

When $y = (2n+1)\pi$, $\cos y = -1$, $\cos(\phi+\delta) = \cos(90^\circ - a)$; therefore either $\delta = 90^\circ - a - \phi$, which is impossible if $90^\circ - a - \phi > \epsilon$, or $< -\epsilon$,

$$\text{i.e. if } \phi < 90^\circ - \epsilon - a, \text{ i.e. } < 48^\circ\ 33' \text{ or if } \phi > 90^\circ + \epsilon - a;$$

or $\delta = a - 90^\circ - \phi$, which is impossible if $a - 90^\circ - \phi < -\epsilon$,

$$\text{i.e. if } \phi > \epsilon + a - 90^\circ.$$

If $\phi < \epsilon + a - 90^\circ$ both values of δ are possible; if $\phi > 90^\circ + \epsilon - a$ both values are impossible. In the latter case $\cos(\phi+\delta)$ is always $< \sin a$ and y is always imaginary. For such values of ϕ the twilight-curve is therefore entirely imaginary.

§ 9. Consider now the slope of the curve.

$$\frac{dy}{dx} = \frac{\sin\epsilon \cos x \sec^3\delta \sec\phi}{\sin y}(\sin\phi + \sin a \sin\delta).$$

In general $\dfrac{dy}{dx} = \infty$ if $y = n\pi$. Exceptions, of course, occur when y cannot $= n\pi$. Also in general $\dfrac{dy}{dx} = 0$ when $x = n\pi + \dfrac{\pi}{2}$, and also when $\sin\phi + \sin a \sin\delta = 0$. The greatest value of δ is ϵ; hence if $\sin\phi \not> \sin\epsilon \sin a$, or $\phi \not> 7^\circ\ 4'$, $\dfrac{dy}{dx} = 0$ when

$$\sin x = -\frac{\sin\phi}{\sin a \sin\epsilon}.$$

Exceptions may arise if both numerator and denominator vanish together.

§ 10. Let $y=2n\pi$, $\delta=\phi-90^\circ-a$; therefore $\phi \nless 90^\circ+a-\epsilon$ i.e. $\nless 84^\circ\ 33'$. $\sin\phi+\sin a\sin\delta$ cannot vanish, for if

$$\sin(90^\circ+a-\epsilon) \ngtr \sin a\sin\epsilon,$$

we get $\cos a\cos\epsilon \ngtr 0$, which is impossible unless a or $\epsilon=90^\circ$.

Let $\cos x=0$, $x=m\pi+\frac{\pi}{2}$, $\delta=(-)^m\epsilon$.

Since δ is negative, m must be odd and $\delta=-\epsilon$, $\phi=90^\circ-\epsilon+a$.

To find the limiting value of $\frac{dy}{dx}$ put $x=(2n+1)\pi+\frac{\pi}{2}+\mu$, where μ is small.

Then

$$\cos x=\sin\mu=\mu,\ \sin x=-\cos\mu=-\left(1-\frac{\mu^2}{2}\right),$$

$$\sin\delta=-\sin\epsilon\left(1-\frac{\mu^2}{2}\right),\ \sec\delta=(1-\sin^2\delta)^{-\frac{1}{2}}=\sec\epsilon(1-\tfrac{1}{2}\mu^2\tan^2\epsilon),$$

$$\cos y=\left\{\sin\epsilon\sin\phi\left(1-\frac{\mu^2}{2}\right)-\sin a\right\}\sec\phi\sec\epsilon(1-\tfrac{1}{2}\mu^2\tan^2\epsilon),$$

$$=1-\frac{\mu^2}{2}\sec\phi\tan\epsilon\sec^2\epsilon(\sin\phi-\sin a\sin\epsilon),$$

$$\text{since}\qquad \phi+\epsilon=90^\circ+a;$$

$$\sin^2 y=\mu^2\sec\phi\tan\epsilon\sec^2\epsilon(\sin\phi-\sin a\sin\epsilon),$$

and
$$\frac{dy}{dx}=-\frac{\sin\epsilon.\mu.\sec^3\epsilon(\sin\phi-\sin a\sin\epsilon)}{\cos\phi.\mu.\sec\epsilon\sqrt{\{\sec\phi\tan\epsilon(\sin\phi-\sin a\sin\epsilon)\}}}$$

$$=\sqrt{\frac{\sin\epsilon\cos a}{\sin(\epsilon-a)}}\cdot\sec\epsilon=\tan 65^\circ\ 19'.$$

§ 11. Let $y=(2n+1)\pi$, $\delta=90^\circ-a-\phi$ or $a-90^\circ-\phi$.

For $\phi=90^\circ-a-\delta$, $\sin\phi+\sin a\sin\delta=\cos a\cos\delta$, and for $\phi=a-\delta-90^\circ$ it is $=-\cos a\cos\delta$.

Hence this expression cannot vanish except in the limiting cases $a=90^\circ$ or $\delta=90^\circ=\epsilon$.

Let $\cos x=0$, $x=m\pi+\frac{\pi}{2}$, $\delta=(-)^m\epsilon$.

If m is even, $\delta=\epsilon$ and $\phi=90^\circ-\epsilon-a$.

Evaluating the limit, we get

$$\frac{dy}{dx}=\sqrt{\frac{\sin\epsilon.\cos a}{\sin(\epsilon+a)}}\cdot\sec\epsilon=\tan 39^\circ\ 30'.$$

If m is odd, $\delta=-\epsilon$ and $\phi=90^\circ+\epsilon-a$ or $\epsilon+a-90^\circ$.

In the first case the twilight-curve is just vanishing and the points $x=2n\pi-\frac{\pi}{2}$, $y=(2m+1)\pi$ are acnodes.

Evaluating the limit when $\phi=\epsilon+\alpha-90°$, we get

$$\frac{dy}{dx}=\sqrt{\frac{\sin\epsilon.\cos\alpha}{\sin(\epsilon+\alpha)}}\cdot\sec\epsilon=\tan 39° 30'.$$

§ 12. The limiting case given by $\sin\phi=\sin\epsilon\sin\alpha$ deserves special investigation. Here the pairs of minima at $\sin x=\frac{-\sin\phi}{\sin\epsilon\sin\alpha}(\pi>y>0)$ coalesce with the interjacent maximum at $\sin x=-1$ to form a point of undulation which is an apparent minimum. It is easy to show that at this point the differential coefficients up to the fourth all vanish, while

$$\frac{d^4y}{dx^4}=3\tan\alpha\tan^2\epsilon\sec\epsilon.$$

§ 13. Let us now collect our results. Plate I. gives, within the limits $x=-\frac{\pi}{2}$ to $\frac{3\pi}{2}$, $y=0$ to π, the different forms of the twilight-curve in combination with the sunset-curve, the upper one being the twilight-curve.

There are seven forms, not including limiting forms. Their evolution is easiest to follow by considering ϕ and ϵ as constant and α as variable.

The critical values of α are

$$\sin^{-1}\left(\frac{\sin\phi}{\sin\epsilon}\right),\ 90°-\epsilon+\phi,\ 90°-\epsilon-\phi,\ \epsilon+\phi-90°,\ 90°+\epsilon-\phi.$$

$90°-\epsilon-\phi$ is only possible if $\phi<90°-\epsilon$,

$\epsilon+\phi-90°$,, ,, $\phi>90°-\epsilon$,

$90°+\epsilon-\phi$,, ,, $\phi>\epsilon$,

$90°-\epsilon+\phi$ and $\sin^{-1}\left(\frac{\sin\phi}{\sin\epsilon}\right)$ are only possible if $\phi<\epsilon$.

Also

$$\sin^{-1}\left(\frac{\sin\phi}{\sin\epsilon}\right)<90°-\epsilon-\phi \text{ if } \tan\phi<\frac{\tan\epsilon}{1+2\tan^2\epsilon},$$

while

$$90°-\epsilon+\phi>90°-\epsilon-\phi, \qquad 90°+\epsilon-\phi>90°-\epsilon-\phi,$$
$$>\sin^{-1}\left(\frac{\sin\phi}{\sin\epsilon}\right), \qquad >\epsilon+\phi-90°,$$
$$>\epsilon+\phi-90°, \qquad \sin^{-1}\left(\frac{\sin\phi}{\sin\epsilon}\right)>\epsilon+\phi-90°.$$

The critical values of ϕ are

$$\epsilon,\ 90°-\epsilon \text{ and } \tan^{-1}\frac{\tan\epsilon}{1+2\tan^2\epsilon},$$

and we have

$$\tan^{-1}\frac{\tan\epsilon}{1+2\tan^2\epsilon}<\epsilon,$$
$$<90°-\epsilon,$$
$$\epsilon<90°-\epsilon \text{ if } \epsilon<45°.$$

Hence we have the following cases :—

(1) $\phi < \tan^{-1}\dfrac{\tan\epsilon}{1+2\tan^2\epsilon}$.

The curve starts as fig. 1 and passes through the form with the points of undulation as a passes through the value $\sin^{-1}\dfrac{\sin\phi}{\sin\epsilon}$ into fig. 2. When a assumes the value $90°-\epsilon-\phi$ the curve becomes nodal and passes into fig. 5. When $a=90°-\epsilon+\phi$ it breaks up into figures of eight, and then becomes closed ovals as in fig. 6. When $a=90°$ the two values of δ, viz. $90°-a-\phi$ and $a-\phi-90°$, both become $-\phi$ and the ovals reduce to points.

These variations are illustrated in Plate II. for values of a positive and negative at intervals of 10° from $-90°$ to $+90°$, and for $\phi=10°$, $\epsilon=30°$. The curves for negative values of a are entirely similar to those for corresponding positive values, only displaced through π along both axes.

(2) $\phi > \tan^{-1}\dfrac{\tan\epsilon}{1+2\tan^2\epsilon}$, and $<\epsilon$ and $<90°-\epsilon$.

The variations are similar to the preceding case with fig. 4 substituted for 2, when $90°-\epsilon-\phi<a<\sin^{-1}\dfrac{\sin\phi}{\sin\epsilon}$.

(3) $\epsilon<\phi<90°-\epsilon$.

Here the curve, after assuming the form of fig. 4, vanishes when $a=90°+\epsilon-\phi$.

(4) $90°-\epsilon<\phi<\epsilon$.

The curve starts as fig. 14, and passes through a nodal form when $a=\phi+\epsilon-90°$ into 4, thence through 5 and 6.

(5) $\phi>\epsilon$ and $>90°-\epsilon$.

Here the curve starts with 14, passes through 4, and vanishes when $a=90°+\epsilon-\phi$.

These variations are illustrated in Plate III. for values of a positive and negative at intervals of 10° from $-50°$ to $+50°$ and for $\phi=70°$, $\epsilon=30°$.

§ 14. The case for $\epsilon=90°$ requires special treatment. Here $\sin\delta=\sin x$;

$$\begin{aligned} \delta &= x && \text{from } \Upsilon \text{ to S.S.,} \\ &= \pi - x && \text{from S.S. to W.S.,} \\ &= x-2\pi && \text{from W.S. to } \Upsilon. \end{aligned}$$

The sunset-curve becomes

$$\cos y = -\tan\phi\tan x \text{ when } x \text{ lies between } 2n\pi\pm\frac{\pi}{2};$$

$$\cos y = \tan\phi\tan x \text{ when } x \text{ lies between } (2n+1)\pi\pm\frac{\pi}{2}.$$

The curve is always a vertical sinuous curve.

For the twilight-curve, if a is positive, the critical values of a, $90°-\epsilon-\phi$ and $90°+\epsilon-\phi$, become impossible, and the others all become ϕ.

When $a=\phi$ the curve assumes a limiting form,

$$\cos y = -(1+\sin x)\sec x \tan a = -\tan a \cot\left(45° - \frac{x}{2}\right)$$

if $x<90°$. y is imaginary if $a>45°-\frac{x}{2}$, i.e. $x>90°-2a$, and when $x=-90°$, $y=90°$, so that the curve consists of closed ovals as in fig. 4.

The curve then starts as fig. 14, and when $a=\phi$ passes instantaneously through the two nodal forms and this limiting form into 6. Actually at the winter solstice in latitude $\phi=a$ the time at which twilight ends is quite indeterminate, just as at the equator the time of sunset is indeterminate, the sun being just on the horizon.

III. Yearly Phenomena of Light and Darkness.

§ 15. Let daylight, twilight, and true night be denoted by D, T, N respectively, and let a combination of these, such as DTN, denote the phenomena which occur from noon till midnight. There are six possible kinds of days: D, T, N, DT, TN, and DTN; DN being impossible unless $a=0$.

These are marked by the following critical phenomena:—

1. The sun does not rise if $\delta<\phi-90°$. This is only possible if $\phi>90°-\epsilon$.
2. The sun does not set if $\delta>90°-\phi$. This is only possible if $\phi>90°-\epsilon$.
3. There is no true night if $\delta<a-\phi-90°$ or $>90°-a-\phi$. The former is possible only if $\phi<\epsilon+a-90°$, the latter only if $\phi>90°-\epsilon-a$.
4. There is no twilight if $\delta<\phi-a-90°$. This is only possible if $\phi>90°-\epsilon+a$.
5. There is perpetual twilight while $90°-a-\phi<\delta<\phi-90°$. This is only possible if $\phi>90°-\frac{a}{2}$. Previous to attaining this value, ϕ must have become $>90°-\epsilon$, hence this is only possible if $a<2\epsilon$.
6. The twilight-curve has two minima instead of one if $\sin\phi<\sin\epsilon\sin a$.
7. The twilight-curve becomes imaginary if $\phi>90°+\epsilon-a$.

§ 16. There are fifteen different forms of yearly phenomena, and they are represented in Plate I. The following is a description of the diagrams, using the notation explained above :—

1, 2	DTN	always.
3	DT	always.
4, 5	DTN	$\delta<90^\circ-a-\phi$,
	DT	$\delta>90^\circ-a-\phi$.
6	DT	$\delta<a-\phi-90^\circ$,
	DTN	$a-\phi-90^\circ<\delta<90^\circ-a-\phi$,
	DT	$\delta>90^\circ-a-\phi$.
7, 8	TN	$\delta<\phi-90^\circ$,
	DTN	$\phi-90^\circ<\delta<90^\circ-a-\phi$,
	DT	$90^\circ-a-\phi<\delta<90^\circ-\phi$,
	D	$\delta>90^\circ-\phi$.
9	T	$\delta<\phi-90^\circ$,
	DT	$\phi-90^\circ<\delta<90^\circ-\phi$,
	D	$\delta>90^\circ-\phi$.
10, 11	TN	$\delta<90^\circ-a-\phi$,
	T	$90^\circ-a-\phi<\delta<\phi-90^\circ$,
	DT	$\phi-90^\circ<\delta<90^\circ-\phi$,
	D	$\delta>90^\circ-\phi$.
12	T	$\delta<a-\phi-90^\circ$,
	TN	$a-\phi-90^\circ<\delta<\phi-90^\circ$,
	DTN	$\phi-90^\circ<\delta<90^\circ-a-\phi$,
	DT	$90^\circ-a-\phi<\delta<90^\circ-\phi$,
	D	$\delta>90^\circ-\phi$.
13	T	$\delta<a-\phi-90^\circ$,
	TN	$a-\phi-90^\circ<\delta<90^\circ-a-\phi$,
	T	$90^\circ-a-\phi<\delta<\phi-90^\circ$,
	DT	$\phi-90^\circ<\delta<90^\circ-\phi$,
	D	$\delta>90^\circ-\phi$.
14	N	$\delta<\phi-90^\circ-a$,
	TN	$\phi-90^\circ-a<\delta<\phi-90^\circ$,
	DTN	$\phi-90^\circ<\delta<90^\circ-a-\phi$,
	DT	$90^\circ-a-\phi<\delta<90^\circ-\phi$,
	D	$\delta>90^\circ-\phi$.
15	N	$\delta<\phi-90^\circ-a$,
	TN	$\phi-90^\circ-a<\delta<90^\circ-a-\phi$,
	T	$90^\circ-a-\phi<\delta<\phi-90^\circ$,
	DT	$\phi-90^\circ<\delta<90^\circ-\phi$,
	D	$\delta>90^\circ-\phi$.

§ 17. The following criteria which follow from the subsequent investigations (see p. 334) are here tabulated for convenience.

$$\left.\begin{array}{r}\mathbf{6}\\\mathbf{12}\\\mathbf{13}\end{array}\right\}\ \phi<\epsilon+a-90^\circ,\ \begin{array}{c}<\\>\\ \ \end{array}90^\circ-\epsilon,\ \begin{array}{c}\ \\<\\>\end{array}90^\circ-\frac{a}{2}.$$

$$\left.\begin{array}{r}\mathbf{1}\\\mathbf{2}\end{array}\right\}\ \phi<90^\circ-\epsilon-a,\ \begin{array}{c}>\\<\end{array}\sin^{-1}(\sin\epsilon\sin a).$$

$$\left.\begin{array}{r}\mathbf{4}\\\mathbf{7}\\\mathbf{10}\\\mathbf{5}\\\mathbf{8}\\\mathbf{11}\end{array}\right\}\ \phi\left\{\begin{array}{l}>\epsilon+a-90^\circ\\>90^\circ-\epsilon-a\\<90^\circ+\epsilon-a\\<90^\circ-\epsilon+a\end{array}\right\}\begin{array}{c}\ \\>\\ \ \\ \ \\<\\ \ \end{array}\sin^{-1}(\sin\epsilon\sin a),\ \begin{array}{c}<\\>\\ \ \\<\\>\\ \ \end{array}90^\circ-\epsilon,\ \begin{array}{c}\ \\<\\>\\ \ \\<\\>\end{array}90^\circ-\frac{a}{2}.$$

$$\left.\begin{array}{r}\mathbf{3}\\\mathbf{9}\end{array}\right\}\ \phi>90^\circ+\epsilon-a,\ \begin{array}{c}<\\>\end{array}90^\circ-\epsilon.$$

$$\left.\begin{array}{r}\mathbf{14}\\\mathbf{15}\end{array}\right\}\ \phi>90^\circ-\epsilon+a,\ \begin{array}{c}<\\>\end{array}90^\circ-\frac{a}{2}.$$

They are further distinguished as follows:

1	$\alpha < \tan^{-1} \frac{1}{2} \cot \epsilon$,	2	$\alpha < 90^\circ - \epsilon$.
5	$\tan^{-1} \frac{1}{2} \cot \epsilon < \alpha < 180^\circ - 2\epsilon$,	6	$\alpha > 90^\circ - \epsilon$.
12	$\alpha > 180^\circ - 2\epsilon$.	7	$\alpha < 2 \sin^{-1} \frac{1}{2} \operatorname{cosec} \epsilon$ or $< 2\epsilon$, according as $\epsilon \gtrless 45^\circ$.
3	$\alpha > 2\epsilon$,		
10	$2\epsilon > \alpha > \frac{2}{3}\epsilon$,		
14	$\alpha < \frac{2}{3}\epsilon$.	11	$\alpha > 2 \sin^{-1} \frac{1}{2} \operatorname{cosec} \epsilon$.
4	$\alpha < \sin^{-1} \cot \epsilon$,	9	$\alpha > \epsilon$.
8	$\sin^{-1} \cot \epsilon < \alpha < 120^\circ - \frac{2}{3}\epsilon$,	15	$\alpha < \epsilon$.
13	$\alpha > 120^\circ - \frac{2}{3}\epsilon$.		

Also for 3, $\epsilon < 45^\circ$, and for 8, 11, 12, 13, $\epsilon > 45^\circ$, the others being possible for all values of ϵ.

§ 18. The critical values of ϕ, obtained from § 15, are as follows:

(*a*) $\phi < \sin^{-1}(\sin \epsilon \sin \alpha)$.	Two minima in the twilight-curve.
(*b*) $\phi < \epsilon + \alpha - 90^\circ$.	Twilight-curve open at winter solstice. (Impossible if $\alpha < 90^\circ - \epsilon$.)
(*c*) $\phi > 90^\circ - \epsilon - \alpha$.	Twilight-curve open at summer solstice. (Always occurs if $\alpha > 90^\circ - \epsilon$.)
(*d*) $\phi > 90^\circ - \epsilon$.	Perpetual day.
(*e*) $\phi > 90^\circ - \frac{\alpha}{2}$.	Perpetual twilight. (Nugatory if $\alpha > 2\epsilon$.)
(*f*) $\phi > 90^\circ - \epsilon + \alpha$.	Perpetual night. (Impossible if $\alpha > \epsilon$.)
(*g*) $\phi > 90^\circ + \epsilon - \alpha$.	Twilight-curve imaginary. (Impossible if $\alpha < \epsilon$.)

Consider the relative magnitudes of these values

$a > b$,	$c < d$.
$a < c$ if $\tan \epsilon \tan \alpha < \frac{1}{2}$,	$c < e$.
$a < d$ if $\sin \alpha < \cot \epsilon$,	$c < f$.
$a < e$ if $\sin \frac{\alpha}{2} < \frac{1}{2} \operatorname{cosec} \epsilon$,	$c < g$.
$a < f$,	$d < e$ if $\alpha < 2\epsilon$. If $d > e$, *e* is nugatory.
$a < g$.	
$b < d$ if $\alpha < 180^\circ - 2\epsilon$,	$d < f$.
$b < e$ if $\alpha < 120^\circ - \frac{2}{3}\epsilon$,	$d < g$ if $\alpha < 2\epsilon$.
$b < f$,	$e < f$ if $\alpha > \frac{2}{3}\epsilon$.
$b < g$,	$e < g$ if $\alpha < 2\epsilon$.

§ 19. The critical values of α are therefore

i. $\frac{2}{3}\epsilon$.	v. 2ϵ.	Impossible if $\epsilon > 45°$.
ii. ϵ.	vi. $120° - \frac{2}{3}\epsilon$.	Impossible if $\epsilon < 45°$ (vi.–ix.).
iii. $90° - \epsilon$.	vii. $180° - 2\epsilon$.	
iv. $\tan^{-1} \frac{1}{2} \cot \epsilon$.	viii. $\sin^{-1} \cot \epsilon$.	
	ix. $2 \sin^{-1} \frac{1}{2} \operatorname{cosec} \epsilon$.	

Consider the relative magnitudes of these values.

i < ii,	iii > v if $\epsilon < 30°$.
i < iii if $\epsilon < 54°$,	iii < vi.
i < iv if $\epsilon < 42° 41'$ approx.,	iii < vii.
i < v,	iii < viii.
i < vi,	iii < ix.
i < vii if $\epsilon < 67° 30'$,	iv > v if $\epsilon < 24° 6'$ approx.
i < viii if $\epsilon < 58°$ approx.,	iv < vi.
i < ix,	iv < vii.
ii < iii if $\epsilon < 45°$,	iv < viii.
ii < iv if $\epsilon < 35° 16'$ approx.,	iv < ix.
ii < v,	vi > vii.
ii < vi if $\epsilon < 72°$,	vi > viii.
ii < vii if $\epsilon < 60°$,	vi > ix.
ii < viii if $\epsilon < 51° 50'$ approx.,	vii > viii.
ii < ix if $\epsilon < 66° 14'$ approx.,	vii < ix.
iii > iv,	viii < ix.

The critical values of ϵ are therefore

$$30°,\ 45°,\ 54°,\ 60°,\ 67° 30',\ 72°,$$

and the solutions of the equations

$$\tan \tfrac{2}{3}\epsilon = \tfrac{1}{2} \cot \epsilon, \qquad \tan \epsilon = \tfrac{1}{2} \cot \epsilon, \text{ i.e. } \epsilon = \tan^{-1}\frac{1}{\sqrt{2}},$$

$$\sin \tfrac{2}{3}\epsilon = \cot \epsilon, \qquad \sin \epsilon = \cot \epsilon, \text{ i.e. } \epsilon = \cos^{-1} \tfrac{1}{2}(\sqrt{5} - 1),$$

$$\sin \frac{\epsilon}{2} = \tfrac{1}{2} \operatorname{cosec} \epsilon, \qquad \tan 2\epsilon = \tfrac{1}{2} \cot \epsilon, \text{ i.e. } \epsilon = \tan^{-1}\frac{1}{\sqrt{5}};$$

or arranged in order

24° 6′, 30°, 35° 16′, 42° 41′, 45°, 51° 50′, 54°, 58°, 60°, 66° 14′, 67° 30′, 72°.

§ 20. The critical values of α therefore assume the following orders for different values of ϵ:

I.	$\epsilon < 24° 6'$	i	ii	iii	vii	iv
II.	$24° 6' < \epsilon < 30°$	i	ii	vii	iii	iv
III.	$30° < \epsilon < 35° 16'$	i	ii	vii	iv	iii
IV.	$35° 16' < \epsilon < 42° 41'$	i	vii	ii	iv	iii

V. $42^\circ\ 41' < \epsilon < 45^\circ$	vii i ii iv iii
VI. $45^\circ < \epsilon < 51^\circ\ 50'$	vii i iv ii viii vi ix v
VII. $51^\circ\ 50' < \epsilon < 54^\circ$	vii i iv viii ii vi ix v
VIII. $54^\circ < \epsilon < 58^\circ$	vii iv i viii ii vi ix v
IX. $58^\circ < \epsilon < 60^\circ$	vii iv viii i ii vi ix v
X. $60^\circ < \epsilon < 66^\circ\ 14'$	vii iv viii i vi ii ix v
XI. $66^\circ\ 14' < \epsilon < 67^\circ\ 30'$	vii iv viii i vi ix ii v
XII. $67^\circ\ 30' < \epsilon < 72^\circ$	vii iv viii vi i ix ii v
XIII. $72^\circ < \epsilon$	vii iv viii vi i ix v ii

§ 21. In these 13 different cases a . . . . g assume the following orders:

I. $\epsilon < 24^\circ\ 6'$

1. $a < \frac{2}{3}\epsilon$	$a\ c\ d\ f\ e$
2. $< \epsilon$	$a\ c\ d\ e\ f$
3. $< 2\epsilon$	$a\ c\ d\ e\ g$
4. $< \tan^{-1} \frac{1}{2} \cot \epsilon$	$a\ c\ g\ d$
5. $< 90^\circ - \epsilon$	$c\ a\ g\ d$
6. $> 90^\circ - \epsilon$	$b\ a\ g\ d$

II. $24^\circ\ 6' < \epsilon < 30^\circ$

1. $a < \frac{2}{3}\epsilon$	$a\ c\ d\ f\ e$
2. $< \epsilon$	$a\ c\ d\ e\ f$
3. $< \tan^{-1} \frac{1}{2} \cot \epsilon$	$a\ c\ d\ e\ g$
4. $< 2\epsilon$	$c\ a\ d\ e\ g$
5. $< 90^\circ - \epsilon$	$c\ a\ g\ d$
6. $> 90^\circ - \epsilon$	$b\ a\ g\ d$

III. $30^\circ < \epsilon < 35^\circ\ 16'$

1. $a < \frac{2}{3}\epsilon$	$a\ c\ d\ f\ e$
2. $< \epsilon$	$a\ c\ d\ e\ f$
3. $< \tan^{-1} \frac{1}{2} \cot \epsilon$	$a\ c\ d\ e\ g$
4. $< 90^\circ - \epsilon$	$c\ a\ d\ e\ g$
5. $< 2\epsilon$	$b\ a\ d\ e\ g$
6. $> 2\epsilon$	$b\ a\ g\ d$

IV. $35^\circ\ 16' < \epsilon < 42^\circ\ 41'$

1. $a < \frac{2}{3}\epsilon$	$a\ c\ d\ f\ e$
2. $< \tan^{-1} \frac{1}{2} \cot \epsilon$	$a\ c\ d\ e\ f$
3. $< \epsilon$	$c\ a\ d\ e\ f$
4. $< 90^\circ - \epsilon$	$c\ a\ d\ e\ g$
5. $< 2\epsilon$	$b\ a\ d\ e\ g$
6. $> 2\epsilon$	$b\ a\ g\ d$

V. $42° 41' < \epsilon < 45°$

1.	$\alpha < \tan^{-1} \frac{1}{2} \cot \epsilon$	*a c d f e*
2.	$< \frac{2}{3}\epsilon$	*c a d f e*
3.	$< \epsilon$	*c a d e f*
4.	$< 90° - \epsilon$	*c a d e g*
5.	$< 2\epsilon$	*b a d e g*
6.	$> 2\epsilon$	*b a g d*

VI. $45° < \epsilon < 51° 50'$

1.	$\alpha < \tan^{-1} \frac{1}{2} \cot \epsilon$	*a c d f e*
2.	$< \frac{2}{3}\epsilon$	*c a d f e*
3.	$< 90° - \epsilon$	*c a d e f*
4.	$< \epsilon$	*b a d e f*
5.	$< \sin^{-1} \cot \epsilon$	*b a d e g*
6.	$< 180° - 2\epsilon$	*b d a e g*
7.	$< 2 \sin^{-1} \frac{1}{2} \operatorname{cosec} \epsilon$	*d b a e g*
8.	$< 120° - \frac{2}{3}\epsilon$	*d b e a g*
9.	$> 120° - \frac{2}{3}\epsilon$	*d e b a g*

VII. $51° 50' < \epsilon < 54°$

1.	$\alpha < \tan^{-1} \frac{1}{2} \cot \epsilon$	*a c d f e*
2.	$< \frac{2}{3}\epsilon$	*c a d f e*
3.	$< 90° - \epsilon$	*c a d e f*
4.	$< \sin^{-1} \cot \epsilon$	*b a d e f*
5.	$< \epsilon$	*b d a e f*
6.	$< 180° - 2\epsilon$	*b d a e g*
7.	$< 2 \sin^{-1} \frac{1}{2} \operatorname{cosec} \epsilon$	*d b a e g*
8.	$< 120° - \frac{2}{3}\epsilon$	*d b e a g*
9.	$> 120° - \frac{2}{3}\epsilon$	*d e b a g*

VIII. $54° < \epsilon < 58°$

1.	$\alpha < \tan^{-1} \frac{1}{2} \cot \epsilon$	*a c d f e*
2.	$< 90° - \epsilon$	*c a d f e*
3.	$< \frac{2}{3}\epsilon$	*b a d f e*
4.	$< \sin^{-1} \cot \epsilon$	*b a d e f*
5.	$< \epsilon$	*b d a e f*
6.	$< 180° - 2\epsilon$	*b d a e g*
7.	$< 2 \sin^{-1} \frac{1}{2} \operatorname{cosec} \epsilon$	*d b a e g*
8.	$< 120° - \frac{2}{3}\epsilon$	*d b e a g*
9.	$> 120° - \frac{2}{3}\epsilon$	*d e b a g*

IX. $58^\circ < \epsilon < 60^\circ$

1.	$a < \tan^{-1}\frac{1}{2}\cot\epsilon$	*a c d f e*
2.	$< 90^\circ - \epsilon$	*c a d f e*
3.	$< \sin^{-1}\cot\epsilon$	*b a d f e*
4.	$< \frac{2}{3}\epsilon$	*b d a f e*
5.	$< \epsilon$	*b d a e f*
6.	$< 180^\circ - 2\epsilon$	*b d a e g*
7.	$< 2\sin^{-1}\frac{1}{2}\operatorname{cosec}\epsilon$	*d b a e g*
8.	$< 120^\circ - \frac{2}{3}\epsilon$	*d b e a g*
9.	$> 120^\circ - \frac{2}{3}\epsilon$	*d e b a g*

X. $60^\circ < \epsilon < 66^\circ\ 14'$

1.	$a < \tan^{-1}\frac{1}{2}\cot\epsilon$	*a c d f e*
2.	$< 90^\circ - \epsilon$	*c a d f e*
3.	$< \sin^{-1}\cot\epsilon$	*b a d f e*
4.	$< \frac{2}{3}\epsilon$	*b d a f e*
5.	$< 180^\circ - 2\epsilon$	*b d a e f*
6.	$< \epsilon$	*d b a e f*
7.	$< 2\sin^{-1}\frac{1}{2}\operatorname{cosec}\epsilon$	*d b a e g*
8.	$< 120^\circ - \frac{2}{3}\epsilon$	*d b e a g*
9.	$> 120^\circ - \frac{2}{3}\epsilon$	*d e b a g*

XI. $66^\circ\ 14' < \epsilon < 67^\circ\ 30'$

1.	$a < \tan^{-1}\frac{1}{2}\cot\epsilon$	*a c d f e*
2.	$< 90^\circ - \epsilon$	*c a d f e*
3.	$< \sin^{-1}\cot\epsilon$	*b a d f e*
4.	$< \frac{2}{3}\epsilon$	*b d a f e*
5.	$< 180^\circ - 2\epsilon$	*b d a e f*
6.	$< 2\sin^{-1}\frac{1}{2}\operatorname{cosec}\epsilon$	*d b a e f*
7.	$< \epsilon$	*d b e a f*
8.	$< 120^\circ - \frac{2}{3}\epsilon$	*d b e a g*
9.	$> 120^\circ - \frac{2}{3}\epsilon$	*d e b a g*

XII. $67^\circ\ 30' < \epsilon < 72^\circ$

1.	$a < \tan^{-1}\frac{1}{2}\cot\epsilon$	*a c d f e*
2.	$< 90^\circ - \epsilon$	*c a d f e*
3.	$< \sin^{-1}\cot\epsilon$	*b a d f e*
4.	$< 180^\circ - 2\epsilon$	*b d a f e*
5.	$< \frac{2}{3}\epsilon$	*d b a f e*
6.	$< 2\sin^{-1}\frac{1}{2}\operatorname{cosec}\epsilon$	*d b a e f*
7.	$< \epsilon$	*d b e a f*
8.	$< 120^\circ - \frac{2}{3}\epsilon$	*d b e a g*
9.	$> 120^\circ - \frac{2}{3}\epsilon$	*d e b a g*

XIII. $\epsilon > 72°$

1.	$a < \tan^{-1}\frac{1}{2}\cot\epsilon$	*a c d f e*
2.	$< 90° - \epsilon$	*c a d f e*
3.	$< \sin^{-1}\cot\epsilon$	*b a d f e*
4.	$< 180° - 2\epsilon$	*b d a f e*
5.	$< \frac{2}{3}\epsilon$	*d b a f e*
6.	$< 2\sin^{-1}\frac{1}{2}\operatorname{cosec}\epsilon$	*d b a e f*
7.	$< 120° - \frac{2}{3}\epsilon$	*d b e a f*
8.	$< \epsilon$	*d e b a f*
9.	$> \epsilon$	*d e b a g*

§ 22. We are now in a position to tabulate the sequences of phenomena that appear for any values of ϵ and a and for values of ϕ varying from 0° to 90°.

It will be seen from an inspection of the results of the last section that there are only 22 distinct cases. These are tabulated as follows. It is to be noted that the sequence which holds for each number which is mentioned, holds also for the corresponding arabic numeral under each of the following roman numerals up to the next mentioned. Thus the sequences for I 6, II 6, . . . V 6 are the same, and those for VIII 3, . . . XIII 3 are the same.

I 1,			
I 2,	V 2,		
I 3,	IV 3,	VIII 3,	
I 4,	II 4,	VI 4,	IX 4,
I 5,	III 5,	VII 5,	XII 5,
I 6,	VI 6,	X 6,	
VI 7,	XI 7,		
VI 8,	XIII 8,		
VI 9.			

§ 23. We have then the following table of sequences of phenomena. The figures in heavy type refer to the diagrams in Plate I.

I. $\epsilon < 24° 6'$

1. $a < \frac{2}{3}\epsilon$	$\phi < \sin^{-1}(\sin\epsilon\sin a)$	**2**
	$< 90° - \epsilon - a$	**1**
	$< 90° - \epsilon$	**4**
	$< 90° - \epsilon + a$	**7**
	$< 90° - \frac{a}{2}$	**14**
	$> 90° - \frac{a}{2}$	**15**

[illegible]

III. $30^\circ < \epsilon < 35^\circ\ 16'$

5. $90^\circ - \epsilon < a < 2\epsilon$	$\phi < \epsilon + a - 90^\circ$	6
	$< \sin^{-1}(\sin\epsilon \sin a)$	5
	$< 90^\circ - \epsilon$	4
	$< 90^\circ - \frac{a}{2}$	7
	$< 90^\circ + \epsilon - a$	10
	$> 90^\circ + \epsilon - a$	9

IV. $35^\circ\ 16' < \epsilon < 42^\circ\ 41'$

3. $\tan^{-1}\frac{1}{2}\cot\epsilon < a < \epsilon$	$\phi < 90^\circ - \epsilon - a$	2
	$< \sin^{-1}(\sin\epsilon \sin a)$	5
	$< 90^\circ - \epsilon$	4
	$< 90^\circ - \frac{a}{2}$	7
	$< 90^\circ - \epsilon + a$	10
	$> 90^\circ - \epsilon + a$	15

V. $42^\circ\ 41' < \epsilon < 45^\circ$

2. $\tan^{-1}\frac{1}{2}\cot\epsilon < a < \frac{2}{3}\epsilon$	$\phi < 90^\circ - \epsilon - a$	2
	$< \sin^{-1}(\sin\epsilon \sin a)$	5
	$< 90^\circ - \epsilon$	4
	$< 90^\circ - \epsilon + a$	7
	$< 90^\circ - \frac{a}{2}$	14
	$> 90^\circ - \frac{a}{2}$	15

VI. $45^\circ < \epsilon < 51^\circ\ 50'$

4. $90^\circ - \epsilon < a < \epsilon$	$\phi < \epsilon + a - 90^\circ$	6
	$< \sin^{-1}(\sin\epsilon \sin a)$	5
	$< 90^\circ - \epsilon$	4
	$< 90^\circ - \frac{a}{2}$	7
	$< 90^\circ - \epsilon + a$	10
	$> 90^\circ - \epsilon + a$	15
6. $\sin^{-1}\cot\epsilon < a < 180^\circ - 2\epsilon$	$\phi < \epsilon + a - 90^\circ$	6
	$< 90^\circ - \epsilon$	5
	$< \sin^{-1}(\sin\epsilon \sin a)$	8
	$< 90^\circ - \frac{a}{2}$	7
	$< 90^\circ + \epsilon - a$	10
	$> 90^\circ + \epsilon - a$	9

7. $180° - 2\epsilon < \alpha < 2\sin^{-1}\frac{1}{2}\operatorname{cosec}\epsilon$	$\phi < 90° - \epsilon$	6
	$< \epsilon + \alpha - 90°$	12
	$< \sin^{-1}(\sin\epsilon\sin\alpha)$	8
	$< 90° - \frac{\alpha}{2}$	7
	$< 90° + \epsilon - \alpha$	10
	$> 90° + \epsilon - \alpha$	9
8. $2\sin^{-1}\frac{1}{2}\operatorname{cosec}\epsilon < \alpha < 120° - \frac{2}{3}\epsilon$	$\phi < 90 - \epsilon$	6
	$< \epsilon + \alpha - 90°$	12
	$< 90° - \frac{\alpha}{2}$	8
	$< \sin^{-1}(\sin\epsilon\sin\alpha)$	11
	$< 90° + \epsilon - \alpha$	10
	$> 90° + \epsilon - \alpha$	9
9. $\alpha > 120° - \frac{2}{3}\epsilon$	$\phi < 90° - \epsilon$	6
	$< 90° - \frac{\alpha}{2}$	12
	$< \epsilon + \alpha - 90°$	13
	$< \sin^{-1}(\sin\epsilon\sin\alpha)$	11
	$< 90° + \epsilon - \alpha$	10
	$> 90° + \epsilon - \alpha$	9

VII. $51° 50' < \epsilon < 54°$

5. $\sin^{-1}\cot\epsilon < \alpha < \epsilon$	$\phi < \epsilon + \alpha - 90°$	6
	$< 90° - \epsilon$	5
	$< \sin^{-1}(\sin\epsilon\sin\alpha)$	8
	$< 90° - \frac{\alpha}{2}$	7
	$< 90° - \epsilon + \alpha$	10
	$> 90° - \epsilon + \alpha$	15

VIII. $54° < \epsilon < 58°$

8. $90° - \epsilon < \alpha < \frac{2}{3}\epsilon$	$\phi < \epsilon + \alpha - 90°$	6
	$< \sin^{-1}(\sin\epsilon\sin\alpha)$	5
	$< 90° - \epsilon$	4
	$< 90° - \epsilon + \alpha$	7
	$< 90° - \frac{\alpha}{2}$	14
	$> 90° - \frac{\alpha}{2}$	15

IX. $58° < \epsilon < 60°$			
	4. $\sin^{-1}\cot\epsilon < \alpha < \tfrac{2}{3}\epsilon$	$\phi < \epsilon + \alpha - 90°$	6
		$< 90° - \epsilon$	5
		$< \sin^{-1}(\sin\epsilon\sin\alpha)$	8
		$< 90° - \epsilon + \alpha$	7
		$< 90° - \dfrac{\alpha}{2}$	14
		$> 90° - \dfrac{\alpha}{2}$	15
X. $60° < \epsilon < 66°\ 14'$			
	6. $180° - 2\epsilon < \alpha < \epsilon$	$\phi < 90° - \epsilon$	6
		$< \epsilon + \alpha - 90°$	12
		$< \sin^{-1}(\sin\epsilon\sin\alpha)$	8
		$< 90° - \dfrac{\alpha}{2}$	7
		$< 90° - \epsilon + \alpha$	10
		$> 90° - \epsilon + \alpha$	15
XI. $66°\ 14' < \epsilon < 67°\ 30'$			
	7. $2\sin^{-1}\tfrac{1}{2}\operatorname{cosec}\epsilon < \alpha < \epsilon$	$\phi < 90° - \epsilon$	6
		$< \epsilon + \alpha - 90°$	12
		$< 90° - \dfrac{\alpha}{2}$	8
		$< \sin^{-1}(\sin\epsilon\sin\alpha)$	11
		$< 90° - \epsilon + \alpha$	10
		$> 90° - \epsilon + \alpha$	15
XII. $67°\ 30' < \epsilon < 72°$			
	5. $180° - 2\epsilon < \alpha < \tfrac{2}{3}\epsilon$	$\phi < 90° - \epsilon$	6
		$< \epsilon + \alpha - 90°$	12
		$< \sin^{-1}(\sin\epsilon\sin\alpha)$	8
		$< 90° - \epsilon + \alpha$	7
		$< 90° - \dfrac{\alpha}{2}$	14
		$> 90° - \dfrac{\alpha}{2}$	15
XIII. $\epsilon > 72°$			
	8. $120° - \tfrac{2}{3} < \alpha < \epsilon$	$\phi < 90° -$	6
		$< 90° - \dfrac{\alpha}{2}$	12
		$< \epsilon + \alpha - 90°$	13
		$< \sin^{-1}(\sin\epsilon\sin\alpha)$	11
		$< 90° - \epsilon + \alpha$	10
		$> 90° - \epsilon + \alpha$	15

§ 24. The case of $\epsilon = 23° 27'$ and $a = 18°$ evidently comes under I. 2, and the sequence is

$\phi < 7° 4'$	2
$7° 4' < \phi < 48° 33'$	1
$48° 33' < \phi < 66° 33'$	4
$66° 33' < \phi < 81°$	7
$81° < \phi < 84° 33'$	10
$84° 33' < \phi$	15

§ 25. The various cases and critical values can be represented graphically, as follows.

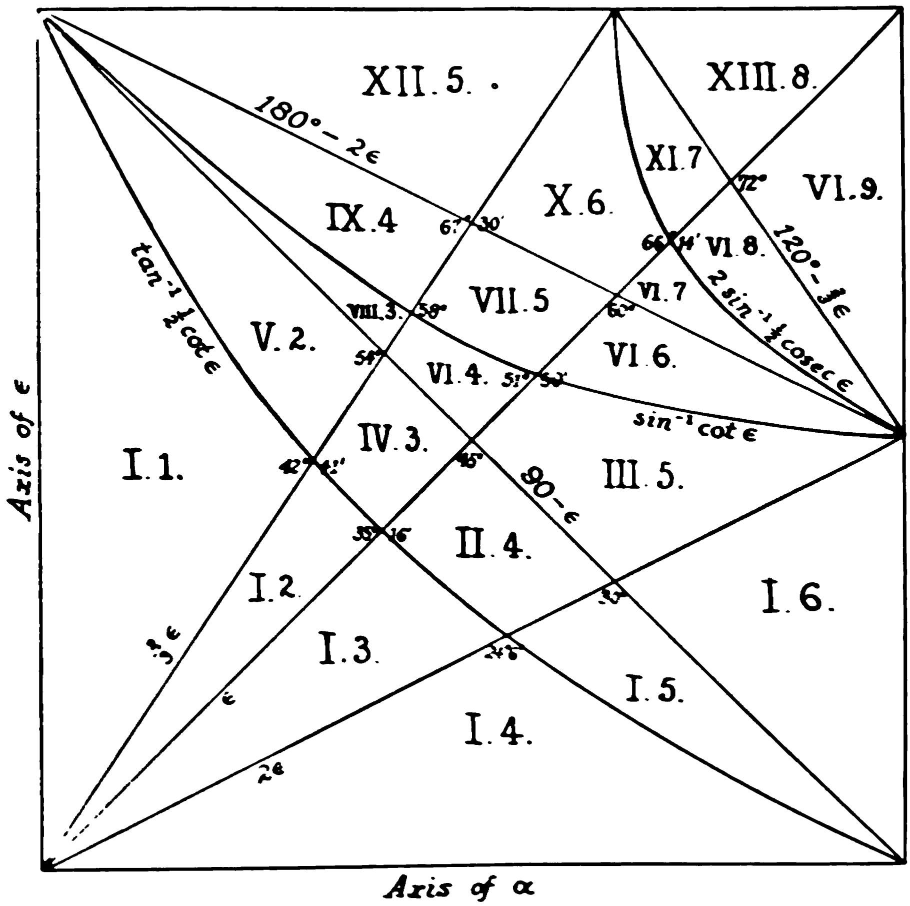


Fig. 3.

There are nine critical values of a in terms of ϵ. Considering ϵ and a as both variable, these will be the equations of nine curves, the points of intersection of which give the critical values of ϵ (and also of a). We are only concerned with the portions of the curves contained within the square $\epsilon = 0°$ to $90°$, $a = 0°$ to $90°$. These curves are represented in the diagram, fig. 3. They divide the square into 22 regions, each of which is characterised by a different sequence of phenomena for varying latitudes. These correspond to the 22 different sequences of phenomena which we have already obtained. This gives us a means of tabulating the results more succinctly for varying values of a in terms of ϵ.

	$a < \tan^{-1}\frac{1}{2}\cot\epsilon$	$\tan^{-1}\frac{1}{2}\cot\epsilon < a < 90° - \epsilon$	$90° - \epsilon < a < \sin^{-1}\cot\epsilon$	$\sin^{-1}\cot\epsilon < a < 180° - 2\epsilon$	$180° - 2\epsilon < a < 2\sin^{-1}\frac{1}{2}\operatorname{cosec}\epsilon$	$2\sin^{-1}\frac{1}{2}\operatorname{cosec}\epsilon < a < 120° - \frac{2}{3}\epsilon$	$a > 120° - \frac{2}{3}\epsilon$
$a < \frac{2}{3}\epsilon$	I. 1	V. 2	VIII. 3	IX. 4	XII. 5		
$\frac{2}{3}\epsilon < a < \epsilon$	I. 2	IV. 3	VI. 4	VII. 5	X. 6	XI. 7	XIII. 8
$\epsilon < a < 2\epsilon$	I. 3	II. 4	III. 5	VI. 6	VI. 7	VI. 8	VI. 9
$a > 2\epsilon$	I. 4	I. 5	I. 6				

In a similar way the critical values of ϕ, which are expressed in terms of both ϵ and a, determine surfaces, or rather portions of surfaces, contained within the cube, ϵ, a, ϕ from $0°$ to $90°$. The projections upon the plane of ϵ, a of the curves of intersection of these surfaces are exactly the curves corresponding to the critical values of a in terms of ϵ. These surfaces divide the cube into 16 regions, but one of the partitions is nugatory, viz. that portion of the plane $\phi = 90° - \frac{a}{2}$ which divides the region bounded by

$$\phi = 90° + \epsilon - a, \quad \phi = 90° - \epsilon, \quad \epsilon = 0 \quad \text{and} \quad a = 90°.$$

Hence these regions correspond to the 15 different yearly phenomena, and the nugatory partition corresponds to the case in which the critical value $\phi = 90° - \frac{a}{2}$ was found to be nugatory, viz. when $a > 2\epsilon$.

The regions of the cube can be described without difficulty. The four planes

$$\phi = \epsilon + a - 90°, \quad \phi = 90° - \epsilon - a, \quad \phi = 90° + \epsilon - a, \quad \phi = 90° - \epsilon + a$$

determine a regular tetrahedron inscribed in the cube. This tetrahedron is cut in two by the surface $\sin\phi = \sin\epsilon \sin a$. We have then seven regions.

$\phi < \epsilon + a - 90°$

$\phi < 90° - \epsilon - a$ and $\gtrless \sin^{-1}(\sin\epsilon \sin a)$

$\phi > \epsilon + a - 90°$, $> 90° - \epsilon - a$, $< 90° + \epsilon - a$, $< 90° - \epsilon + a$, and $\gtrless \sin^{-1}(\sin\epsilon \sin a)$

$\phi > 90° + \epsilon - a$

$\phi > 90° - \epsilon + a$

The first and the fourth and fifth are each cut in three by $\phi = 90° - \frac{a}{2}$ and $\phi = 90° - \epsilon$. The last is cut in two by $\phi = 90° - \frac{a}{2}$, and the sixth is cut in three by $\phi = 90° - \epsilon$ and $\phi = 90° - \frac{a}{2}$ but the latter partition is nugatory.

The seven primary regions correspond to the seven different forms of the twilight-curve by itself, not taken in conjunction with the sunset-curve.

As one ascends vertically from the plane $\phi = 0$, the regions passed through give the sequence of phenomena for varying latitudes, and for the values of ϵ and a corresponding to the starting-point.

(*Issued separately May* 14, 1908.)

PLATE I.

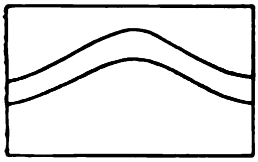


FIG. 1.

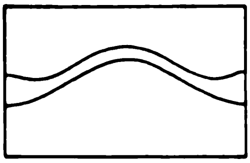


FIG. 2.

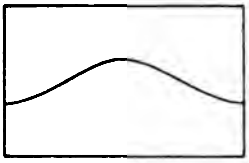


FIG. 3.

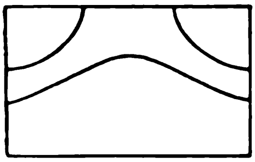


FIG. 4.

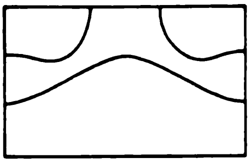


FIG. 5.

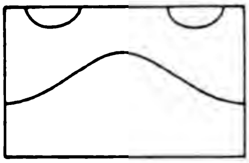


FIG. 6.

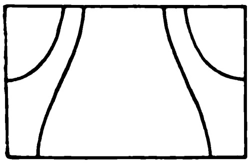


FIG. 7.

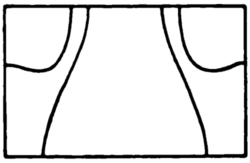


FIG. 8.

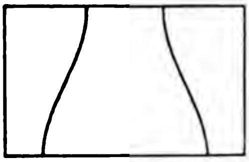


FIG. 9.

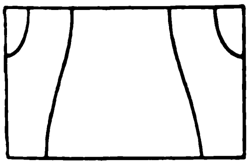


FIG. 10.

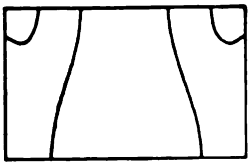


FIG. 11.

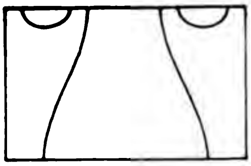


FIG. 12.

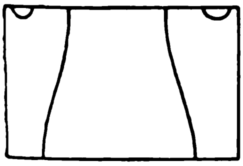


FIG. 13.

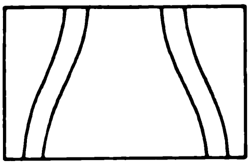


FIG. 14.

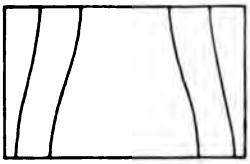


FIG. 15.

PLATE II.

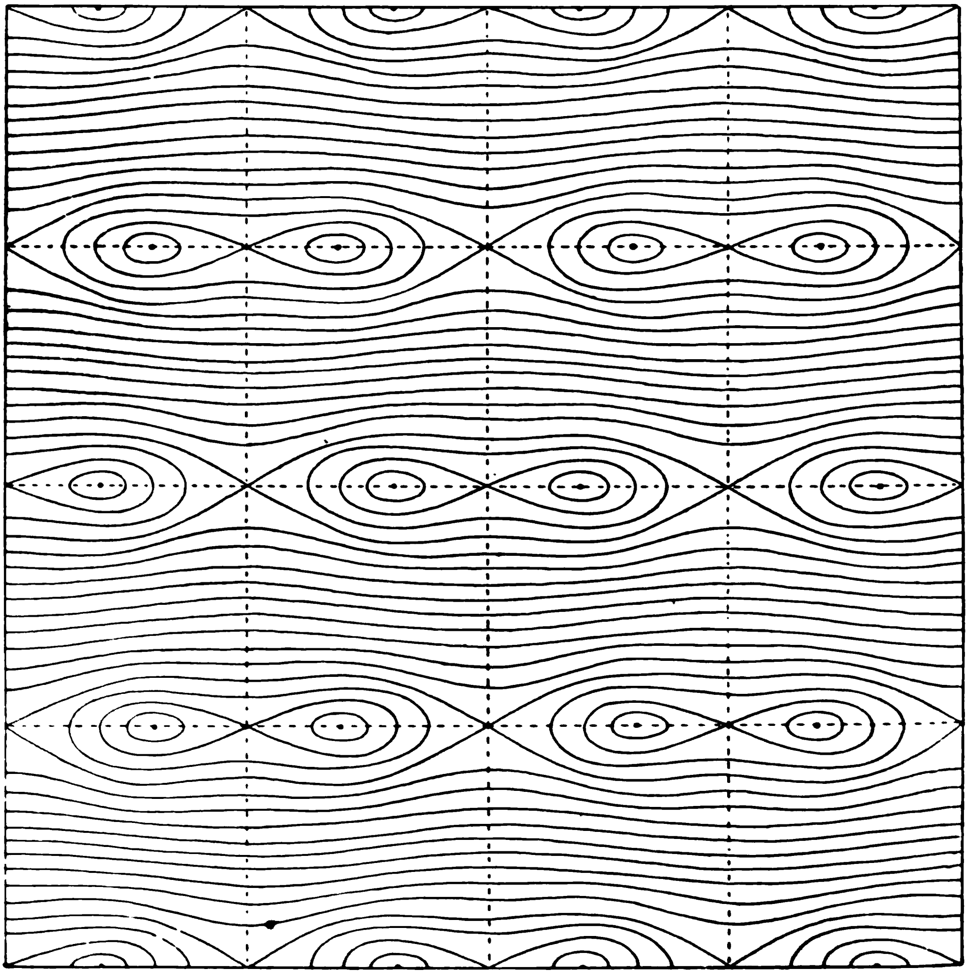
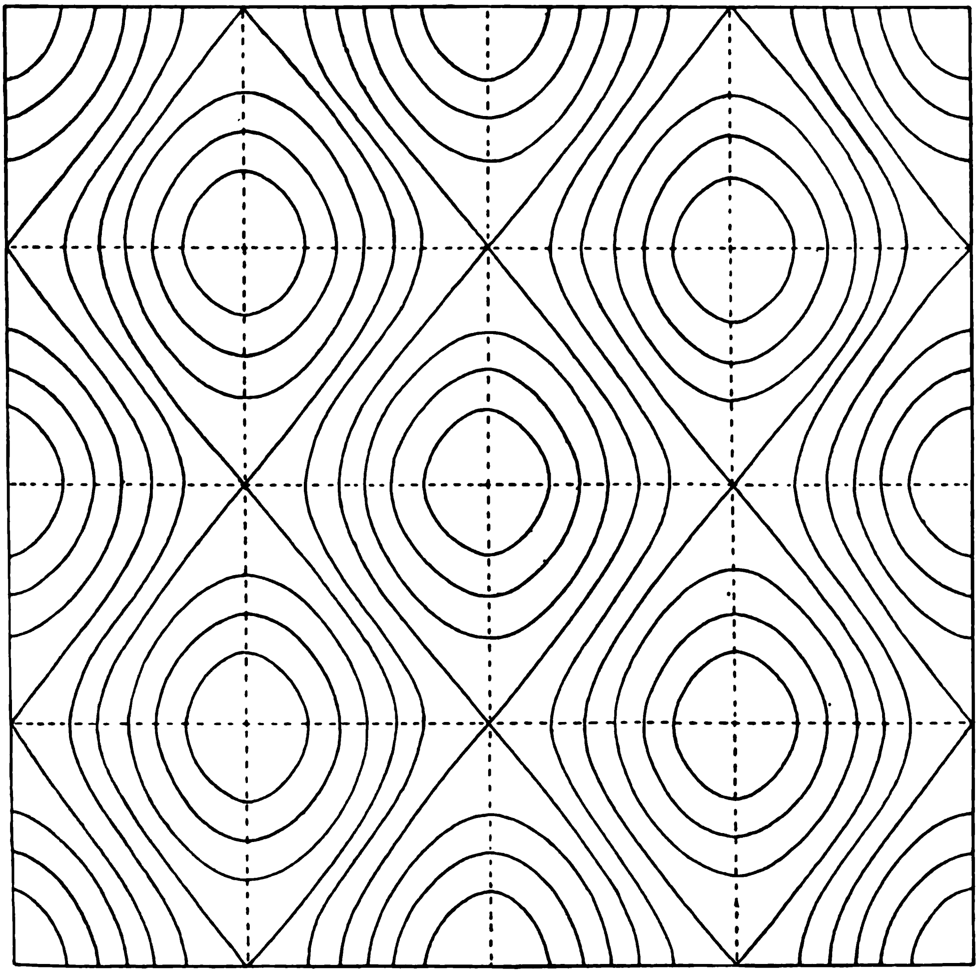


PLATE III.



XIX.—A Preliminary Notice of Five New Species of Iron-Bacteria.

By **David Ellis**, Ph.D. (Marburg), D.Sc. (London), F.R.S.E., Lecturer in Botany and Bacteriology, Glasgow and West of Scotland Technical College, Glasgow. (With Three Plates.)

(Read March 2, 1908. MS. received March 10, 1908.)

For the past two years I have been engaged in the investigation of the iron-water of Scotland, England, and Wales, and altogether have examined over a hundred samples. This publication is a preliminary notice of some new species of iron-bacteria which I have found in the course of this investigation. At present the number of known forms belonging to this class is six. In this paper I wish to outline the main characteristics of five new species, reserving for later papers the detailed accounts of their life-histories.

1. *Spirosoma ferrugineum* (Ellis).—The generic name Spirosoma has

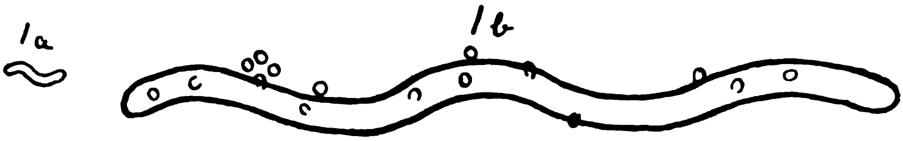


Fig. 1.—*Spirosoma ferrugineum*. In 1*b* the beginnings of conidia-formation are diagrammatically shown.

been introduced by Migula to include seven species, the main characteristics of which are a spiral form and an absence of motion. I propose to call this, the eighth species, *Spirosoma ferrugineum*.

Its general appearance is seen in Plate I. Each individual consists of a wavy thread. The waves are regular in size, each wave-length being on the average twice or three times the amplitude. All stages are seen, from small threads (fig. 1*a*) which have just germinated, to large thick threads (fig. 1*b*). According to the condition of growth, the size varies from a few μ to about 100μ and more. Reproduction is effected more commonly by the formation of conidia (Plate I. and fig. 1*b*), and more rarely by the process of splitting such as is found in other iron-bacteria, *e.g.*, *Leptothrix ochracea* and *Spirophyllum ferrugineum*.

With regard to distribution, I find that in point of numbers it bears the same relation to *Leptothrix ochracea* as do *Gallionella ferruginea* and *Spirophyllum*, viz., that whereas in most places *Leptothrix ochracea* is overwhelmingly predominant, in one or two places the supremacy is gained by this organism. I have found it in Lanarkshire, Renfrewshire, Clackmannanshire, Perthshire, Dumbartonshire, Ayrshire, and Peeblesshire, in small quantities. It is the predominant organism in some iron-water near Kingswell in Ayrshire. It seems to be absent from the extreme northern counties of Scotland, and I have not yet been able to identify it in the English iron-waters. I am in possession of the complete life-history of this organism, which will form the subject of a later publication.

2. *Nodofolium ferrugineum* (Ellis).—The second new species is in shape quite unlike anything hitherto known, and demands the creation of a new genus for its insertion. I propose to call the genus Nodofolium, and this species *Nodofolium ferrugineum.* Its general appearance may be described as that of a flat band, in which constrictions at regular intervals

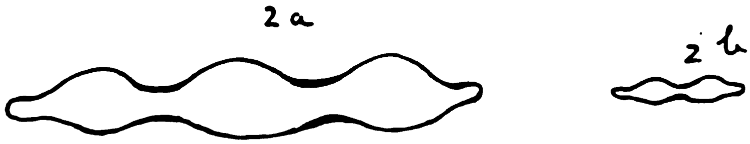


Fig. 2.—*Notofolium ferrugineum* (diagrammatic).

are to be found. The band between two such constrictions is in appearance like a stretched string sounding its fundamental note. The constrictions are further emphasised by the fact that at these places the individual is slightly humped up so that an arch is formed. An example is shown in fig. 2*a*.

This species was found in samples derived from the central and western parts of Scotland, but seems to be wanting in the iron-waters of the north of Scotland, and also in those south of the Border. These statements cannot, however, be accepted as final until a more extensive examination has been made. With regard to the size of these organisms, we find all gradations. Thus in fig. 2*b* we see one about 10μ long, consisting of two loops. A band of this length is $1–1\frac{1}{2}\mu$ at its broadest and about $\frac{3}{4}\mu$ at its narrowest part; other examples may have as many as twelve loops, each about $10–12\mu$ long, and doubtless some individuals are still longer even than this.

Its method of reproduction consists in the formation of a large number of conidia, which are formed in the same way as those of *Leptothrix* *ochracea*, so that the whole structure becomes in a similar manner greatly swollen out, being sometimes, in consequence of this conidia-formation, double its normal thickness (Plate II.). I am in possession of the main facts of the life-history of this organism, which I intend to publish later. I may here note that its life-history very closely resembles that of *Leptothrix ochracea* and its allies.

3. *Leptothrix Meyeri* (Ellis).—Whilst I am in possession of the salient facts in the life-history of the two preceding forms, with regard to this species and the two that follow I can only record their presence in the iron-waters of Scotland, and hope that I shall later be fortunate enough to find a sample in which one or other of these three preponderates. This third species, in its general characteristics, is very similar to *Leptothrix ochracea* (fig. 3, Plate III.), and differs from it in an entire absence of shapely-contoured walls, and in the nature of the iron deposit, which is at first very

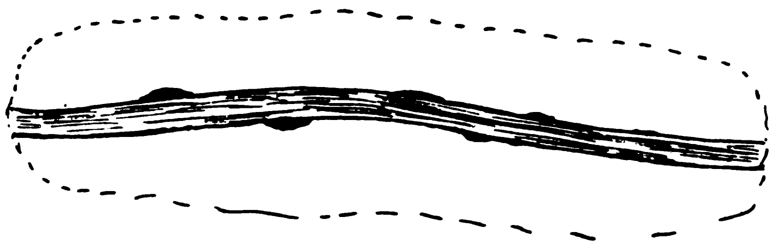


FIG. 3.—*Leptothrix Meyeri*. Dark excrescences indicate spots which have already become opaque.

transparent, so that the walls are easily visible even though the thickness of the individual, owing to the deposit of iron, has been trebled. Later, the density of the deposit gets greater, and consequently the individuals become more opaque. The threads measure from 2μ to 3μ in breadth, and in length from 40μ to 70μ, and, exceptionally, still longer threads may be noticed. The transparent nature of the deposit is probably due to the presence of a large amount of mucilage outside the walls, formed by the degeneration of the latter. The iron penetrates this mucilage, gradually colouring the latter, and thus rendering it visible. As the amount of iron increases, naturally the transparency decreases.

In the case of *Leptothrix ochracea*, speaking generally, large increases in thickness are due to the laying down of iron in the spaces between the numberless conidia that have been formed, but in this case the thickness of the deposit has nothing to do with conidia-formation. I have not as yet been able to observe its methods of reproduction, owing to its scarcity in the iron-water samples which I have up to now examined.

4. *Spirophyllum tenue* (Ellis).—As each individual of this species consists of a spirally twisted *flat* band, it belongs to the genus Spirophyllum, of which only one species, viz., *Spirophyllum ferrugineum*, has hitherto been described.

I have seen this species in one place only, viz., near Alexandria in Renfrewshire. The band is about 1μ in width. Its general appearance is represented in fig. 4. As there may be 200 and more spiral turns, the total

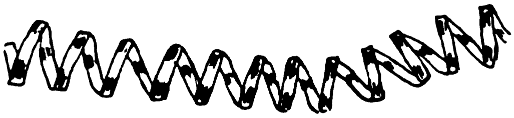


Fig. 4.—*Spirosoma tenue.*

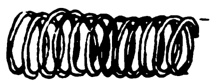


Fig. 5.—*Spirosoma solenoide.*

length is considerable, in spite of the fact that the spirals are very close together. Each individual consists of a gently undulating, somewhat loose solenoidal structure about 200–300μ long. I am not yet in possession of further details as to its life-history. There was a very slight deposit of iron on its surface,—so small, indeed, that it was scarcely visible when examined under the microscope,—but treatment with ammonium sulphide and with potassium ferrocyanide established unmistakably the presence of the iron. I hope to be able to elucidate its life-history when more material for observation is at hand.

5. *Spirosoma solenoide* (Ellis).—This species differs from *Spirosoma ferrugineum* in that the windings are very close together, the whole having the appearance of a solenoid. In general appearance it bears a close resemblance to *Spirophyllum tenue*, but a closer observation shows that the spiral is made up of a *thread*, and not a *flat band*. In spite of the close winding, a length of 100μ and more is often attained; in fact, the average individual is very seldom less than 70μ. As the thread is about $\frac{1}{2}\mu$ in diameter, and the distance between two turns very seldom more than 1μ, an average individual has about fifty windings (fig. 5).

The threads are always formed in very small numbers, though fairly well distributed over the west and middle of Scotland. I have not yet obtained any futher details as to its life-history.

The investigation, of which these are the preliminary results, is being carried on with the aid of a grant from the Royal Society of London, to whom my thanks are due.

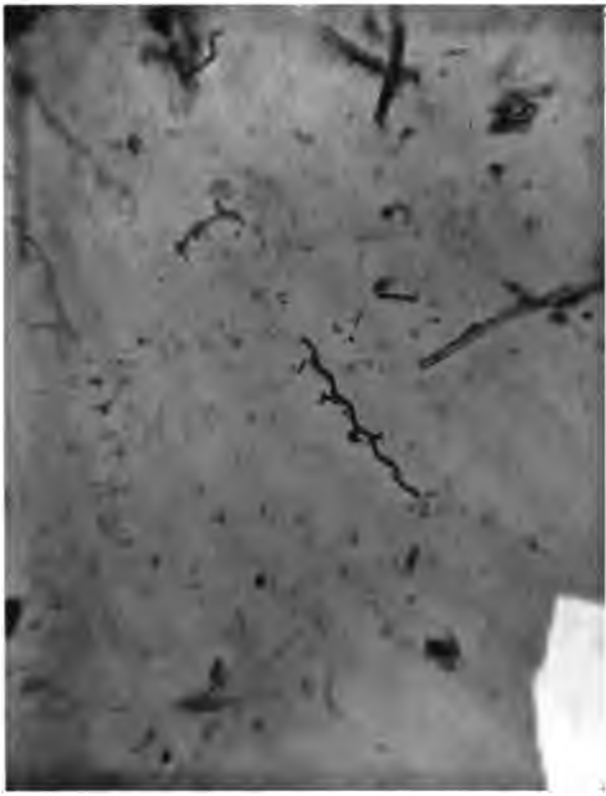
EXPLANATION OF PLATES.

Plate I. *Spirosoma ferrugineum*, showing the beginnings of conidia-formation.

Plate II. *Nodofolium ferrugineum*, with thick deposit of iron peroxide on surface.

Plate III. *Leptothrix Meyeri*. The individuals are seen to be covered with a thick mass of iron peroxide. In this particular case the deposit has lost its transparency.

(*Issued separately June* 13, 1908.)



DR DAVID ELLIS. [Plate II.

XX.—The Arterial Pressure in Man. I.: Methods. By G. A. Gibson, M.D., D.Sc., LL.D., F.R.C.P.Ed.

(Read January 20, 1908.)

In approaching the consideration of the problems afforded by the arterial pressure in man, it is unnecessary to enter deeply into the history of the subject. Tigerstedt (1), Hill (2), Vaschide and Lahy (3), and Janeway (4) have fully analysed the voluminous literature which has grown up around it. It will be sufficient to mention the original experiment of Hales (5), in which the arterial pressure was measured by the height to which the blood rose in a vertical tube connected with the artery, and of Poisseuille (6), and Ludwig (7), who introduced and improved the method of estimating the arterial pressure by means of the mercurial manometer, which has since been modified by many subsequent observers. To obviate some of the disadvantages of the mercurial manometer, which will engage attention at a later stage, Chauveau and Marey (8), and Fick (9), almost simultaneously introduced elastic or spring manometers. These have undergone many alterations at the hands of numerous followers.

The arterial pressure has been obtained in man by the kymograph on a few occasions during operations. Faivre (10) may be referred to as having carried out investigations of this sort which must be condemned as absolutely unjustifiable.

The earliest attempt at the clinical estimation of the arterial pressure was that of Vierordt (11), who employed for the purpose his own sphygmograph, weighted in order to ascertain the amount of pressure necessary to obliterate the pulsation of the radial artery. Marey (12), Waldenburg (13), von Basch (14), Hoorweg (15), Potain (16), Hürthle (17), Bloch (18), Mosso (19), Oliver (20), Riva Rocci (21), Hill and Barnard (22), Gärtner (23), Stanton (24), Cook (25), Erlanger (26), and Janeway (4) have, since his time, introduced different forms of clinical sphygmomanometers, which may be grouped in respect of their mode of application or with regard to the principles involved.

The sphygmomanometers of von Basch and Potain, as well as the earlier instrument devised by Oliver and the smaller instrument suggested by Hill and Barnard, estimate the pressure by application directly to the radial artery. The three first mentioned register the pressure by means of an aneroid, and the other by the resistance of the air compressed in the upper part of the tube. The apparatus of Riva Rocci, Stanton, Cook, Erlanger, and Janeway obtain readings by circular compression of one of the limbs, so as to obliterate one of the larger arteries and estimate the effects by examination of itself or one of its branches further from the heart; the instruments of Marey, Hürthle, and Mosso act by embracing the extremities after the manner of the plethysmograph. The tonometer of Gärtner gauges the pressure, after the removal of the compression of a finger, by the return of colour to the skin. It may be said at once that, with the exception of those sphygmomanometers which employ the method of circular compression of one of the larger limbs, there is none which need be seriously discussed. All the others are only of historic interest. The larger instrument of Hill and Barnard registers its results by means of an aneroid, and the later instrument of Oliver (20) by the use of a tube of spirit. Both of these also employ the method of circular compression.

The methods of Riva Rocci, Martin, Cook, Stanton, and Janeway yield an accurate determination of the systolic pressure, and allow of an approximate estimate of diastolic pressure. The larger instrument of Hill and Barnard and the earlier apparatus of Oliver, in which the arterial pressure is gauged by the maximum excursions recorded, were intended to register the mean pressure within the vessel; the theory being that when the pressure surrounding the vessel and that within it are equal, oscillations attain their maximum. It has, however, been proved by Howell and Brush (27) that the maximum oscillations give diastolic readings. The sphygmomanometer of Erlanger is based upon the same principles as that of Riva Rocci, and gives systolic and diastolic readings; it is the earliest attempt to furnish graphic records of arterial pressure in clinical investigation. A full description of the instrument is given by Janeway (4). It yields tracings of the oscillations of the column of air between the compressing band and the column of mercury in the manometer, but it does not furnish any record of the height of the column of mercury by which the pressure may be estimated. It is, therefore, necessary to watch the manometer and note the oscillations. The return of the pulse at the wrist is found to coincide with the commencement of large oscillations, and there can be no difficulty, therefore, in the estimation of the systolic pressure; while, according to Erlanger, the point at which the large oscillations suddenly begin to diminish is an indication of the diastolic pressure. My own observations with this instrument, which have been very numerous, lead me to concur with Erlanger in these views.

The sphygmomanometer recently introduced by me (28) also takes ad-

vantage of the principle of circular comp
maximum or systolic arterial pressure, wh
minimum or diastolic pressure. It has a
of which is exactly that of the ordinary
air contained in the armlet can be increase
therefore elevated, by means of a large sy
raised quickly or slowly according to requir
the pressure may also be lowered quickly
the mercury, surrounded, as is usual in t
alcohol, and an upright rod of aluminium l
writes on the revolving cylinder. In ord
fixed arm traces the abscissa upon the c
clockwork placed horizontally, as in the
pulsations of the artery below the point
means of a transmission sphygmograph.
brought into contact with the brachial or
convenient, by a pelotte resting upon the ve
by means of a spring provided with a sc
into communication by rubber tubing wit
ments of which are recorded on the cyli
movements of the kymograph. The best
tambour in contact with the artery is la
the recording lever, by means of which the
whole apparatus is shown in fig. 1.

In using the instrument the pressure
gradually or quickly, the latter being th
slowly raised, the tracing of the kymograph
with small oscillations, but as it rises th
more marked, and the excursion of the i
maximum point of amplitude is attained,
and gradually disappear. Simultaneously t
records a gradual diminution in the ampl
finally cease. When all the movements of t
sphygmograph, have come to an end, the pre
escape of air from the valve, and the events
of those just described. Such a tracing is sh
the systolic pressure on the ascending curve,
of the pulsations of the transmission sphyg
descending curve, measured by the reappea
exactly the same. The diastolic pressure,

FIG. 3.—Tracing t

aken in the usual way by swift inflation and gradual escape. It shows the systolic pressure t
and the diastolic to be by my own method 110, by Masing's 90.

of the greatest amplitude of pulsation on the ascending curve, was 120, while on the descending curve it was also the same. It will be observed that in this tracing there are great variations in the amplitude of the oscillations, both in the ascending and descending curves, but more especially on the curve of descent. These will require more careful consideration afterwards.

In the tracings which accompany this paper the lowest curve is that registered by the transmission sphygmograph. The abscissa is immediately above it, and the kymographic tracing starts from this level. Since the manometer has a double column of mercury, the height of the tracing above the abscissa must be doubled in reading the record; if the tracing at any point should be 60 mm. from the abscissa, the pressure at that point must be recorded as 120 mm. Hg. As usually employed, the pressure is raised with one steady forcible expulsion of the air contained in the syringe until a high level is reached—150 or 160 mm. Hg. being sufficient under ordinary circumstances. The valve already set at the slow escape allows the pressure to fall gradually. If sufficient pressure has been employed, the tracing from the artery, after a few oscillations due to the inertia of the mercury, shows an entire absence of all movements, or only very small pulsations, and the curve resulting from the gradual lowering of the column of mercury is therefore almost destitute of any fluctuations. The instant that the arterial pressure overcomes the resistance, small waves begin to appear in the arterial tracing. Sometimes, as was indeed noticed by von Recklinghausen (29), Janeway (4), and Masing (30), in taking tracings either with the sphygmograph or plethysmograph along with the Riva Rocci sphygmomanometer, one or two little waves show themselves before the appearance of definite pulsation. In most cases, however, the return of the pulsation is quite unmistakable. The usual appearances are shown in fig. 3, taken with slow revolution of the cylinder, and in fig. 4 with quick movement.

In the interpretation of the tracings there is one point which may always be depended upon with a reasonable degree of certainty: the point at which the pulsation returns in the vessel below the seat of compression is approximately the systolic pressure. This has been admitted ever since the observations of Vierordt (11), and von Basch (14). The middle point of the kymographic curve at this point is therefore chosen as the index of systolic pressure. It is perfectly true that it is not absolutely the end pressure. The top of the first wave which appears gives the maximum systolic pressure indeed, but it is the lateral and not the end pressure which is recorded, and therefore the method of circular compression, as was shown by Masing (30),* is not the absolute maximum. The determination of the minimum or diastolic pressure is not such an easy problem. Marey (12) originally suggested that the point at which the largest swing of the instrument occurred was an index of the mean pressure, and this was adopted and amplified by Roy and Adami (31). It has been proved experimentally, however, by Howell and Brush (27) that this does not indicate the mean, but really records the diastolic, pressure. Masing (30), who obtained some tracings from the artery below the seat of compression, believes that the greatest movement of the sphygmographic tracing marks the diastolic pressure. This must, however, be an error, as the greatest amplitude of oscillation of the sphygmographic curve is very commonly found after the pressure in the armlet has been allowed to fall nearly, if not quite, to zero, and the results obtained by this method of estimation are unmistakably erroneous. My own method of obtaining the diastolic pressure is to ascertain where the greatest amplitude of oscillation occurs in the kymographic curve, and to take the middle point of this as the expression of the diastolic pressure.

In fig. 3, for example, while the systolic pressure by every observer would be reckoned as 180 mm. Hg., the diastolic pressure by my estimate would be 110, and by the method of Masing 90 mm. Hg. In fig. 4 the systolic pressure is 170 mm. Hg., and the diastolic by my computation 106, while according to Masing it would be 90. In some other tracings this is even more striking, as the maximum excursions of the transmission sphygmograph occur after the pressure in the armlet has been almost entirely removed. Fig. 5 is a good example of such a tracing.

At first the lowest point of the greatest swing seemed to me that which might be considered as the index. There is one objection to this, however, which was suggested long ago by Marey (12), and which has been urged upon me by Dr Janeway in private correspondence. He points out that if the pressure in the apparatus is allowed to fall continuously, the inertia of the mercury will be apt to carry the lowest point of oscillation below the true level. This has been carefully tested by arresting the escape for a few seconds, after each five or ten millimetres of descent. It has been observed that if the escape was recommenced at an instant when the curve was falling, the inertia sometimes caused the curve to fall as much as 10 mm. If, on the contrary, the pressure recommenced its escape during an ascent of the kymographic index, the result never exceeded 5 mm., and was usually 2 or 3 mm. There is, therefore, real weight in the criticism, and it

* Masing allows that the difference between the lateral and terminal pressure in one of the larger arteries is inconsiderable.

Fig. 5.—Tracing taken in the usual way, showing that the maximum pulsations of the transmission sphygmograph occur when the pressure has fallen to zero.

has led me to modify my first method. By taking the mean instead of the lowest point of the curve, the inertia is as far as possible compensated for.

Fig. 6 is a curve obtained by the method of intermittent escape, and illustrates this point quite distinctly.

In order to ascertain whether the local condition of the circulation below the point of constriction might not be demonstrated better by means of the plethysmograph than by the transmission sphygmograph, a good many tracings have been taken by means of it. One of these is shown in fig. 7. It will be seen that on increasing the pressure in the armlet, the plethysmograph tracing at once rose to a point above the abscissa, and, instead of showing a curve of the pulsations, it only revealed a respiratory curve. As the pressure was allowed to fall, the plethysmograph curve fell below the abscissa, and at a certain point began to show definite pulsations. At this point, however, the curve began to rise to such a degree as to render the exact record somewhat difficult, and as, in every case, the same difficulty presented itself, the use of the plethysmograph was discontinued.

There are, undoubtedly, considerable difficulties in the estimation of diastolic pressure, since the amplitude of the pulsations, as revealed by the kymograph, is subject to so many influences. It is probable that the state of the arterioles, as regards contraction and relaxation, is the most powerful factor. But there can be no doubt that fluctuations in the energy of the heart itself play an important part. Some of the curves which have been obtained afford excellent illustrations of these difficulties. Of these, that which is shown in fig. 8 is very striking. It will be seen that there are, on the descending curve, at least four considerable fluctuations, as well as a few smaller ones. These are often very much like Traube-Hering curves, and certainly owe their origin to the same influences.

For practical purposes the observations which are made, whether with such a recording apparatus as has been described or any of the ordinary sphygmomanometers, should be noted on a graphic chart. It is difficult, without such a record, to watch the course of any investigations. Two charts of the arterial pressure and pulse rate are to be found in fig. 9 and fig. 10. In the former of these, with a normal pulse rate there is an extremely high arterial pressure. The facts were obtained from a patient with arterial sclerosis, cardiac hypertrophy, and interstitial nephritis. The latter records the observations in a case of Addison's disease, and it will be seen that, with a rather high pulse rate, there is an extremely low curve of pressure. They are taken from an address recently delivered by me (32).

To Professor Schäfer it is an agreeable duty to express my obligations for kind help and valuable advice in regard to many of the matters with which

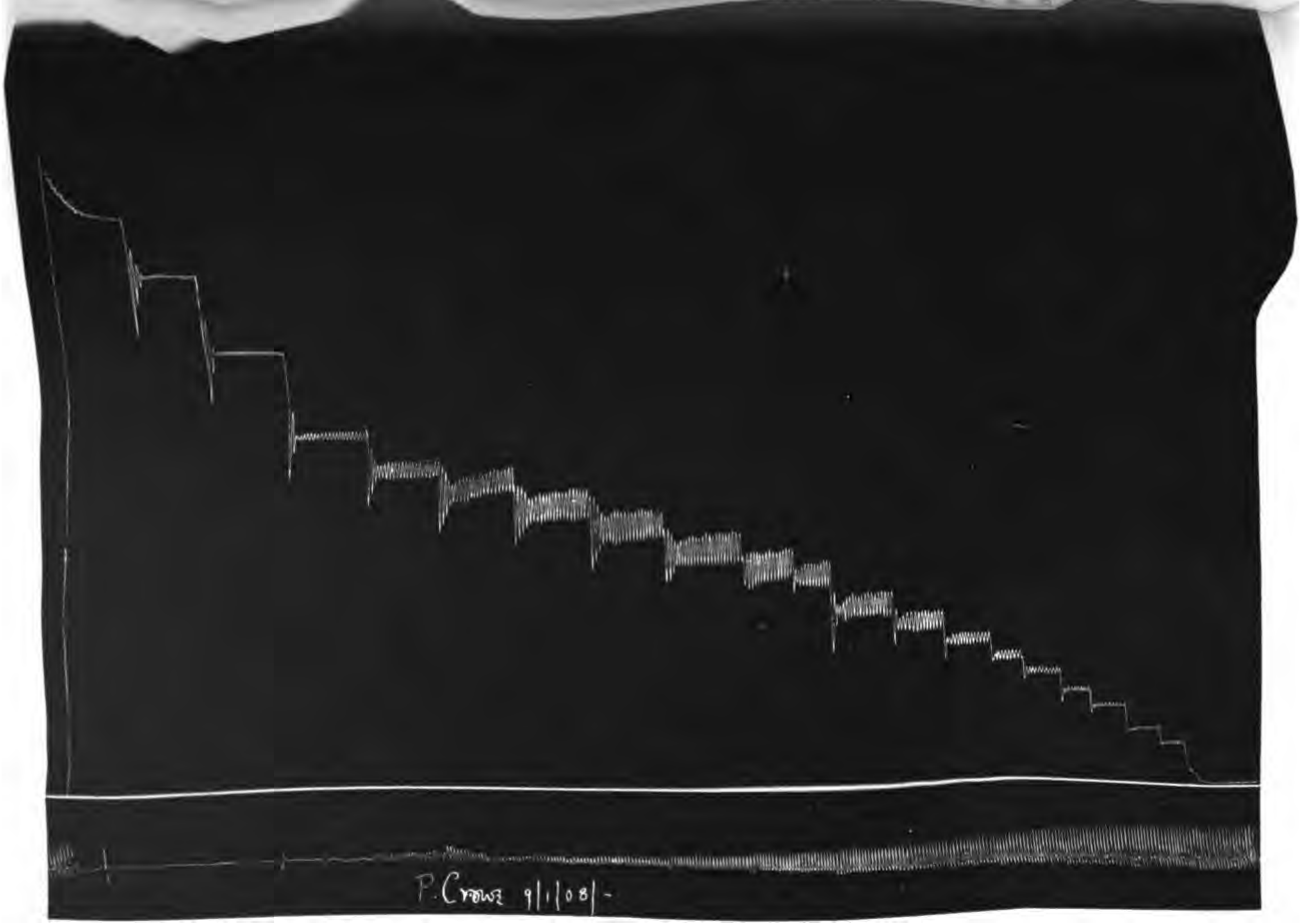


FIG. 6.—Tracing taken by rapid inflation and gradual escape, interrupted approximately after each descent of 5 or 10 mm. Hg. It shows a systolic pressure of 118, and a diastolic pressure of 74. The varying effects of [illegible] are observed, as described in the text

this paper is concerned; and it is also a sincere pleasure to acknowledge my indebtedness to my house physicians, Dr G. A. Gordon, Dr A. I. Shepheard Walwyn, and Dr G. Henderson, for their unvarying devotion and untiring exertions in assisting me with the investigations upon which it is based.

The instrument has been constructed by Mr Ednie, of the Physiological Laboratory, University of Edinburgh, who has spent much time and great ingenuity on overcoming difficulties, for which he deserves my warm thanks.

In a subsequent paper the practical results obtained from the study of arterial pressure will be described, and a detailed analysis of the different factors involved in it will be attempted.

LIST OF AUTHORITIES.

(1) Tigerstedt, *Lehrbuch der Physiologie des Kreislaufes*, Leipzig, 1893, S. 321; *Ergebnisse der Physiologie*, 1907, vi. Jahrgang, S. 265.

(2) Hill, *Text-book of Physiology*, edited by E. A. Schäfer, Edinburgh and London, 1900, vol. ii. p. 77.

(3) Vaschide and Lahy, *Archives générales de Médecine*, 1902, pp. 349, 480, and 602.

(4) Janeway, *The Clinical Study of Blood-pressure*, New York and London, 1904, pp. 43, 89, 93; *New York University Bulletin of the Medical Sciences*, 1901, p. 105.

(5) Hales, *Statical Essays*, London, 1733, vol. ii. p. 1.

(6) Poisseuille, *Recherches sur la force du cœur aortique*, Paris, 1828, p. 23.

(7) Ludwig, *Archiv für Anatomie, Physiologie, und wissenschaftliche Medicin*, Leipzig, 1847, S. 261.

(8) Chauveau and Marey, *Mémoires de l'Académie de Médecine*, Paris, 1863, tome xxvi. p. 268.

(9) Fick, *Archiv für Anatomie und Physiologie*, 1864, S. 583.

(10) Faivre, *Gazette Médicale de Paris*, 1856, p. 727.

(11) Vierordt, *Die Lehre vom Arterienpuls*, 1855, S. 164.

(12) Marey, *La Méthode graphique*, 1878, p. 610; *Travaux de Laboratoire*, 1876, tome ii. p. 316; *La Circulation du Sang*, 1881, p. 179.

(13) Waldenburg, *Berliner klinische Wochenschrift*, 1877, S. 17.

(14) Von Basch, *ibid.*, 1877, S. 179, 206, 225, 244, and 285.

(15) Hoorweg, *Archiv für die gesammte Physiologie*, 1889, S. 166.

(16) Potain, *Archives de Physiologie*, 1890, v. Série, tome ii. p. 300 and p. 681; and *La Pression artérielle*, Paris, 1902, p. 3.

(17) Hürthle, *Archiv für die gesammte Physiologie*, 1890, S. 1; 1891, S. 29 and 104; 1892, S. 281 and 323; 1893, S. 319; also *Deutsche medizinische Wochenschrift*, 1896, S. 574.

(18) Bloch, *Comptes rendus de la Société de Biologie de Paris*, 1888, p. 84.

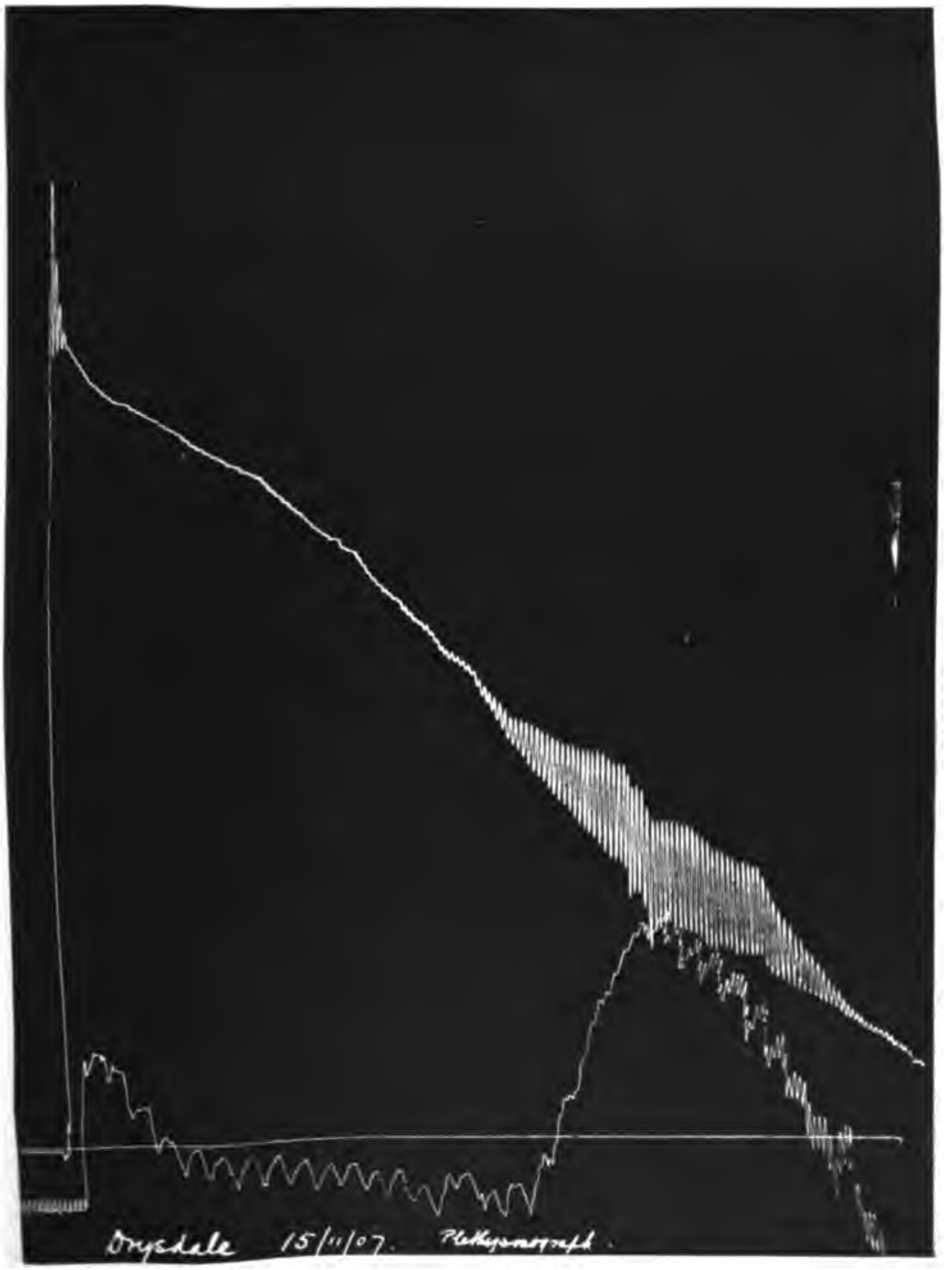


Fig. 7.—Tracing taken in the usual way, along with plethysmograph curve. It shows systolic pressure of 120, and diastolic pressure of 68.

(19) Mosso, *Archives italiennes de Biologie*, 1895, p. 177.

(20) Oliver, *Journal of Physiology*, 1897–98, p. 51; *A Contribution to the Study of the Blood and Blood-pressure*, London, 1901, p. 104; *Studies in Blood-pressure*, 1906, p. 12.

(21) Riva Rocci, *Gazzeta medica di Torino*, 1896, p. 981 and p. 1001.

(22) Hill and Barnard, *British Medical Journal*, 1897, vol. ii. p. 904.

(23) Gärtner, *Wiener medizinische Wochenschrift*, 1899, Band xlix. S. 1412.

(24) Stanton, *University of Pennsylvania Medical Bulletin*, 1903, p. 466.

(25) Cook, *Journal of the American Medical Association*, 1903, p. 1199.

(26) Erlanger, *American Journal of Physiology*, 1904, vol. x.; *Proc. Amer. Phys. Soc.*, p. xiv.

(27) Howell and Brush, *Boston Medical and Surgical Journal*, 1901, p. 146.

(28) Gibson, *Quarterly Journal of Medicine*, 1907, vol. i. p. 103.

(29) Recklinghausen, *Archiv für experimentelle Pathologie und Pharmacologie*, 1901, S. 78.

(30) Masing, *Deutsches Archiv für klinische Medicin*, 1902, Bd. lxxiv. S. 253, 258, 263.

(31) Roy and Adami, *The Practitioner*, 1890, vol. xlv. p. 32.

(32) Gibson, *Edinburgh Medical Journal*, 1908, new series, vol. xxiii. p. 17.

(*Issued separately June* 22, 1908.)

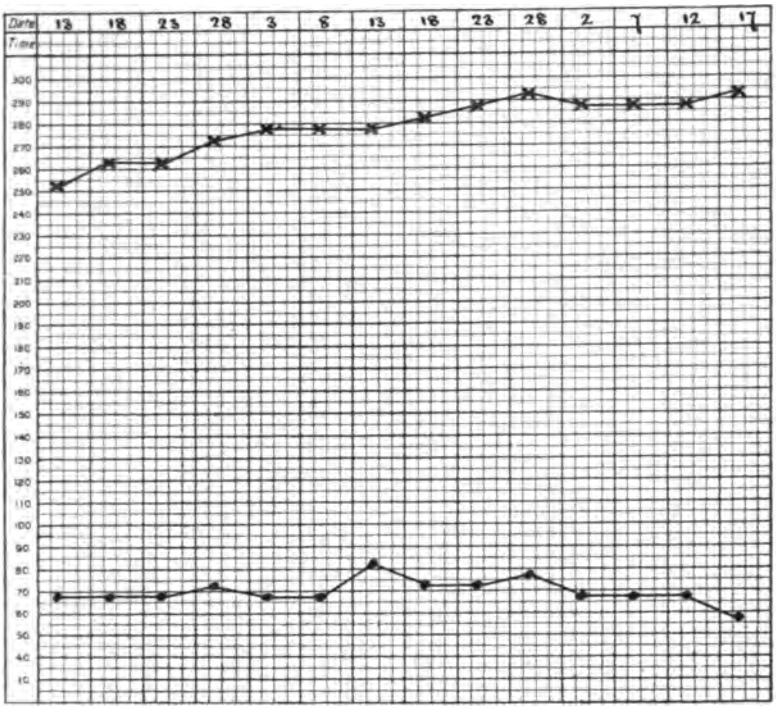


Fig. 9.—Chart of pulse rate and arterial pressure from a case of arterial sclerosis, cardiac hypertrophy, and interstitial nephritis. It shows a continuous high pressure in spite of all measures adopted. The record gives the figures every fifth day at 11 a.m. from 13th September to 17th November. The crosses give the pressure and the dots the rate.

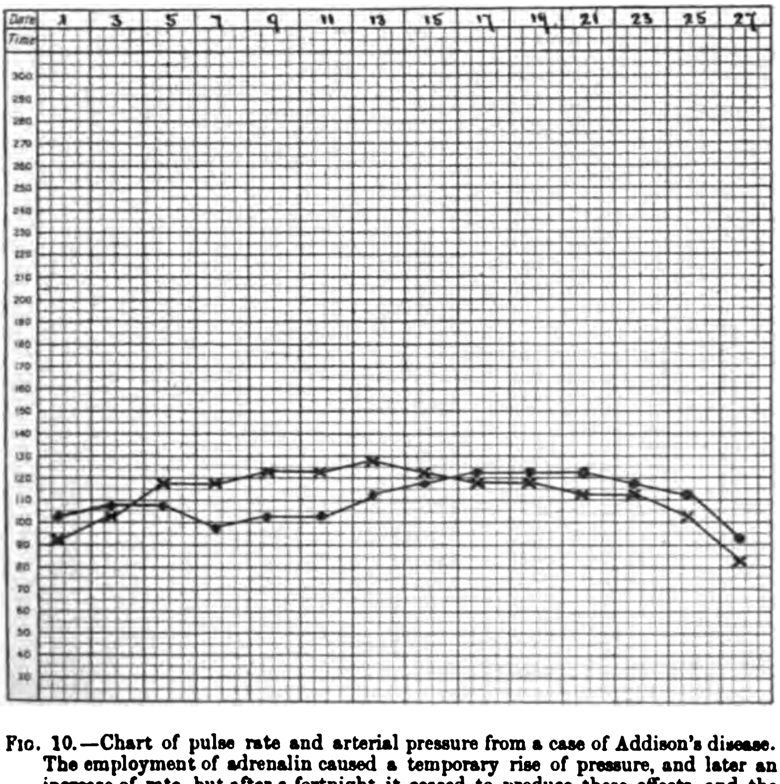


Fig. 10.—Chart of pulse rate and arterial pressure from a case of Addison's disease. The employment of adrenalin caused a temporary rise of pressure, and later an increase of rate, but after a fortnight it ceased to produce these effects, and the patient gradually sank and died. The pressure is recorded every second day, taken about 11 a.m., from 1st to 27th October.

XXI.—Preliminary Statement on the Morphology of the Cone of Lycopodium cernuum and its bearing on the affinities of Spencerites.

By **William H. Lang**, M.B., D.Sc., Lecturer in Botany, Glasgow University. *Communicated by* Professor F. O. Bower, Sc.D., F.R.S.L. and E.

(MS. received May 5, 1908. Read May 5, 1908.)

It is a remarkable fact that in spite of the primitive features which appear to be preserved in the existing genus *Lycopodium*, there is little or no evidence pointing to ancient forms to which this genus itself can be related. Some of the smaller palæozoic Lycopodiales, impressions of which are named *Lycopodites*, may perhaps have been eligulate, homosporous forms, but in many of the better known examples they appear to have been heterosporous, and suggest comparison rather with *Selaginella* than with *Lycopodium*. In the course of a re-examination of the morphology and structure of *Lycopodium cernuum*, and a comparison of it with other species of *Lycopodium*, features of interest in the morphology of the cone were disclosed which indicated a remarkable similarity in plan of construction between this *Lycopodium* and the cone of *Spencerites*. A preliminary account of the cone of *L. cernuum* will be given here, leaving the consideration of the anatomy and the comparison with other species until the full account is published.*

The material of *L. cernuum* which has served for this work was generously placed at my disposal by Professor Bower, for whose use it had been collected in Ceylon by Dr J. C. Willis. The cones attained a length of

* While this preliminary statement was in preparation, a paper by Miss M. G. Sykes has appeared which deals with the subject ("Notes on the Morphology of the Sporangium-bearing Organs of the Lycopodiaceæ," *New Phytologist*, vol. vii. p. 41). In this the cones of a number of species of *Lycopodium*, including *L. cernuum*, are described, and the distal position of the sporangium, the presence of lignified tissue in the sporangial stalk, and other points of interest are recorded for the first time. Miss Sykes suggests the derivation of such a plant as *L. cernuum* from *Lepidodendron* or one of its allies, with *Spencerites* as an ancient connecting link. The genus *Lycopodium* is regarded as exhibiting a reduction series, and the sporophyll in *L. cernuum* (from which that of *L. inundatum* is derived by reduction) as an axial structure, terminated by a single sporangium and bearing a single leaf. It will be evident from the following description that I have been unable to accept as adequate the account of the morphology of the cone of *L. cernuum* given by Miss Sykes. Further, while recognising an affinity between *Spencerites* and *Lycopodium*, I do not see sufficient ground for the interpretation of the morphology of the cone adopted by this author, nor for necessarily regarding the genus *Lycopodium* as a reduction series.

12 mm. and a diameter of 3 mm. Their general external appearance is represented in the figure given by Pritzel in *Die natürlichen Pflanzenfamilien*,* in which the abrupt transition from the vegetative leaves on the branchlets bearing the cones to the sporophylls is shown. The sporophylls, like the vegetative leaves, are borne in alternating whorls of five, and thus form ten vertical ranks on the axis of the cone. Each sporophyll consists of a horizontal base and a nearly erect lamina with a fimbriate margin. The large sporangium is attached by a relatively small area of insertion or stalk close to the distal limit of the horizontal sporophyll-base, and lies against the upper surface of the latter. The lower portion of the sporophyll-base is occupied by a large mucilage cavity. The mutual relations of the sporophylls in the cone are more complicated than appears from any of the accounts hitherto given. This will be evident from a study and recombination of figs. 1–4, which represent radial longitudinal, tangential, and transverse sections of mature cones; these figures are outlines, based on camera lucida drawings, and not diagrammatic.

The radial section (fig. 1) passes on either side through a vertical series of sporophylls, the two series belonging of necessity to alternate whorls. In the case of one sporophyll on the left, which has been cut in an absolutely median plane, the course of the slender vascular trace can be followed from the stele into the lamina. The leaf-trace originates from the stele at the level of the insertion of the whorl of sporophylls below, or, what comes to the same thing, at the lower limit of the base of the sporophyll which it supplies. Structures standing at the same horizontal level in the alternating whorls can be readily ascertained by a comparison of the two sides of this section.

Each sporophyll shows in section the thick horizontal base and the obliquely erect lamina already referred to. The lower portion of the sporophyll-base is occupied by a large mucilage cavity (*m*). The mucilaginous change extends to the surface involving the epidermis, so that this portion of the sporophyll-base may be described as consisting of a mass of mucilage bounded below by a structureless membrane.† The upper portion of the sporophyll-base consists of persistent tissue, traversed by the vascular bundle; and the outer (abaxial) surface of the sporophyll-base is also composed of persistent tissue. The lamina has no mucilage cavity and is continuous with the persistent tissue of the base. The vascular bundle traverses the pedicel-like portion of the base and continues throughout

* Theil i., Abth. iv., p. 603, fig. 379.

† The extension of the mucilaginous change to the surface is shown in Pl. xix., fig. 13, of Mr T. G. Hill's paper in the *Annals of Botany*, vol. xx., but is not commented on.

the greater part of the length of the lamina. The epidermal cells of the outer surface of the lower half of the lamina and the vertical outer face of the sporophyll-base are large and have thick, lignified walls which show curious undulations and infoldings.* The sporangium is attached to the upper surface of the sporophyll-base immediately over the vascular bundle and close to the commencement of the upward curve of the lamina. It is thus in the mature condition removed a considerable distance from the axis, and the large sporangium extends between the attachment and the latter, and only projects to a relatively slight extent to the distal side of the attachment. The stalk is short, and consists of tracheide-like cells with lignified walls

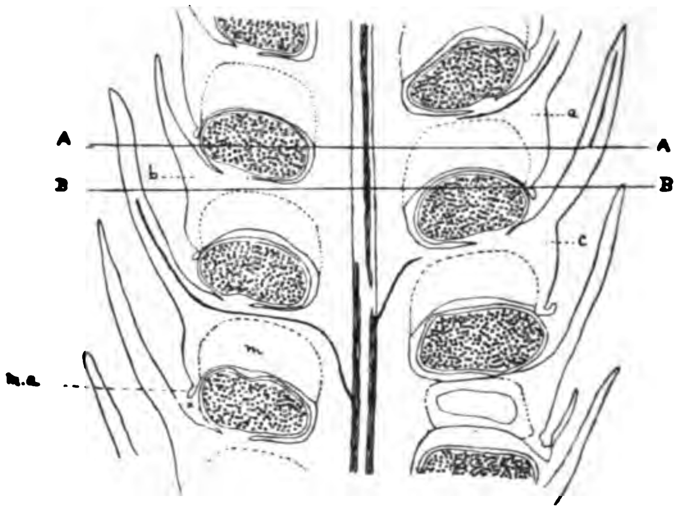


FIG. 1.

and a close, spiral thickening.† The basal cells of the sporangial wall have the same characters, and this tissue constantly extends for a short distance distally as a modification of the epidermal cells of the lower part of the lamina. The wall of the sporangium consists of two layers of cells, the inner being thin-walled, while those of the epidermal layer have bands of thickening on the inner periclinal wall and all the anticlinal walls. While these thickenings can be traced all round the sporangium, they are best developed and most strongly lignified over the distal convexity; it is in the

* Miss Sykes suggests (*loc. cit.*, p. 49) that these cells play a part in bending back the lamina to allow of spore dispersal.

† This feature was first recognised by Miss Sykes, and is figured by her for *L. cernuum* (*loc. cit.*, Pl. iii., fig. 8). Similar lignified cells in this position are recorded for *L. inundatum* and *L. carolinianum*, and slight indications of lignification were found in the stalks of some other species. The significance of this feature is at present obscure.

middle of this face that dehiscence takes place. The point of dehiscence is immediately opposite the small space between the sporophylls, and is marked by an *x* in the lowest sporangium on the left-hand side of fig. 1.

The disappearance of the mucilaginous mass from the sporophyll-base explains the strongly peltate form of the sporophyll in Pritzel's figure.* To this extent the sporophyll might better be termed *pseudo-peltate*. In *L. cernuum* the sporophyll is also truly peltate though only slightly so. In the radial section (fig. 1) a small downgrowth of the outer margin of the sporophyll-base (*m.a.*) can be seen lying against the surface of the sporangium immediately below. In a young stage of development this marginal growth fitted in between the sporangium and its sporophyll, but has become removed from this position, which originally determined its form, by the further growth of the cone. The marginal appendage, unlike the persistent tissue above, is bounded on both sides by an epidermis, with usually a single layer of mesophyll; its cells are throughout thin-walled. The sporangium at its origin is situated close to the axis of the cone, and its distal position when mature is due to subsequent growth of the basal region of the sporophyll; this takes place after the mucilage cavity has been formed.

The structure described above is characteristic of all the sporophylls of any mature cone except those at the base and those at the summit. In the basal sporophylls the mucilaginous change does not extend to the surface of the sporophyll-base and the mucilage cavity is bounded by one or more layers of cells. In this respect, though not in their general form, these sporophylls are intermediate between the sporophylls above and the vegetative leaves. Towards the apex of the cone the sporophyll-bases are inserted at an increasingly acute angle, and on the upper sporophylls arrested sporangia are constantly found. In these abortive sporangia the stalk is normally or even excessively developed, while the small cavity is occupied by a degenerated mass of spore-mother-cells. The sporophylls in this region, being arrested in earlier stages of their development, often have the mucilage cavity still limited by a layer of cells.

A consideration of the relative positions of the parts in the sporophylls of the alternating whorls on the opposite sides of the axis in fig. 1 will make it clear that the leaf-bases of any one whorl (*a*) must hang down between the sporangia of the alternating whorl below (*b*) and reach to the level of the sporophylls of this whorl. What the radial section does not by itself demonstrate is the fact that the sporophyll-base is coherent with the margins of the two sporophylls between which it lies. This is seen in the tangential section of the cone (fig. 2). If the outline of any one leaf-

* *Loc. cit.*, fig. 379, E.

horizontal portion, the section here passing through the junction of the lamina and sporophyll-base. One sporophyll has been cut across at a slightly higher level, and is not traversed by the section throughout its length. It shows the transversely widened pad of tracheidal tissue forming the sporangial stalk (*sp. a*). Between these sporophylls of the whorl *b* appear the summits of sporangia belonging to a lower alternating whorl (*c*). Reference to the plane of section B B indicated on fig. 1 will show that these sporangia are bulging up into the mucilaginous bases of the whorl of sporophylls *a*. These sporophyll-bases have been traversed at the level of their lower horizontal surfaces. The outer persistent portions of these sporophyll-bases (*a*) are cut through, but being here traversed at the level common to them and to the sporophylls of the alternating whorl below (*b*), are not free as in fig. 3. In fig. 4 they are seen to be continuous with the sporophylls between which they stand. If the tissue of the lower surface of the sporophyll-base *a* had persisted, this continuity would be apparent as a web of tissue connecting the sporophyll-bases of the whorl *b* from their proximal to their distal end. Since, however, only the outer margin of the sporophyll-base *a* is composed of persistent tissue, the continuity appears as a band connecting the sporophylls below (*b*) at the junction between the sporophyll-base and the lamina.

The evidence afforded by the sections described shows that the sporophyll-bases in *L. cernuum* are coherent to a short distance further out than the place of attachment of the sporangia. This conclusion has been fully confirmed by the study of sections of cones of different ages. A little consideration will make it clear that, when dealing with a mature cone, only three planes of section will afford evidence of continuity of persistent tissue (as contrasted with that shown by the membrane bounding the mucilaginous sporophyll-base). These three sections are:—

1. A transverse section at the level of the pedicel-like persistent portions of the sporophyll-bases of one whorl (*b*) and the lower limit of the sporophyll-base of the whorl above (*a*). (*Cf.* the plane B B in fig. 1, and fig. 4.)

2. A tangential section close to the periphery of the cone and passing through the persistent tissue of the abaxial surface of the sporophyll-bases.

3. Sections slightly removed from the radial plane, showing the margin of the lamina of one sporophyll connected with the lower margin of the sporophyll-base of the whorl above. (*Cf.* the lowest complete sporophyll on the right-hand side of fig. 1, where this continuity is *nearly* shown.)

Were the mucilage wholly removed from the cone, the fact that the sporophyll of *L. cernuum* is not composed of a free subcylindrical pedicel bearing a large peltate lamina would only be shown by the continuity of the middle of the mucilage cavity, and, except at the periphery, which corresponds to the persistent portion of the pseudo-peltate sporophyll-base, shows only a membrane limiting a mass of mucilage. The vascular bundles on

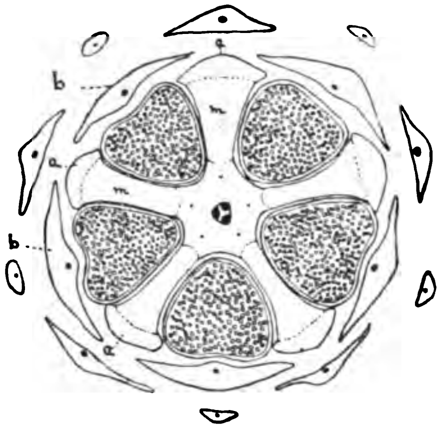


FIG. 3.

their way out to supply the sporophylls of this whorl (*a*) are seen on the corresponding radii in the cortex.

The second transverse section (fig. 4) follows the plane marked by the

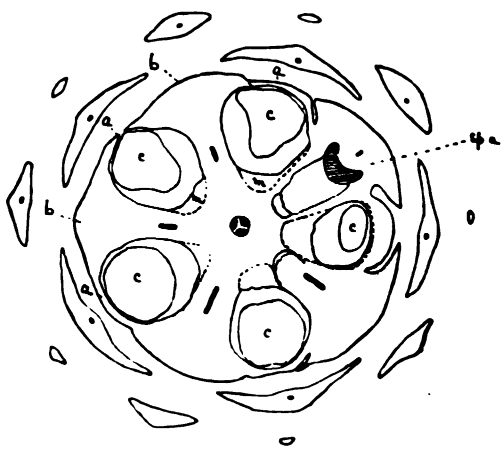


FIG. 4.

line B B in fig. 1. It cuts the cone nearer to its base in the plane of the "pedicels" of the sporophylls (*b*), to which the sporangia shown in fig. 3 belong. The five sporophylls are seen radiating from the axis, and within each is its leaf-trace. The sporophyll widens out at the extremity of the horizontal portion, the section here passing through the junction of the lamina and sporophyll-base. One sporophyll has been cut across at a slightly higher level, and is not traversed by the section throughout its length. It shows the transversely widened pad of tracheidal tissue forming the sporangial stalk (*sp. a*). Between these sporophylls of the whorl *b* appear the summits of sporangia belonging to a lower alternating whorl (*c*). Reference to the plane of section B B indicated on fig. 1 will show that these sporangia are bulging up into the mucilaginous bases of the whorl of sporophylls *a*. These sporophyll-bases have been traversed at the level of their lower horizontal surfaces. The outer persistent portions of these sporophyll-bases (*a*) are cut through, but being here traversed at the level common to them and to the sporophylls of the alternating whorl below (*b*), are not free as in fig. 3. In fig. 4 they are seen to be continuous with the sporophylls between which they stand. If the tissue of the lower surface of the sporophyll-base *a* had persisted, this continuity would be apparent as a web of tissue connecting the sporophyll-bases of the whorl *b* from their proximal to their distal end. Since, however, only the outer margin of the sporophyll base *a* is composed of persistent tissue, the continuity appears as a band connecting the sporophylls below (*b*) at the junction between the sporophyll base and the lamina.

The evidence afforded by the sections described shews that the sporophyll bases in *[illegible]* are coherent to a short distance further out than the place of attachment of the sporangia. This conclusion has been fully confirmed by the study of sections of cones of different ages. A little consideration will make it clear that, when dealing with a mature cone, only three planes of section will afford evidence of continuity of persistent tissue [illegible] the membrane bounding the [illegible]. These three sections are —

1. A [illegible] persistent portions [illegible]

2. [illegible]

3. [illegible]

[illegible]

the persistent peripheral tissues in one or other of these three planes of section.

So far as is known, the cone of *L. cernuum* is the most complicated in the genus *Lycopodium*. The biological interest of the very complete protection afforded to the sporangium by the pads of mucilage which bound it on all sides except at the small distal opening, is obvious in the case of a plant flourishing in exposed situations in the tropics. A similar result is obtained in the case of *L. inundatum*,* where, however, the sporophyll-bases are not coherent, and in varying degrees in other species. In *L. cernuum* alone, so far as is known, does the mucilaginous change extend to the surface of the leaf-base and render the sporophyll, as seen in accurately median section, distinctly pseudo-peltate. Sporangia inserted at a greater or less distance from the axis are found in *L. carolinianum*, *L. clavatum*, and some other species, but this feature is never so marked as in *L. cernuum*. The extension of the sporangium between its place of attachment and the axis of the cone is also a distinctive feature of the latter plant.

The special interest attaching to the cone of *Lycopodium cernuum* lies in the light which it appears to throw on the morphology of the cone of *Spencerites* and on the affinities of that plant. The comparison may be for the present confined to *S. insignis*, the better known species, and it will suffice to refer generally to the works of Williamson,† Scott,‡ and Berridge § for the descriptions and figures of the cone.

In *Spencerites* the cone consisted of an axis bearing sporophylls which were usually arranged in alternating whorls. The sporophyll is thus described by Dr Scott (*Progressus*, p. 170): "The sporophyll in *S. insignis* consists of a narrow pedicel bearing an upturned lamina with a dorsal lobe; at the base of the lamina is a massive ventral outgrowth to which the distal end of the sporangium is attached by a narrow neck." Elsewhere the sporophyll, owing to the extension downwards of the "dorsal lobe," is described as peltate. All these points are well represented in the reconstructed radial section given in Miss Berridge's paper, which also shows the small marginal appendage of the dorsal lobe. This figure should be compared with fig. 1 of the present paper.

* The relations between the sporophyll-bases and the sporangia in *L. inundatum* were described and figured by Glück (*Flora*, 80, 1895, p. 359. Pl. v. figs. 1-3), whose account I can confirm. The sporophyll does not possess the peltate form shown in Miss Sykes' figure (*loc. cit.*, text-fig. 5). This invalidates the argument advanced on p. 54 of her paper in favour of the *L. inundatum* type of cone being reduced from that of *L. cernuum*.

† "Organisation of the Fossil Plants of the Coal Measures," *Phil. Trans.*, Part ix., 1878; Part x., 1880; Part xvi., 1889; Part xix., 1893.

‡ *Phil. Trans.*, vol. 189 (1897), Ser. B., p. 83; *Progress. Rei. Botanicæ*, Bd. I., p. 170.

§ *Annals of Botany*, vol. xix. (1905), p. 273.

Almost every feature in the sporophyll of *Spencerites*, as seen in radial section, except the protuberance of tissue bearing the sporangium, is paralleled by the radial section of *L. cernuum*, if we assume the removal or disappearance of the structureless mucilage from the latter. The relation of the pedicel-like portion of the sporophyll to the distal portion, the shape of the latter, and the marginal appendage, all appear to correspond. The position of the sporangium is essentially similar, though it is somewhat less distal in *L. cernuum*. The close correspondence between the two sporophylls appears to justify the suggestion that in *Spencerites*, as in *L. cernuum*, the peltate appearance of the sporophyll is in the main of secondary origin, and is due to the disappearance of a mucilage cavity from a large sporophyll-base, leaving only the pedicel and the outer abaxial portion (the dorsal lobe), with which the lamina is continuous. The similarity of the marginal appendage shown in Miss Berridge's reconstruction to that seen in *L. cernuum* (fig. 1, *m.a*), further suggests that in *Spencerites*, as in the recent plant, this fitted in between the sporangium and sporophyll vertically below. The fact that in the fossil it appears to have rested in the mature cone, not against the sporangium but on the ventral outgrowth, may indicate that the origin of this outgrowth was subsequent to that of the sporangium in the ontogeny. If this interpretation of the sporophyll of *Spencerites* be correct, the sporophyll would be in great part pseudo-peltate, and the sporangial insertion would be distal on the sporophyll-base and not on the lamina, which would bring *Spencerites* into line with other Lycopodiales in this respect.

The view just expressed, that the sporophyll of *Spencerites* owes its shape to the disappearance of a mucilaginous portion of the sporophyll-base before fossilisation, might be advanced on the comparison of the radial sections in the fossil and in *L. cernuum*, even if the sporophylls in *Spencerites* were throughout free from one another. The justification of the comparison is, however, increased by the existence of evidence pointing to the sporophylls of *Spencerites* having been coherent in a similar manner and degree to those of *L. cernuum*. This possibility of coherence of the sporophylls is not entertained in the later accounts given by Scott and Berridge, but was stated by Williamson in his first account of this cone.* He there figures a cross-section of a strobilus (*loc. cit.*, Pl. 22, fig. 53), and says regarding it: "The section, fig. 53, appears to have been made near one extremity of the strobilus, where some of the discs seen at the right-hand part of the figure are in close contact with each other (53, *c*), and in some cases these sporangiophores are actually confluent (53, *c'''*)." This section (Williamson Coll., 626)

* "Organisation," *Phil. Trans.*, Part ix., 1878, p. 341.

was re-examined by Scott, who figures several sporangia from it, but he does not refer to Williamson's statement that the sporophylls may be coherent, only stating, in reference to the abortive sporangia found on one side of the section, "the sporophylls in this part are crowded, and apparently somewhat displaced." *

Further study of this section leads me to entertain Williamson's view, that the sporophylls are coherent distally, as highly probable if not actually established. The full grounds for this cannot be given here, and require the refiguring of the section, but a brief description with reference to fig. 53 of Williamson's memoir will bring out the most important facts. It should be mentioned that the peripheral structures of the left-hand side of the cone are omitted in this very accurate figure.

The section has the stele of the axis displaced to one side by a large Stigmarian root, and this stele is cut obliquely. It does not appear to have been hitherto recognised that the whole section is an oblique one, the obliquity of the section of the stele not being due to its displacement. As Williamson and Scott both state, the section is cut near one end of the strobilus. I would add that it is cut near to the summit of a mature cone, and that it affords evidence that in *Spencerites*, as in *L. cernuum*, the sporophylls were here inserted at a more acute angle on the axis. The right-hand side of the section in fig. 53 is the higher on the cone, and the section follows the plane of the pedicels or sporophyll-bases on the right-hand side. Passing obliquely across the cone, it cuts the structures round the upper and lower sides at a lower level, and on the left-hand side the section necessarily passes across sporangia and sporophylls of two or sometimes three whorls.†

On the right-hand side of the figure, when the section has passed in the plane of a whorl of sporophyll-bases, the broad pedicels of these are seen widening out at their distal extremities where the laminæ are, as usual, wanting. Intervening between the sporophylls are the sporangia belonging to the alternating whorl lower on the cone (*cf.* fig. 4, above).

These sporangia are all in an arrested condition of development, and their structure is exactly paralleled by the abortive sporangia found constantly near the summit of the mature cone of *L. cernuum*. Immediately outside these intervening sporangia, at one level traversed by the section, the distal ends of the whorl of pedicels appear to be connected by bands of tissue. In the light of the cone of *L. cernuum* we can recognise these

* *Phil. Trans*, B. 189, p. 94.

† An example of this is seen in a figure in Dr Scott's paper (*Phil. Trans.*, B. 189, Pl. xiv. fig. 13), which comes from this side of the section; the sporophyll-base here belongs to the whorl vertically above that to which the sporangium and its insertion belong.

connections as belonging to the sporophyll-bases of the alternating whorl above. On the right-hand side the plane of section appears to have passed through the sporangial attachments, and the connections between the sporophylls are best seen a short distance round from this on the lower side (at c''' and further round). The section is comparable with one of the three sections which have been seen above to afford evidence of the cohesion of sporophyll-bases in the mature cone of *L. cernuum*. I have obtained corroborative evidence of the connection of the distal parts of the sporophyll-bases, in a section from Dr Kidston's collection, passing in a corresponding plane through the lower region of a cone of *Spencerites*; in it the pedicels are horizontal and the sporangia normal. If continuity of the distal portions of the sporophyll-bases in *Spencerites*, corresponding to what is found in *L. cernuum*, can be established, as seems probable, we are forced to assume a similar disappearance of the connection of the proximal portions in the two cases. The evidence derived from these transverse sections thus supports the interpretation suggested by the corresponding shape of the sporophylls in the two cones as seen in radial section. There is reason to think that at c''' in Williamson's fig. 53, the connection between the proximal portions of the sporophylls is persistent and preserved.

I have confined myself in this preliminary statement to giving an account of the general morphology of the cone of *Lycopodium cernuum* and of showing that a number of features in the cone of *Spencerites* can be interpreted in the light of what is known of the living plant. The discussion of the difficulties in the way of the homologies suggested above, and of the general questions which they raise, must be deferred until a full description can be published. One difficulty which I have felt strongly is the apparently definite limitation of the surface of those regions of the pedicel and axis in *Spencerites* that are assumed to have abutted on a mucilage cavity and not to have formed part of the original, free, outer surface of the plant. It can only be said here that little or no distinction can be made in *L. cernuum* between the tissues in the same relative positions.

The facts adduced, though not amounting to a demonstration, appear to afford a *prima facie* case for regarding the cone of *Spencerites* as having been constructed on essentially the same plan as that of *L. cernuum*. This involves the view that in the mature cones of *Spencerites*, which are alone known, the inner and lower portions of the dependent sporophyll-bases had been removed or destroyed before petrifaction. The pedicels would thus be the equivalent of the persistent upper portion of the sporophyll-base in *L. cernuum*, and the dependent "dorsal lobe" of the outer (abaxial) portion of the sporophyll-base. The disappearance of the presumably mucilaginous portion of the sporophyll-base would destroy the indications of the continuity of this with the alternating sporophyll-bases to either side, and, as in the mature cone of *L. cernuum*, this is only to be found in the relations of the distal portion of the leaf-base to the alternating sporophylls lower on the axis of the cone. The sporangium in *Spencerites* is inserted on a special outgrowth which on this view springs from the upper surface of the distal region of the radially extended sporophyll-base. It is, I think, open to question whether any special morphological significance should be attached to this outgrowth. There does not appear to be any evidence to justify the interpretation of the region between the insertion of the sporangium and the axis of the cone as being of axial nature either in the recent or the extinct plant. This region in *L. cernuum* is intercalated late in the ontogeny, and the same may have been the case in *Spencerites*.

Allowing for the smaller size and simpler structure of the cone of *L. cernuum*, the resemblances between it and the cone of *Spencerites* appear to be so close that it is difficult to avoid the conclusion that the plants are related. Taking the probability of *Spencerites* having been eligulate and homosporous into consideration, the resemblances appear to point to the inclusion of *Spencerites* and *Lycopodium* in the same great subdivison of the Lycopodiales. There does not appear to be any evidence for deriving this eligulate group from *Lepidodendron* or any other member of the ligalate Lycopodiales. To what extent *Spencerites* is more especially related to *Lycopodium cernuum* itself or presents a similar development, as regards the complexity of the cone, from a common type; to what extent the genus *Lycopodium* is a natural one, and whether the simpler existing forms are reduced or simply persistent types, are matters for critical inquiry in the light of all the available evidence, not merely that afforded by the morphology of the cone.

I have pleasure in acknowledging the assistance I have received both in discussion and in the provision of material from Professor Bower and Dr Kidston. To the authorities of the Natural History Department of the British Museum I am indebted for facilities in consulting the specimens in the Williamson Collection.

DESCRIPTIONS OF FIGURES.

Fig. 1. Outline drawing of a radial longitudinal section of the middle region of a mature cone of *Lycopodium cernuum*. The dotted outline indicates the boundary of the mucilage cavity (*m*) against the persistent tissue of the sporophyll-base and axis. *m.a.*, marginal appendage. The letters *a*, *b*, *c* indicate corresponding whorls of sporophylls in all the figures. Further explanation in text.

Fig. 2. Tangential section of a mature cone of *L. cernuum* in the plane of the sporangial insertion. Lettering as in fig. 1. Explanation in the text.

Fig. 3. Transverse section of a mature cone of *L. cernuum* at the level A A in fig. 1. Lettering as in fig. 1. Explanation in the text.

Fig. 4. Transverse section of a mature cone of *L. cernuum* at the level B B in fig. 1. *sp.a.*, sporangial attachment. Other lettering as in fig. 1. Explanation in the text.

(*Issued separately June* 22, 1908.)

XXII.—Note on the Electrical Resistance of Spark Gaps. By Robert A. Houstoun, Ph.D., D.Sc. *Communicated by* Professor A. Gray, F.R.S.

(MS. received May 4, 1908. Read June 1, 1908.)

Of recent years a considerable amount of work has been done in measuring the electrical resistance of spark gaps under different conditions. The subject is important, both for its application to wireless telegraphy and on account of its bearing on the mechanism of the spark gap. Spark spectra are, as is well known, more complex than arc spectra. Formerly this was attributed to a very high temperature in the spark, but it is now regarded as due to a disintegration of the atom produced in the spark gap. Or, as Baly puts it in his book on *Spectroscopy*, in the spark gap the atom is in a state of assisted radio-activity. If the current density in the spark gradually increases until it is strong enough to disintegrate a particular system in the atom, we should expect a new spectrum to be produced then, and probably a change in the resistance of the spark gap. My experiments have shown me that our methods of measuring the resistance of spark gaps are not nearly accurate enough to show such a change if it existed.

The object of this note is to record the effect of changing the material of the electrode on the resistance of the spark gap for eight different metals, the other conditions being always the same. This has as yet been done only by Slaby's method, in which an additional spark gap is inserted in the circuit. (*Cf.* Fleming's *The Principles of Electric Wave Telegraphy*, p. 184.) I employed the method of taking a resonance curve first used by Bjerknes, which according to Eickhoff* is much to be preferred to the former method.

The apparatus consisted of an oscillator and a resonator (*cf.* fig.). The oscillator consisted of a condenser K, the opposite sides of which were connected by short thick copper wires to the electrodes E, between which the spark passed. The electrodes, no matter what the metal was, were cylindrical, 4 mm. diameter, about 3 cms. long, and had rounded ends. They were connected to the secondary of a large induction coil. The condenser in the primary circuit was formed of six zinc plates 20×20 cms., placed at a distance of 2 cms. from one another, and hence had a capacity of 80 cms. The resonator was made in Drude's form. It consisted of two

* Wilh. Eickhoff, *Phys. Zeits.*, Aug. 1, 1907, p. 494.

parallel tubes in which rods slid, the rods and tubes being mounted vertically 2 cms. apart. The lower ends of the tubes were attached to the opposite plates of a condenser C, the capacity of which was 50 cms. The

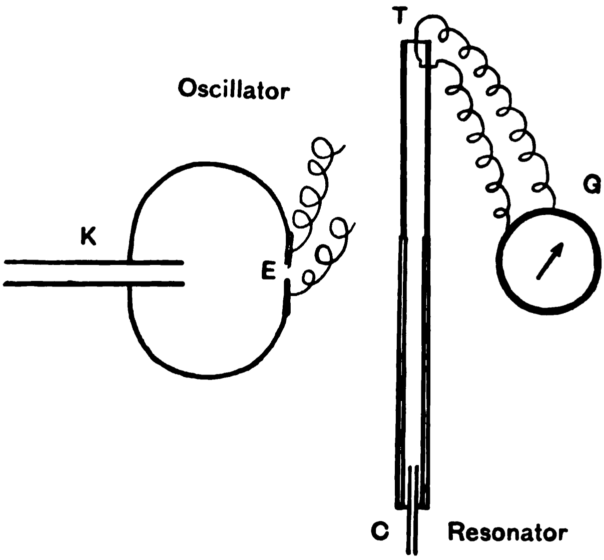


FIG. 1.

upper ends of the rods were joined by a thermo-couple T, from which wires led to a sensitive d'Arsonval galvanometer G.

The thermo-couple, which fulfilled its purpose very well, is shown in

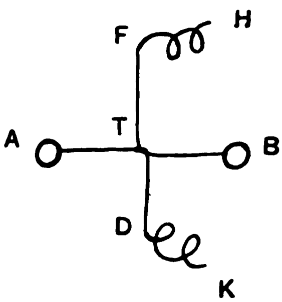


FIG. 2.

the second diagram. A, B are the ends of the rods viewed from above. ATD is a short piece of thin platinum wire, BTF a short piece of thin constantan wire, both of which are held together by a very small drop of solder at T. FH and DK are the leads to the galvanometer. When the current passed from A to B, the junction T heated and a current passed in the galvanometer circuit.

The resonator was thus a long narrow rectangle, the length of which could be varied from 58 to 100 cms. by sliding the rods inside the tubes. The self-inductance of the resonator was found to be strictly proportional to its length. The latter was read off on a scale.

The procedure was as follows:—When the induction coil was started it charged the condenser. The latter then discharged itself through the spark gap in a train of diminishing oscillations and induced oscillations in the resonator. These heated the thermo-couple and produced a deflection on the galvanometer scale. This deflection was proportional to the mean square of the induced current. The self-inductance—*i.e.* the length—of the resonator was altered until this deflection was a maximum. The resonator was then in resonance with the oscillator. The deflection of the galvanometer was then read for different lengths on both sides of resonance, and deflection plotted against length gave the "resonance curve."

If J_m denotes the maximum value of the deflection, a_m the corresponding length of the rectangle, J the value of the deflection for another length, $a_m \pm \delta a$ where δa is small in comparison with a_m, then Drude has shown * that

$$\gamma_1 + \gamma_2 = \pi \frac{\delta a}{a_m} \sqrt{\frac{J}{J_m - J}},$$

where γ_1 and γ_2 are respectively the logarithmic decrements of the vibrations in the oscillator and the resonator, the logarithmic decrement being defined as the natural logarithm of the ratio of two successive oscillations in the same direction.

$$\gamma = \pi R \sqrt{\frac{C}{L}},$$

where R denotes the resistance, C the capacity, and L the self-inductance in the circuit. Drude's formula is derived on the suppositions (1) that the magnetic coupling of the two circuits is weak; (2) that the resistance of the spark gap is constant; and (3) that the oscillator condenser is not charged again until it has been completely discharged. In other words, there are no partial oscillations in the secondary of the induction coil. The second supposition is open to serious objections. Still it is obvious that $\gamma_1 + \gamma_2$ measures the flatness of the resonance curve, and that from general considerations the less the vibrations in the oscillator are damped, the sharper and better defined the resonance curve must be.

* *Ann. d. Phys.*, xv., 1904, p. 709.

The resistance of the resonator was about 0·54 ohms, a_m was 88 cms., and the diameter of the rods 5 mm. Its self-inductance, therefore, when in resonance with the oscillator was 730 cms. γ_2 was thus found to be 0·015. $\gamma_1+\gamma_2$ was determined experimentally, γ_2 subtracted, and the values of γ_1, the logarithmic decrement of the vibrations in the oscillator, determined for different materials of the electrodes and various lengths of spark gap.

Variation of Logarithmic Decrement γ_1 with Metal, and Length of Spark Gap.

Zinc—						
Gap . . .	0·4	0·85	1·1	1·4	1·8	2·2 mm.
Decrement . .	0·32	0·24	0·27	0·23	0·24	0·29
Aluminium—						
Gap . . .	0·4	0·8	1·05	1·5	2·0 mm.	
Decrement . .	1·08 ?	0·53	0·44	0·27	0·24	
Cadmium—						
Gap . . .	0·45	0·8	1·1	1·6	2·4 mm.	
Decrement . .	0·26	0·25	0·22	0·18	0·22	
Tin—						
Gap . . .	0·45	0·8	1·1	1·5	2·4 mm.	
Decrement . .	0·45	0·41	0·40	0·25 ?	0·28	
Copper—						
Gap . . .	0·4	0·7	1·1	1·5	1·9 mm.	
Decrement . .	0·30	0·23	0·24	0·29	0·21	
Iron—						
Gap . . .	0·45	0·75	1·1	1·45	1·7 mm.	
Decrement . .	0·33	0·25	0·30	0·19	0·31	
Brass—						
Gap . . .	0·35	0·8	1·1 mm.			
Decrement . .	0·20	0·19	0·20			
Nickel—						
Gap . . .	0·45	0·8	1·15	1·45	1·7 mm.	
Decrement . .	0·39	0·28	0·26	0·23	0·21	

The resistance of the spark gap in ohms can be obtained approximately by multiplying the above values of γ_1 by 23. The smallest resistance recorded is therefore about 5 ohms. The period of the oscillator was $4\ 10^{-8}$ seconds.

As will be seen from the table, the values of the decrements for different spark gaps do not agree well. The resistance of the spark gap itself seems to vary irregularly, and I could not get a steady deflection of the galvanometer. Four induction coils were tried, also different breaks; the ordinary hammer break, the Wehnelt electrolytic break, and the Foucault mercury break were used. The break giving the most regular results was one driven by a motor, and consisting of a platinum strip pressing against the rim of a brass wheel, in which pieces of slate had been set. All the values given in the table were obtained with this break. In place of the thermo-couple a quadrant electrometer with uncharged needle was connected to the plates of the resonator condenser, but was not so convenient to work.

Zinc always gave a larger decrement when freshly cleaned, but cleaning had no definite effect on the others. When the Wehnelt interrupter was used much lower values of the decrement were obtained, 0·12 and thereabouts.

The decrease of the decrement with increase in the length of the spark gap in the case of aluminium and nickel is very striking. Drude has shown in the case of zinc that as the length of the spark gap increases the decrement decreases to a minimum and then increases again, and the minima in the case of aluminium and nickel probably lie beyond the range that could be investigated with the means at my disposal.

The measurements were made in the Physical Laboratory of the University of Glasgow.

(*Issued separately July* 20, 1908.)

XXIII.—On the Cohesion of Steel, and on the Relation between the Yield Points in Tension and in Compression. By G. H. Gulliver, B.Sc., A.M.I.Mech.E., Lecturer in Engineering in the University of Edinburgh.

(MS. received March 5, 1908. Read May 18, 1908.)

1. *Direction of Internal Sliding in a Prismatic or Cylindrical Bar of Homogeneous Isotropic Material when subjected to a Single Axial Load.*

In works on the Strength of Materials, it is shown that the normal stress on an oblique section of a uniform prismatic or cylindrical bar, subjected to a simple, longitudinal, compressive or tensile load, is $p \cdot \sin^2 \theta$, where p is the stress on a normal cross section—that is, the total load on the bar divided by the area of the normal cross section—and θ is the inclination of the oblique section to the axis of the bar. It is shown also that the tangential or shearing stress along the same section is $p \cdot \sin \theta \cdot \cos \theta$, and this shearing stress is therefore a maximum on an oblique section inclined at 45° to the axis of the bar; so that if the metal gave way by shearing, and there were no internal friction, fracture would take place along such a surface. Since there is a resistance to the movement of the metallic particles over each other, the surfaces, not necessarily planes, along which slipping actually occurs, do not coincide with those over which the shearing stress reaches its maximum value. The resistance to sliding is, at least partially, of the nature of a simple frictional resistance. Whether it is entirely of this character is open to question; but, on the supposition that it is so, and that the coefficient of friction is independent of the load, the following results are obtained.

Let $\mu = \tan \phi$ be the coefficient of friction.

(a) *Compression.* (Fig. 1.)

The intensity of tangential stress along a surface inclined at an angle β to the axis is $p \cdot \sin \beta \cdot \cos \beta$.

The intensity of normal stress on the same surface is $p \cdot \sin^2 \beta$.

The frictional resistance per unit of area along this surface is $\mu \cdot p \cdot \sin^2 \beta = p \cdot \sin^2 \beta \cdot \tan \phi$.

The frictional resistance increases with the load applied to the bar, because the pressure between the surfaces of sliding is increased. Slipping occurs along the surface where the difference between the intensity of tangential stress and the frictional resistance is greatest, that is, where $p.\sin\beta.\cos\beta - p.\sin^2\beta.\tan\phi$ is a maximum; that is, where $\beta = 45° - \phi/2$.

(b) *Tension.* (Fig. 2.)

The intensity of tangential stress along a surface inclined at an angle α to the axis is $p.\sin\alpha.\cos\alpha$.

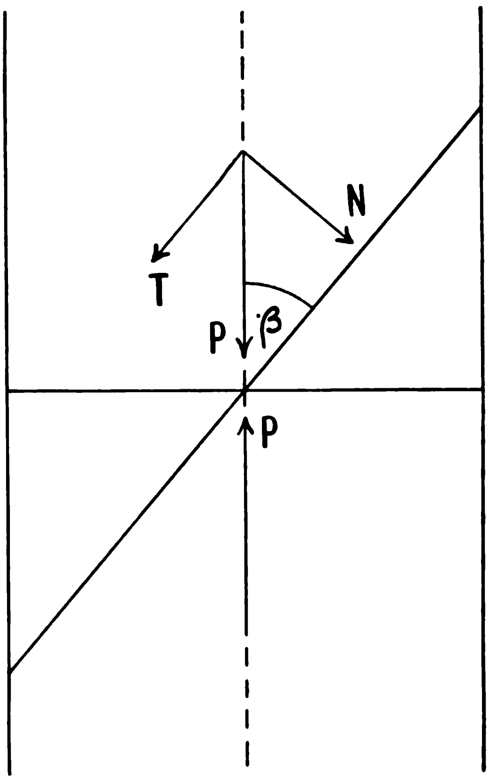


FIG. 1.

The frictional resistance per unit of area along this surface is $\mu.p.\sin^2\alpha = p.\sin^2\alpha.\tan\phi$.

The frictional resistance decreases with increase of load, because the pressure between the surfaces of sliding is decreased. Hence, if increase of load is regarded as positive, the change in friction is a negative increase. Slipping occurs along the surface where $p.\sin\alpha.\cos\alpha - (-p.\sin^2\alpha.\tan\phi)$ is a maximum; that is, where $\alpha = 45° + \phi/2$. It follows that $\alpha + \beta = 90°$.

Evidences of slipping in directions such as have just been indicated are afforded by the inspection of iron and steel bars which have been submitted to loads exceeding the elastic limit of the material. As stated elsewhere,* the author has found, from a large number of tensile tests of mild steel such as is used for structural purposes, that the usual value of the angle α is 50°, and that deviations from this inclination are generally quite small. The corresponding value of the angle β obtained in compressive tests should be 40°, but this has not been verified so satisfactorily. The value of the angle of friction is, then, $\phi = 2(\alpha - 45°) = 10°$, and the coefficient of friction is $\mu = \tan\phi = 0{\cdot}176$. The last figure agrees very closely with the coefficients obtained by Morin for clean, dry, plane, metallic surfaces.

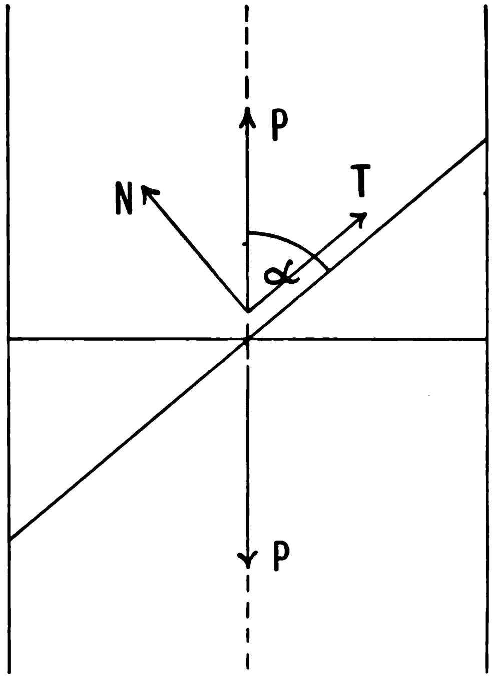


Fig. 2.

2. *Relation between the Yield Points in Tension and Compression.*

A relation between the values of the stresses corresponding with the yield points in tension and compression can be found on the assumption that yielding takes place when the internal friction of the metal is just overcome.

Let t be the stress at which permanent yielding occurs under a tensile load.

Let c be the corresponding stress under a crushing load.

If the limiting friction per unit of surface is the same in each case,

* *Proc. Inst. Mech. Eng.*, Feb. 1905, p. 141.

then the normal stress on the surface of slipping at the instant when yielding begins must also be the same for each, since this is $1/\mu$ times the limiting friction.

Let the value of the normal stress at the moment when slipping commences be n.

For tension, the surface of slipping is inclined to the axis of the bar at an angle $(45^\circ + \phi/2)$, and the normal stress on this surface at the yield point is

$$n = t \,.\, \sin^2(45^\circ + \phi/2) = t/2 \,.\, (1 + \sin\phi).$$

For compression, the surface of slipping is inclined at an angle $(45^\circ - \phi/2)$, and the normal stress on this surface at the yield point is

$$n = c \,.\, \sin^2(45^\circ - \phi/2) = c/2 \,.\, (1 - \sin\phi).$$

Hence $$t/2 \,.\, (1 + \sin\phi) = c/2 \,.\, (1 - \sin\phi).$$

And $$\frac{t}{c} = \frac{1 - \sin\phi}{1 + \sin\phi}.$$

Or, since $$\tan\phi = \mu,\ \sin\phi = \frac{\mu}{\sqrt{1 + \mu^2}}.$$

And* $$\frac{t}{c} = \frac{\sqrt{1 + \mu^2} - \mu}{\sqrt{1 + \mu^2} + \mu}.$$

Since ϕ is a small angle, $\sin\phi = \tan\phi$ nearly, and the result may be simplified to

$$\frac{t}{c} = \frac{1 - \mu}{1 + \mu}.$$

As mentioned above, the value of μ for mild steel is 0·176. The corresponding ratio t/c is 0·705, or, if the approximate expression be used, 0·701.

Some measurements were taken on six pieces cut from a ¾-inch round bar of mild steel. Three of the pieces gave a yield point in tension of 18·5 tons per square inch, and the other three a yield point in compression of 19·9 tons per square inch. The values corresponding with these figures are

$$t/c = 0{\cdot}93$$
$$\mu = 0{\cdot}036$$
$$\phi = 2^\circ\ 4'$$
$$\alpha = 46^\circ, \text{ and } \beta = 44^\circ, \text{ approximately,}$$

which do not accord well with those just deduced.

* Since the above was written, the author has found that the same result has been obtained by Mesnager, *Comptes Rendus*, vol. cxxvi., p. 515.

Other experiments with thin, flat steel bars, 3 inches wide and $\frac{1}{8}$ inch thick, gave $t=17{\cdot}6$ and $c=21{\cdot}9$ tons per square inch. These correspond with

$$t/c=0{\cdot}804$$
$$\mu=0{\cdot}109$$
$$\phi=6^\circ\ 13'$$
$$\alpha=48^\circ, \text{ and } \beta=42^\circ, \text{ approximately.}$$

In this case it was necessary to give lateral support to the bars tested in compression in order to prevent buckling.

3. *Cohesion.*

For a bar under a crushing load which gives rise to an axial stress, c,—

The normal stress on a surface of sliding is $c.\sin^2(45^\circ-\phi/2)$ $=c/2.(1-\sin\phi)$.

The frictional resistance along this surface due to the normal stress is $\mu.c/2.(1-\sin\phi)$.

The tangential stress along the surface is $c.\sin(45^\circ-\phi/2).\cos(45^\circ-\phi/2)$ $=c/2.\cos\phi$.

It is easy to show that, whatever the value of ϕ, $\cos\phi$ cannot be less than $\mu(1-\sin\phi)$, and therefore the tangential stress can never be less than the frictional resistance due to the normal stress.

Since ϕ is a small angle, $c/2.\cos\phi$ is considerably greater than $\mu.c/2.(1-\sin\phi)$. But, on the previous hypothesis, at the moment when sliding commences the tangential stress only just overcomes the frictional resistance, and there must, therefore, be some resistance to sliding in addition to that already considered. If cohesion be regarded as a force acting in a direction perpendicular to the surface of each particle of the metal, a cohesive force acting normally to the surface of sliding will have the effect of increasing the normal stress on this surface due to the compressive load, without altering the tangential stress.

Whether it is legitimate to consider the frictional effect of cohesion, an internal molecular force, as comparable with that of an externally applied load, is open to serious question, but that some form of internal frictional resistance exists can be scarcely doubted. An arbitrary assumption has been made as to the nature of this resistance; and by substituting figures found from experiments, a numerical value of the assumed cohesive force has been found. This value is compared finally with a number of results obtained from tests of steel bars, with which it accords fairly well. Thus the original assumptions are in a sense justified, though it is unlikely that they represent the actual behaviour in its complete aspect. They give, however, a fair working hypothesis.

Let K be the cohesion of the metal, expressed in the same units as the stresses.

Then μK is the additional frictional resistance due to cohesion. If c is the axial stress corresponding with the yield point of the material in compression, then at this load we have

$$c/2 \,.\, \cos\phi = \mu \,.\, c/2 \,.\, (1 - \sin\phi) + \mu \,.\, \mathrm{K}.$$

Or

$$\mathrm{K} = \frac{c}{2\mu} \,.\, \left\{ \cos\phi - \mu \,.\, (1 - \sin\phi) \right\}.$$

Putting $\sin\phi = \frac{\mu}{\sqrt{1+\mu^2}}$, and $\cos\phi = \frac{1}{\sqrt{1+\mu^2}}$,

the expression becomes

$$\mathrm{K} = \frac{c}{2\mu} \cdot \left(\sqrt{1+\mu^2} - \mu \right).$$

In the case of a tensile load which gives rise to an axial stress, t,—

The normal stress on a surface of sliding is $t \,.\, \sin^2(45^\circ + \phi/2) = t/2 \,.\, (1 + \sin\phi)$.

The decrease in frictional resistance along this surface due to the normal stress is $\mu \,.\, t/2 \,.\, (1 + \sin\phi)$.

The tangential stress along the surface is $t \,.\, \sin(45^\circ + \phi/2) \,.\, \cos(45^\circ + \phi/2) = t/2 \,.\, \cos\phi$.

But in this case the normal stress acts in the direction opposite to the cohesion; the latter force must be the greater if the metal is unbroken. Hence, if the yield point in tension corresponds with the axial stress t, we have

$$t/2 \,.\, \cos\phi = \mu \,.\, \mathrm{K} - \mu \,.\, t/2 \,.\, (1 + \sin\phi).$$

Or

$$\mathrm{K} = \frac{t}{2\mu} \,.\, \left\{ \cos\phi + \mu \,.\, (1 + \sin\phi) \right\}$$

$$= \frac{t}{2\mu} \cdot \left(\sqrt{1+\mu^2} + \mu \right).$$

For steel, taking as before $\phi = 10^\circ$, $\mu = 0{\cdot}176$, we obtain

$$\mathrm{K} = 2{\cdot}384\ c = 3{\cdot}384\ t.$$

The ratio t/c is obviously the same as that found in paragraph 2.

If it be the case that fracture of a metal bar cannot occur until the normal tensile stress on a plane cross section becomes equal to the cohesion, we can see, either by finding the value of a which makes $\mathrm{K} = t$ in the tension equation, or by noting that the surface of least total cohesion is the surface of greatest normal stress, that the bar must begin to break along a plane normal to the axis. Rupture should commence when the axial tensile stress is equal to the cohesion K, but one would expect the value of this final stress to be somewhat less than K on account of the fact, of which evidence has been given by the writer elsewhere,* that fracture usually begins in the centre of the cross section of the bar, suggesting some concentration of stress here.

In the table given below, some values of K, calculated from the yield points obtained for steel bars tested in tension, are compared with the final stresses carried by the bars at the instant of fracture, the final stress being calculated for a normal cross section taken through the place where rupture commenced. These figures have been chosen at random from a number of results, and the bars were not tested specially to check the value of K. The values of both yield point and breaking stress were not taken with extreme accuracy; in fact, to obtain a true value of the breaking stress is not an easy matter, as the lever of the testing machine must be exactly balanced at the instant of rupture. Moreover, the measurement of

No.	Yield point $=t$. Tons per sq. in.	$K=3\cdot384\,t$.	Stress at rupture. Tons per sq. in.	Reduction of Area per cent.
		Round Bars, originally $\frac{3}{4}$ inch Diameter.		
6060	17·0	57·5	54·1	61·3
6170	16·0	54·2	56·7	63·5
6321	17·0	57·5	59·7	64·0
6323	17·0	57·5	53·6	59·7
6324	18·0	60·9	56·9	61·6
6337	17·0	57·5	56·8	63·0
6729	20·0	67·7	59·7	62·7
6730	20·0	67·7	65·5	61·5
6765	19·5	66·2	56·8	62·9
6942	23·0	77·9	55·2	41·3
6944	23·0	77·9	65·0	54·2
7346	18·5	62·5	51·0	54·4
7354	19·0	64·3	58·2	64·6
		Flat Bars, originally $1\frac{1}{2}$ inches wide and $\frac{1}{2}$ inch thick.		
6052	16·0	54·2	52·1	47·2
6062	17·0	57·5	50·9	52·6
6202	16·0	54·2	53·0	58·0
6215	13·0	44·0	46·5	51·0
6437	14·0	47·4	50·5	57·0
6440	16·5	55·9	50·8	54·3
6441	16·0	54·2	41·6	45·8
6731	16·0	54·2	46·4	51·1
6732	17·0	57·5	54·0	56·5

* *Loc. cit.*, p. 146.

the cross-sectional area of the specimen at the fractured part presents difficulties, especially in the case of flat bars. Having regard to all these circumstances, the value of K corresponds fairly closely with the breaking stress; but, as expected, the latter is somewhat lower, the average value of K being 59·5 tons per square inch, while that of the breaking stress is 54·3. In only four cases, namely for round bars 6170 and 6321, and for flat bars 6215 and 6437, is the breaking stress greater than K. A column showing the reduction of area of each bar in the fractured region is also included in the table.

Summary.

(*a*) On the assumptions that resistance to deformation is due to simple friction, and that the coefficient of friction is independent of the load, the ratio of the yield point in tension to the yield point in compression, for what is ordinarily known as mild steel, is calculated as 2·384 to 3·384, or as 0·705 to 1. Experimental results so far obtained do not agree well with these figures, the value for the tensile yield point being relatively high, and that for compression relatively low.

(*b*) On the further assumption that a cohesive force acting between the metallic particles gives rise to a frictional resistance which may be added (algebraically) to that due to the effect of the external load, the value of this cohesive force is deduced as equal to 3·384 times the stress which corresponds with the tension yield point, or to 2·384 times that corresponding with the compression yield point. Experimental results from a large number of tests agree very fairly with the calculated figures for the case of tension.

NOTE (*June* 11, 1908).—The value of t/c may be obtained in a more simple manner than that given on page 377; for since

$$\alpha = (45° + \phi/2), \text{ and } \beta = (45° - \phi/2),$$

it follows that

$$t \sin^2 \alpha = c \sin^2 \beta = c \cos^2 \alpha$$

and

$$\frac{t}{c} = \cot^2 \alpha = \cot^2 50° = (0{\cdot}839)^2 = 0{\cdot}704.$$

(*Issued separately July* 20, 1908.)

XXIV.—The Electromotive Force of Iodine Concentration Cells in Water and Alcohol.

By **A. P. Laurie**, M.A., D.Sc., Principal of the Heriot-Watt College, Edinburgh.

(Read May 4, 1908. MS. received May 21, 1908.)

In a recent paper (*Zeitschrift für Electrochemie*, 1906, page 265) Mr Maitland has redetermined with great care the electromotive force of an iodine solution with a platinum electrode against a mercury-calomel electrode. As, however, no observations seem to have been published of the electromotive force of iodine against iodine of different concentrations, it seemed advisable to determine those values before proceeding to the determination of the values for alcohol solutions.

If a cell be made up consisting of a platinum wire inserted in a strong solution of iodine in potassium iodide—potassium iodide—platinum wire in a weak solution of iodine in potassium iodide, the potassium iodide being of the same strength throughout the cell, an electromotive force can be obtained between the platinum wires, the strong solution of iodine forming the positive pole and the weak solution of iodine forming the negative pole.

The reactions in the cell are due to the transference of iodine from the strong to the weak solution, through the potassium iodide, the iodine ions travelling from the strong to the weak solution, and the potassium ions moving the opposite way. The transference of one atom of iodine from the strong to the weak solution involves the formation of one molecule of potassium iodide in the strong solution, and the decomposition of one molecule of potassium iodide in the dilute solution.

The method of measuring the E.M.F. adopted in this paper was as follows:—The electromotive force was measured by means of a Dolezalek electrometer, which was arranged to have a sensitiveness which enabled charges of ·0003 volts to be determined. An accumulator was connected through a resistance box, and a portion of the resistance used to measure against the E.M.F. of the cell to be measured. The accumulator was kept always connected to the resistance box when not in use, through some 5000 ohms, and in the course of several weeks slowly lost about ·01 volt. The arrangement was as shown in fig. 1.

By this arrangement no current is drawn from the cell being measured. The accumulator was standardised by replacing C by a standard cadmium cell, the E.M.F. of which was 1·0196 volts at 17° C., according to the report of the National Physical Laboratory.

The method used was a null one, the resistance ratio being altered till no deflection was obtained on the electrometer.

The standard cell and the cells being measured were kept in a thermostat, the temperature of which, with a corrected thermometer, was 20·4° C. for the measurements with the water cells, but afterwards raised to 25° C. for the alcohol measurements, as this is the temperature usually selected.

The room in which the experiments were made is fairly constant in temperature, and the thermostat was not found to alter as much as $\frac{1}{50}$th of a degree. A special form of cell was devised for the experiments. A glass tube about ½ centimetre in diameter had a stopper ground into one end and the other end was drawn out, and a platinum wire fused into it, so as to form a little vessel. Two of these vessels were used, the iodine solutions being introduced into the vessels and stoppered up, and the vessels

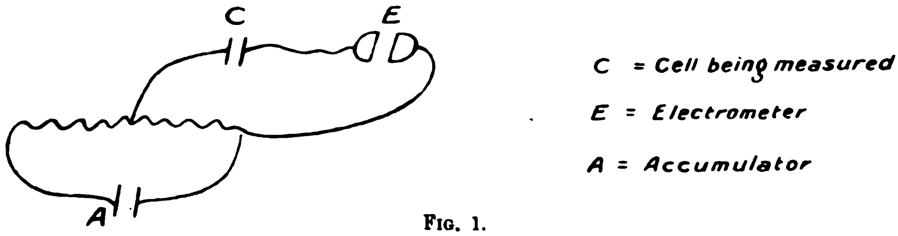


Fig. 1.

then introduced upside down into a solution of potassium iodide. The complete arrangement is shown in fig. 2.

If the stoppers are free from grease, the layer of moisture round them forms an efficient conductor for a quadrant electrometer. To prepare fresh platinum wires for use, they were heated red hot and then steeped for some hours in strong iodine solution. No difference could be detected on reversing the relation of the wires in the cells, the readings obtained from platinum wires in even very dilute solutions of iodine being evidently quite reliable. The wires were never touched or handled in any way, being simply washed or left in contact with the iodine solutions, after having been once prepared in the way described above.

The solutions were prepared in freshly boiled redistilled water from Merck's guaranteed pure iodine and pure potassium iodide. The potassium iodide solution showed no signs of liberating iodine from an aqueous solution kept for some weeks, and the addition of a minute quantity of iodine tinted the solution, a tint which persisted for weeks.

The freshly made strong solutions were titrated against sodium thiosulphate solution which agreed exactly with a decinormal iodine solution, which had been standardised against pure recrystallised barium thiosulphate.

The dilute solutions were made by diluting measured volumes of the strong solutions in standard Charlottenburg apparatus. They were then titrated again against the thiosulphate solution.

In order to decide whether such extremely dilute solutions really contain the amount of active iodine added and determined by titration, the dilution was carried one step further and the following experiments made. The actual solutions used ranging from ·1 to ·001 normal iodine, a ·0001

FIG. 2.—Iodine Concentration Cell, showing Stoppered Electrodes.

normal iodine solution was therefore prepared and was titrated against a ·0001 thiosulphate solution, which was made by dilution in the same apparatus. 25 c.c. of the iodine solution were decolorised by 23·5 c.c. of the thiosulphate solution. It was also found that 1·5 c.c. of the iodine solution were required to give a perceptible colour with starch. This result was then compared with the E.M.F. readings. The E.M.F. readings for a solution of ·1 normal iodine against ·001 normal iodine, and of ·1 normal iodine against ·0001 normal iodine, were taken, the following values being obtained —all solutions containing ·115 molecules of potassium iodide in 1000 c.c. :—

·1–·001 = ·0749 volts,

·1–·0001 = ·1054 volts,

giving us the E.M.F. between ·001 and ·0001 solution ·0305 volts. Now, this value, with so much potassium iodide present, as will be shown later, should have approximated very closely to ·0291 volts if ·0001 of iodine was present.

If we calculate the amount of iodine present in the solution, on the assumption that the correct reading is ·0291 volts, from this E.M.F. reading, the result is to show that the percentage of iodine present is ·00009 instead of ·0001.

It is, however, quite possible that at such extreme dilution some of the iodine is lost owing to slight traces of ammonia, the alkalinity of the glass, and what not, although the usual precautions were taken, both by boiling out the glass and preparing with great care the distilled water. Moreover, it was found that, although boiled distilled water had been used to make the dilution of the hyposulphite used in this titration, yet, after keeping a few days, the solution had become completely oxidised at this extreme dilution, and therefore might be slightly oxidised in the process of diluting and using.

It is evident, then, that the titration of such a solution is likely to give too high a value, owing to slight oxidation of the thiosulphate, while it is probable that calculation from the E.M.F. reading is too low, as there is a very small correction for the mass equation, and that its real value lies between ·0001 and ·00009. It is, however, evident that even at these extreme dilutions the E.M.F. readings and the titration results show a close agreement, so that we may safely accept the solutions made up for ·001 normal iodine and upwards as containing that amount of iodine.

The following readings were taken with two separately prepared sets of solutions of iodine and potassium iodide:—

	Volts.		Difference, Volts.
	1st Solution.	2nd Solution.	
·115 *potassium iodide.*			
(1) ·1 : ·01 normal iodine . .	E = ·0445	·0442	·0003
·01 : ·001 „ . .	E = ·0304	·0303	·0001
·23 *potassium iodide.*			
(2) ·1 : ·01 normal iodine . .	E = ·0350	·0354	·0004
·01 : ·001 „ . .	E = ·0298	·0296	·0002
·344 *potassium iodide.*			
(3) ·1 : ·01 normal iodine . .	E = ·0327	·0324	·0003
·01 : ·001 „ . .	E = ·0294	·0296	·0002

In calculating the E.M.F. of this cell, we have to take into account the following conditions:—

In the first place, an E.M.F. is produced owing to the different osmotic pressures of the iodine at the two poles of the cell, which can be calculated from the formula $E = \cdot 0291 \log \frac{P}{P'}$ at 20·4° C., where P, P′ are the pressures due to the iodine solutions. This equation is true when the concentration of iodine ions is the same at both ends of the cell, and consequently the osmotic pressure of the iodine ions need not be considered. But, owing to the fact that iodine in presence of potassium iodide forms complexes, such as KI_3, both the amount of free iodine present and the amount of potassium iodide actually present in the solution are affected. If we consider the mass equation $\frac{KI.I_2}{KI_3} = C$, it is evident that the formation of KI_3 not only diminishes the amount of free iodine present, but also the amount of potassium iodide present. If, then, we compare the E.M.F. of two solutions to which the same amount of potassium iodide has been added, but which contain different quantities of added iodine, then in the solution containing the larger proportion of iodine the amount of free potassium iodide will be less than on the side containing the weak solution of iodine; consequently the osmotic pressure of the iodine ions must be taken into account at both sides of the cell, the concentration of the ions being equal to the concentration of the free potassium iodide molecules, with a correction for the amount of dissociation. Consequently, if p, p' represent the iodine ion concentrations, the complete formula for the E.M.F. will be $E = \cdot 0291\left(\log \frac{P}{p^2} - \log \frac{P'}{p'^2}\right)$. When the potassium iodide is in great excess as compared with the iodine on both sides of the cell, the E.M.F. of the cell will be very nearly proportional to the iodine added. But when the amount of iodine added on one side sensibly equals the amount of potassium iodide, then the E.M.F. will be very sensibly increased, owing not only to the free iodine being no longer proportional to the iodine added, but also owing the larger diminution of free KI on the strong iodine side. This increase, as we shall find, exists in practice, but is not so great as is to be expected from the mass equation and equation for calculating the E.M.F. given above.

Taking, then, the mass equation $\frac{KI.I_2}{KI_3} = C$, and accepting the results obtained by Jakowkin, we can take the value of C as $= \cdot 0014$ with sufficient nearness for these calculations.

If, then, a be the total number of potassium iodide molecules added to the solutions, and b the total number of iodine molecules added to the solutions, and x the number of free iodine molecules present, then

$$\frac{x(a-b+x)}{b-x}=\mathrm{K}.$$

Expanding this quadratic, we get

$$x=\frac{\mathrm{K}b}{a-b+\mathrm{K}}-\frac{\mathrm{K}^2b^2}{(a-b+\mathrm{K})^{3}}+\text{etc.}$$

Within the range of the quantities here used, all but the first term can be neglected, and we get

$$x=\frac{\mathrm{K}b}{a-b+\mathrm{K}}.$$

Calculating, then, the amount of free iodine and of free KI molecules present from this equation, and making a correction for the dissociation of the KI molecules, and putting the values into the equation for calculating the E.M.F., we get the following calculated values, which are compared in this table with the mean of the two experimental determinations:—

TABLE I.

	Calculated E.M.F.	Measured E.M.F.
·344 *molecules potassium iodide in* 1000 *c.c.* ·005 molecules of iodine in 1000 c.c. against ·0005 molecules .	·0295	·0295
·23 *molecules potassium iodide in* 1000 *c.c.* ·005 molecules of iodine in 1000 c.c. against ·0005 molecules .	·0298	·0297
·115 *molecules potassium iodide in* 1000 *c.c.* ·005 molecules of iodine in 1000 c.c. against ·0005 molecules .	·0305	·0304

The E.M.F. for iodine against iodine, if the mass law had not been acting, would have been ·0291. It is evident that for those cases where the correction for the mass law is small the calculated and measured values agree very closely.

If we now compare the calculated and observed values for cells where the proportion of iodine on one side is sensibly of the same order as the proportion of potassium iodide, we find the E.M.F. increasing, but sensibly lower in value than that calculated from the mass equation.

TABLE II.

	Calculated E.M.F.	Observed E.M.F.
·344 *molecules potassium iodide in* 1000 *c.c.* ·05 molecules of iodine in 1000 c.c. against ·005 molecules .	·0341	·0326
·23 *molecules potassium iodide in* 1000 *c.c.* ·05 molecules of iodine in 1000 c.c. against ·005 molecules .	·0371	·0352
·115 *molecules potassium iodide in* 1000 *c.c.* ·05 molecules of iodine in 1000 c.c. against ·005 molecules .	·0471	·0444

It is evident from these results that the actual ratios between iodine and potassium iodide in these solutions differ sensibly from those calculated from the formula $\frac{KI.I_2}{KI_3} = \cdot 0014$, although the law is being approximately followed.

It might be suggested that these solutions are too strong to obey the gas laws accurately. I therefore took the solutions ·05 iodine, ·115 KI, and ·005 iodine and ·115 KI, and diluted them both to one-tenth the strength. The E.M.F. readings were not sensibly affected by this dilution.

Jakowkin shows that for stronger solutions of potassium iodide than those so far considered the constant alters for the stronger concentration of iodine, probably owing to the formation of sensible quantities of higher complexes. It seemed, therefore, of interest to see how far the E.M.F. for such solutions would agree with those calculated from the same equations, but allowing for the changing constant. In the table selected Jakowkin makes the amount of potassium iodide normal, and the amount of iodine ranges from ·5066 molecules per litre to ·0141 molecules per litre, the constant changing from ·000773 to ·001365. If, then, the weakest solution of iodine, namely, ·0141, be taken for one electrode, and the other stronger solutions be taken in turn for the other electrode, the electromotive force can be calculated by applying the appropriate constant to each solution. The table on p. 389 gives the results of these calculations and experiments.

Here also it will be noted that the results are very close where the amount of potassium iodide is in large excess, but only approximately for the other solutions.

It seemed of interest to investigate the question whether KI_3 was stable at higher temperatures, and this can be done as follows:—

If we consider the cell ·115 KI, ·05 I_2 against ·115 KI, ·0005 I_2, it is evident that if no change on warming takes place in the distribution, according to the mass equation, the E.M.F. can be calculated from the formula $E = \frac{\cdot000198}{2} T \log \left(\frac{P}{p^2} - \frac{P'}{p'^2} \right)$; but if KI_3 is dissociated there will be a considerable change of E.M.F., the dissociation in the ·0005 solution merely increasing the free iodine, but in the ·05 per cent. solution both increasing the free iodine and the amount of KI, the E.M.F. tending to fall, therefore, to the value depending on the iodine concentrations alone.

Table of E.M.F.s for Various Percentages of Iodine in Normal Potassium Iodide Solution.

(a) Percentage of Iodine used by Jakowkin in molecules per litre.	(b) Corresponding Values of K.	(c) Percentage used for E.M.F. in molecules per litre.	(d) E.M.F. calculated from (b) and (c).	(e) E.M.F. measured.
·5066	·000773	·508	·0636	·0588
·3615	·000949	·302	·0527	·0491
·2784	·001031	·279	·0461	·0434
·2088	·001105	·209	·0398	·0376
·1097	·001220	·109	·0284	·0273
·0563	·001292	·056	·0185	·0184
·0321	·001315	·0322	·0108	·0108
·0141	·001365	·0139	...	...

Such a cell, when raised to the temperature 75° C., gave the value E = ·0848, while the calculated E.M.F., on the assumption that no change took place in the iodine distribution in the cell, gave the value E = ·0880; so there is no indication of any serious change in the $\frac{KI.I_2}{KI_3}$ distribution up to this temperature.

Electromotive Force of Iodine Concentration Cells in Alcohol.

The evidence obtained from the absorption spectra of solutions of iodine in potassium iodide and alcohol points to the conclusion that KI_3 is formed in the solutions, but the value of the constant for the mass equation is not known.

We might venture to calculate this constant from the E.M.F. results if we knew accurately another figure, namely, the relative amounts of dissociation of potassium iodide in alcohol of different strengths of solution. As we have neither of these values with certainty, it is not possible to calculate out the values of the respective E.M.F.s for alcohol as was done for water. These matters may be treated of in a subsequent paper; in the meantime I confine myself to testing how accurately the osmotic pressure equation for E.M.F. applies to alcoholic solutions of iodine, and whether the E.M.F.s obtained indicate the formation of KI_3 in the solutions.

It is evident that, as in the case of water, if the potassium iodide is present in great excess the E.M.F. should be calculable from the iodine concentrations alone, and that, as the amount of iodine and potassium iodide become sensibly the same on one side of the cell, the E.M.F. should be higher than that due to the iodine concentrations alone if KI_3 is formed in the solutions. Moreover, if the ratio between iodine and KI is made sensibly the same as that formerly employed, we should be justified in calculating the E.M.F. of the cell with the largest ratio of iodine to KI on both sides, from the iodine concentrations alone.

Four independent sets of solutions were made of potassium iodide and iodine in alcohol. The alcohol used was Kahlbaum's absolute alcohol, which contains some ·2 per cent. or ·3 per cent. of water; and while care was taken to expose the solutions as little as possible to the air while being prepared, yet probably traces of water were absorbed. When once the solutions had been transferred to the little stoppered electrodes immersed in absolute alcohol, no further absorption of water was to be feared. The probable errors introduced by traces of water are discussed at the end of the paper. The following were the results obtained at 25° C.:—

	Number of Iodine Molecules in 1000 c.c.	E.M.F. Reading for Solutions				Mean E.M.F.
		a.	*b.*	*c.*	*d.*	
Solution containing ·075 molecules of KI in 1000 c.c.	·01 against ·001	·0356	·0365	·0356	·0353	·0358
	·01 „ ·0001	·0652	·0661	·0653	·0649	·0654
Solution containing ·05 molecules of KI in 1000 c.c.	·01 against ·001	·0393	·0391	·0384	·0388	·0389
	·01 „ ·0001	·0695	·0689	·0691	·0684	·0690
Solution containing ·025 molecules of KI in 1000 c.c.	·01 against ·001	·0505	·0499	·0499	...	·0501
	·01 „ ·0001	·0802	·0812	·0806	...	·0807

If we subtract the values obtained for ·01 against ·001 from those obtained for ·01 against ·0001, and arrange the resulting figures beginning with those containing the largest proportion of KI to I_2, we get the following table, corresponding to the one obtained for water solutions:—

KI.	Iodine.	E.M.F.
·075	·001 to ·0001	·0296
·05	·001 to ·0001	·0301
·025	·001 to ·0001	·0306
·075	·01 to ·001	·0358
·05	·01 to ·001	·0389
·025	·01 to ·001	·0501

The close resemblance of this series to that obtained for water is obvious, the E.M.F., however, rising rather higher as the ratio of iodine is increased.

If we assume that the E.M.F. obtained for ·001 to ·0001 iodine with ·075 KI present is entirely due to the iodine concentrations, as we found to be very nearly true in the case of the water solutions, then the calculated value at 25° C. is ·0295. These experiments prove, then, how accurately the Nernst equation applies to iodine solutions in alcohol, and also indicate the existence of KI_3 in the solutions.

A series of solutions was made up containing different percentages of alcohol and water, but containing iodine and potassium iodide in the same ratios as measured above, and the following values were obtained, which were taken at the two temperatures, 25° C. and 0° C.:—

Number of Molecules of Potassium Iodide = ·025 in 1000 c.c. and Number of Iodine Molecules ·01 against ·001 in 1000 c.c.

Percentage of Alcohol.	Percentage of Water.	E.M.F. at 25° C.	E.M.F. at 0° C.	Difference in E.M.F.
100	0	·0505	·0463	·0042
90	10	·0485	·0442	·0043
80	20	·0476	·0435	·0042
70	30	·0457	·0419	·0048
60	40	·0454	·0405	·0049
50	50	·0447	·0398	·0049
40	60	·0445	·0395	·0050
30	70	·0439	·0388	·0051
20	80	·0435	·0392	·0043
10	90	·0434	·0393	·0041
0	100	·0428	·0392	·0036

In the first place, if the E.M.F. of the alcohol cell is calculated for 0° C. from the value for 25° C., on the assumption that the E.M.F. is proportional to the absolute temperature, the calculated value is ·0462, while the value obtained is ·0463. In the second place, it will be noted on comparing the differences that they tend to increase for mixtures of alcohol and water, and diminish again for either pure alcohol or pure water. This result would be accounted for if we assume that the dissociation of KI for mixtures of alcohol and water at 25° C. is larger than the dissociation of KI at 0° C. in mixtures of alcohol and water. Now, Jones has shown (*Am. Ch. J.*, xxviii, No. 5) from conductivity measurements that this is the case, so that these E.M.F. measurements at 0° C. and 25° C. seem to distinctly confirm the results obtained from the conductivity measurements, and to show that mixtures of alcohol and water at 0° C. have a smaller dissociating power on potassium iodide than pure water or pure alcohol.

The Electromotive Force between Iodine in Alcohol and Iodine in Water.

If solutions of iodine and potassium iodide in water and in alcohol respectively, containing equal quantities of these substances, are placed in opposition round two platinum electrodes, a considerable E.M.F. is obtained, the water solution being positive to the alcohol solution. The action of the cell evidently involves, therefore, the transference of iodine from the water to the alcohol, and the simultaneous transference of potassium iodide from the alcohol to the water. The following values were obtained:—

	E.M.F. at 0° C. Volts.	E.M.F. at 25° C. Volts.
·025 KI + ·001 I_2 in water against ·025 KI + ·001 I_2 in alcohol	·1897	·1988
·025 KI + ·0001 I_2 in water against ·025 KI + ·0001 I_2 in alcohol	·1884	·1979

The electromotive force is, therefore, independent of the strength of the iodine on both sides and of the ratio between the potassium iodide and iodine, these readings being the same within the experimental error. The combination has a large temperature coefficient in that direction, which shows that a considerable amount of heat is being absorbed in the cell, but that this is considerably less than that required for a pure osmotic pressure cell. As it seemed possible that this E.M.F. was merely a contact E.M.F. and not a permanent source of electrical energy, I made up a small cell with a parchment-paper diaphragm and two platinum plates, with ·01 of I_2 with ·025 KI in alcohol on the one side, and the ·01 of I_2 with ·025 KI in water on the other side. This, when connected with a galvanometer of 500 ohms resistance, gave a steady current, falling off a little through polarisation, but continuing for over an hour.

It is evident that during the flow of this current iodine is being transferred from the water to the alcohol solution, and potassium iodide from the alcohol to the water solution, and the main source of the energy of the current is probably the different solution pressures of iodine in water and alcohol respectively. In both cases, namely, the iodine and the KI, they are being transferred from a solution in which they dissolve with difficulty to a solution in which they dissolve easily.

Errors due to Traces of Water.

The main source of error in taking the E.M.F.s of alcohol cells is probably due to the presence of and absorption of water, and it was with a view to determining the probable amount of these errors that the experiments with mixtures of alcohol and water and the readings of alcohol against water were originally undertaken. The absolute alcohol used was that prepared by Kahlbaum, and is supposed to contain about ·2 per cent. of water. Alcohol is, however, very hygroscopic, and though precautions were taken to expose the alcohol as little as possible to air while the solutions were being made up, it is highly probable that some water was absorbed during transference from one vessel to another, and also that the water absorbed was not necessarily of equal amount.

If, in the first place, we assume the water absorbed at both ends of the cell to be of *equal* amount, then the E.M.F.s obtained for water-alcohol admixtures give a very fair idea of the maximum error probable from this source. If these values are plotted as a curve, and the curve continued, it is evident that the presence of 1 per cent. of water on both sides would lower the E.M.F. about ·0002 volts, and therefore this source of error may be neglected. If, however, the amount of water absorbed at the two ends was different, the errors would be more serious. The addition of 10 per cent. of water on the strong iodine side, the strength of the solution remaining the same, caused a lowering of E.M.F. of ·02 volts, agreeing closely with the results obtained and already quoted for alcohol against water, thus making a possible error for an excess of 1 per cent. of water on one side of about ·002 volts. Such a source of error, however, would not be constant, but would appear sometimes on the one side and sometimes on the other, and would therefore not affect the mean of several readings. This probably accounts for the observed differences in readings for independently prepared solutions, which are greater than those for water cells. For these reasons the probability is that the readings given for alcohol against water are probably from ·001 to ·002 volts too low.

In conclusion, I wish to thank Dr Denison and Mr King for the assistance they have given me in this research.

(*Issued separately July* 22, 1908.)

XXV.—Inversion Temperatures, and the Form of the Equation of State. By Prof. W. Peddie.

(MS. received May 25, 1908. Read July 6, 1908.)

1. RECENT experiments by Olszewski on hydrogen (*Phil. Mag.*, May 1902), and on air and nitrogen (*Phil. Mag.*, June 1907), have exhibited results which are not in accordance with theoretical deductions based upon Van der Waal's equation of state (Dewar, *Proc. Roy. Soc.*, March 1904; Porter, *Phil. Mag.*, April 1906; Hamilton Dickson, *Phil. Mag.*, January 1908). An inversion temperature is not necessarily limited by the condition, which was satisfied in the porous-plug experiment of Joule and Kelvin, that the initial and final pressures should be nearly equal. In Olszewski's experiments that condition was widely departed from. Taking account of large differences of pressure, Dickson shows that Van der Waal's equation leads to the result that the inversion temperature must fall when the initial pressure increases, the final pressure being kept constant. Olszewski's experimental results are opposed to this conclusion; and Dickson suggests, as a possible cause of the discrepancy, an appreciable difference between the initial and final values of the kinetic energy. The following reasoning seems to indicate that such a difference would not alter the nature of the theoretical conclusion.

2. Accented letters referring to the final state, the quantity

$$Q=\int_v^{v'} p\,dv+pv-p'v'-\int_v^{v'} t\frac{dp}{dt}dv+T'-T, \quad . \quad . \quad . \quad (1)$$

where T is the kinetic energy of mass motion, represents the amount of heat given out in the passage of unit mass from the initial to the final state. Using Van der Waal's equation

$$\left(p+\frac{a}{v^2}\right)(v-b)=Rt, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and postulating

$$T'-T=k(p-p'), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where k is a constant, the condition $Q=0$ gives

$$(1+k)Rbt=a\left(2+\frac{k}{v}+\frac{k}{v'}\right)\left(1-\frac{b}{v}\right)\left(1-\frac{b}{v'}\right), \quad . \quad . \quad (4)$$

as an expression for the temperature of inversion.

In Olszewski's observations on air, the final pressure was one atmosphere, while the initial pressure varied from 40 to 160 atmospheres. We may therefore neglect terms involving the reciprocal of v' and write (4) in the form

$$(1+k)b\mathrm{R}t = a\left(2+\frac{k}{v}\right)\left(1-\frac{b}{v}\right), \quad . \quad . \quad . \quad . \quad (5)$$

whence

$$(1+k)b\mathrm{R}\frac{dt}{dv} = \frac{a}{v^2}\left[(2b-k)+\frac{2kb}{v}\right]. \quad . \quad . \quad . \quad . \quad (6)$$

Writing now $k=\lambda v$, so that the value of λ cannot exceed unity, it appears that the sign of dt/dv is positive or negative, respectively, according as

$$v \lessgtr 2b\frac{1+\lambda}{\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Again, from (2), (5), and (6),

$$\frac{dp}{dv} = \frac{2a}{v^3}\left(1-\frac{v+k}{b(1+k)}\right), \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is positive or negative according as

$$v \lessgtr \frac{b}{1-\lambda(b-1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now, as Dickson shows, the least value of v in Olszewski's experiments on air is about 10 c.c., when $p=160$ atmosphéres and $t=259°$ C., and the value of b is 1·528 c.c. Therefore (9) shows that dp/dv is negative. Again, (7) gives approximately $\frac{3}{7}$ as the value of λ for which $dt/dv=0$; and, from the description of Olszewski's apparatus and process (*Nature*, April 1902) it seems to be practically certain that λ cannot have so large a value as $\frac{3}{7}$ths, so that dt/dv is positive, and dt/dp is negative.

Equations (6) and (8), when k is made zero, verify Dickson's conclusion that Van der Waal's equation leads to a result which is opposed to Olszewski's observations if we presume that the initial and final values of the kinetic energy are practically equal. The above reasoning seems to indicate that the discrepancy remains, even if we attribute to the difference of the initial and final kinetic energies a value much greater than any that seems to be possibly admissible. If Olszewski's results are not affected by some other source of error, it appears that Van der Waal's equation is inapplicable to the case of air under inversion conditions.

If, on the other hand, various equations of state lead to the same result as Van der Waal's, it may be reasonable to presume that some other experimental condition has influenced Olszewski's observations. This question is discussed in the present paper, in all cases on the presumption that there is no appreciable difference of kinetic energy in the initial and the final states of the gas.

3. Under the latter condition it is not possible for Van der Waal's equation to give a positive value of dt/dp for any suitable values of v and v'. For we have

$$\mathrm{R}\frac{dt}{dv} = \frac{2a}{v^2}\left(1 - \frac{b}{v'}\right),$$

which is always positive; and

$$\frac{dp}{dv} = -\frac{2a}{v^2}\left[\frac{1}{b} - \left(\frac{1}{v} + \frac{1}{v'}\right)\right],$$

which is always negative so long as v and v' exceed $2b$.

4. The modified form of Van der Waal's equation given by Clausius is

$$p = \frac{\mathrm{R}t}{v-b} - \frac{a}{t(v+\alpha)^2}.$$

From this, along with (1) and the condition $\mathrm{Q}=0$, we obtain the expression

$$t^2 = \frac{a}{\mathrm{R}b} \cdot \frac{(v-b)(v'-b)\{3vv' + 2\alpha(v+v') + \alpha^2\}}{(v+\alpha)^2(v'+\alpha)^2},$$

where t is the inversion temperature. Hence

$$2t\frac{dt}{dv} = \frac{a}{\mathrm{R}b}(v'-b)\frac{(v'+\alpha)[(v-b)\alpha + (\alpha+b)v] + (2v'+\alpha)(\alpha+b)(v+\alpha)}{(v+\alpha)^3(v'+\alpha)^2},$$

so that dt/dv is essentially positive.

We have also

$$\frac{dp}{dv} = -\frac{\mathrm{R}}{v-b}\left(\frac{t}{v-b} - \frac{dt}{dv}\right) + \frac{a}{t(v+\alpha)^2}\left(\frac{2}{v+\alpha} + \frac{1}{t}\frac{dt}{dv}\right).$$

The first two terms on the right-hand side of the equation determine the sign of dp/dv. Evaluation of them gives

$$-\frac{b}{a}\frac{v-b}{v'-b}t\frac{dp}{dv} = \frac{1}{2(v+\alpha)^3(v'+\alpha)^2}\Big[(v+\alpha)(3vv' + 2\alpha v' + 2\alpha v + \alpha^2) + (v-b)(3vv' + \alpha v' + 2\alpha v)\Big],$$

so that dp/dv is essentially negative, provided that the two terms involving a do not affect this result, and it is easily shown that the ratio of their sum to the sum of the first two terms is less than unity under conditions of volume and temperature such as those used by Olszewski.

Thus Clausius' modification of Van der Waal's formula also makes dt/dp negative under these conditions when we can neglect the difference of the initial and the final kinetic energies of the gas.

5. Reinganum's equation of state is

$$p=\frac{Rtv^3}{(v-b)^4}-\frac{a}{v^2},$$

and the expression involving the temperature of inversion is, as given by Dickson,

$$Rt=\frac{2a}{b}\frac{(v-b)(v'-b)}{vv'}\frac{(v-b)^3(v'-b)^3}{[2vv'-b(v+v')]\{2v^2v'^2-2bvv'(v+v')+b^2(v^2+v'^2)\}}.$$

From this we obtain

$$R\frac{dt}{dv}\cdot\frac{b}{2a}\cdot\frac{vv'[\ ]\{\ \}}{(v'-b)^4(v-b)^3}=\left(2-\frac{v-b}{v}-\frac{(v-b)(2v'-b)}{2vv'-b(v+v')}+2bv\frac{(v'-b)^2}{\{\ \}}\right).$$

Now the quantity [ ] is positive since v and v' are greater than b; and { } is also positive, for $2\{\ \}=[\ ]^2+b^2(v'-v)^2$. Also the third term in the bracket ( ) is less than unity, for $[\ ]>2v'(v-b)$. Therefore dt/dv is essentially positive.

Again,

$$\frac{dp}{dv}=\frac{2a}{b}\frac{v(v'-b)^4}{v'[\ ]\{\ \}}\left(3-4\frac{v}{v-b}+\frac{bv'[\ ]\{\ \}}{v^4(v'-b)^4}\right.$$

$$\left.+\frac{v}{v-b}\left(2-\frac{v-b}{b}-\frac{(v-b)(2v'-b)}{[\ ]}+2bv\frac{(v'-b)^2}{\{\ \}}\right)\right),$$

the sign of which is determined by the quantity in the large bracket. Writing $v=kb$ and taking v' as being large relatively to v and b, the quantity becomes

$$-4k^7+16k^6-44k^5+66k^4-56k^3+28k^2-8k+1,$$

which is negative for all values of k greater than unity.

Hence the sign of dp/dv is negative under the specified conditions, and therefore dt/dp is negative.

6. Berthelot's equation

$$p=\frac{Rt}{v-b}-\frac{a}{tv^2}$$

leads to the expression

$$Rt^2=3\frac{a}{b}\frac{(v-b)(v'-b)}{vv'}.$$

From this we get

$$2Rt\frac{dt}{dv}=3a\frac{v'-b}{v^2v'},$$

which is always positive. Also

$$t\frac{dp}{dv}=-\frac{3a}{2b}\frac{1}{vv'}\frac{v'-b}{v-b}\left(2-\frac{b}{v}\right),$$

which is always negative. Thus dt/dp is always negative.

7. Callendar's equation

$$p=\frac{Rt}{v-b+c\left(\frac{t_0}{t}\right)^n}$$

leads to the expression

$$bt^n=(n+1)ct_0^n,$$

and so requires a single inversion temperature independent of the pressure.

8. If we take the equation

$$\left(p+\frac{a}{v^n}\right)(v-b)=Rt,$$

and regard v' as being large relatively to v, the temperature of inversion satisfies the conditions

$$Rt=\frac{a}{b}\frac{n}{n-1}\frac{v-b}{v^{n-1}},$$

$$R\frac{dt}{dv}=-\frac{a}{b}\frac{n}{n-1}\frac{1}{v^n}[v(n-2)-b(n-1)]:$$

and we also have

$$\frac{dp}{dv}=-\frac{na}{bv^{n+1}}(v-b),$$

which is always negative.

The case $n=2$ is that of Van der Waal's. The case $n=5/3$ is one adopted by Dieterici, and it also gives dt/dp negative.

9. Thus no one of these six formulæ leads to a positive value of dt/dp under the specified conditions; and it can scarcely be supposed that any of them are more than roughly inapplicable in the region considered. This furnishes considerable evidence in favour of the idea that Olszewski's observations have been affected by a source of error not yet accounted for.

10. It is of interest to compare the forms which the different equations of state give to the expression for dt/dp when v' is large. Van der Waal's gives

$$\frac{dt}{dp}=-\frac{b}{R}\frac{v}{v-b},$$

which is negative infinite when $v=b$, has the value $-2b/R$ when $v=2b$, and approximates to $-b/R$ when v is large. Berthelot's gives

$$\frac{dt}{dp}=-\frac{b}{R}\frac{v-b}{2v-b},$$

which is zero when $v=b$, has the value $-b/3R$ when $v=2b$, and approximates to $-b/2R$ when v is large. Dieterici's equation leads to the value

$$\frac{dt}{dp}=-\frac{1}{2R}\frac{v}{v-b}(v+2b),$$

which is negative infinite when $v=b$, has the value $-4b/R$ when $v=2b$, and approximates to negative infinity again as v becomes large. Its smallest numerical value is $b(2+\sqrt{3})/R$. When $v=10b$, the value is about double the minimum. When $v=100b$, the value is fully 14 times the minimum.

The equation of Clausius gives

$$\frac{dt}{dp}=-\frac{v-b}{4R}\cdot\frac{(v-b)a+(a+b)(3v+2a)}{(v-b)(3v+a)+(v+a)(3v+2a)}.$$

This vanishes when $v=b$, is always less than $(v-b)/4R$ numerically, and approximates to $-(4a+3b)/24R$ as v increases without limit.

Reinganum's equation leads, in the previous notation, to the expression

$$\frac{dt}{dp}=-\frac{1}{R}k(k-1)^3\frac{6k^3-6k^2+3k-1}{4k^7-16k^6+44k^5-66k^4+56k^3-28k^2+8k-1}.$$

This vanishes when $k=1$, that is when $v=b$. It has the value $-58/787R$ when $k=2$, $-2784/3713R$ when $k=3$, $-32292/31327R$ when $k=4$, and approximates to $-3/2R$ as k tends to infinity.

11. Experiments made upon inversion temperatures, when a gas expands from a measured volume to a volume large in comparison with it, may lead to a decisive test amongst various equations of state.

In Van der Waal's equation, the constants b and R have, respectively, the values 1·528 and 2·835 when the unit of pressure is one atmosphere and the unit of volume is 1 c.c. Hence the fall of the inversion temperature per atmosphere of increased pressure is about half a degree centigrade when v is large, and increases to about 1° C. when $v=2b$.

Taking $\bar{\theta}$, the critical temperature of air, as 133° C., and calculating $\bar{p}$ and $\bar{v}$ from the above value of b along with $a=1682$, we get $\bar{p}=1682/27(1{\cdot}528)^2$, $\bar{v}=4{\cdot}584$. Using these values of the critical constants to calculate R in Dieterici's equation of state from the condition $R=8\bar{p}\bar{v}/3\theta$, we find that this equation gives $dt/dp=-2^\circ{\cdot}35$ when $v=2b$, and that it is equal to $-3^\circ{\cdot}12$ when $v=10b$, increasing without limit as v increases.

Similarly, calculation of the constants in Berthelot's equation from the conditions $b=\bar{v}/4$, $a=27R^2\bar{\theta}^3/64\bar{p}$, $R=32\bar{v}\bar{p}/9\bar{\theta}$, gives $-dt/dp$ roughly equal to $\frac{1}{4}$ of a degree centigrade, increasing to about $\frac{1}{2}$ of a degree centigrade when v is large.

Again, as above, Callendar's equation requires that dt/dp should be zero.

(*Issued separately July* 22, 1908.)

XXVI.—Treatment of Aneurysm by Electrolysis. By Dr Dawson Turner.

(MS. received June 1, 1908. Read same date.)

Many attempts have been made to utilise electrolysis in aneurysms, especially in those thoracic ones that are not amenable to ordinary surgical treatment; it has been hoped that the clotting which occurs around the poles might serve as a nucleus for further coagulation and deposits of fibrin, and that the aneurysm cavity might in this way become partially filled up. Such attempts have not met with much success hitherto, and the purpose of this research has been to endeavour to determine by experiments on blood serum outside the body what the actual effect of electrolysis is so far as regards clotting. Various methods of electrolysing the blood in an aneurysm have been used by surgeons. Ciniselli introduced needles connected with both poles, and reversed the direction of the current every five minutes; of 38 cases so treated, 27 were ameliorated, but none were cured. In the unipolar method one pole only was introduced, and the other was connected with an indifferent pad placed in the vicinity. The difficult question was which pole to introduce. The positive pole gave the firmest clot, but it was thought that it might be difficult to withdraw, and that hæmorrhage or even rupture of the vessel might follow. The negative pole gave a large frothy clot made up of hydrogen bubbles, which was not only of little value in setting up a stable coagulation, but was thought to be dangerous from the risk of emboli. Dr John Duncan used the bipolar method, and introduced both poles, keeping them both well in the middle of the blood stream, but later on took a further step and gently cauterised the inner wall of the sac with the positive electrode. He was kind enough to write to me the following letter in regard to the results of his twenty-six years' experience of electrolysis:—"In aneurysm I have made a new departure which I think promises well. I had found that while electrolysis might be used with complete success in small external aneurysm and in the secondary sacs of aortic aneurysm, I did not obtain with it those occasional brilliant cures which had been observed. In taking away the risks of hæmorrhage and inflammation of the sac wall, I had also greatly diminished the curative power of the agent. In short, I came to see that the cure had been in most cases due to the very cauterisation of the sac which had been thought to be so dangerous. I had no deaths, but none of the old sudden cures; I had trusted for cure to the coagulation, the effect of which is slight, and had eliminated cauterisation. I determined, therefore, about three years ago, to gently cauterise with the positive electrode the inner wall of the sac, and I have now had two very satisfactory results by this method." Dr John Duncan was therefore of opinion that the coagulating effect of the electrolysis was slight, and that it was necessary to cauterise (really to prick and slightly injure, for no cauterising effect worth mentioning would be produced under the conditions of his experiment) the wall of the sac as well in order to favour the process of clotting. A more recent method is that known as the Moore-Corradi, and it consists essentially in the introduction into the aneurysm of several feet of wire and of the passage of a current through the wire into the blood; the other electrode is placed on an indifferent spot. The wire is always made the positive pole for the two reasons already mentioned. The wire used may be of silver, gold, platinum, or iron. Dr Stewart of Philadelphia (*Philadelphia Medical Journal*, November 12, 1898) writes:—"Silver, gold, or platinum wire is undoubtedly the preferable material. It should be introduced through a thoroughly insulated needle, and be so drawn as to assume snarled spiral coils inside the sac." Dr Stevenson (*Electrolysis in Surgery*, p. 10) writes:— "In the electrolysis of aneurysms additional advantage is obtained by the needles being composed of steel, for a chloride of iron is formed if the needle be made positive, and in addition to the coagulating effect of the positive pole we have the coagulating influence of the chloride of iron." Professor S. Leduc (*Les Ions et les Médications ioniques*, p. 28), however, states that "zinc is the best coagulant and very superior to platinum"; he showed the writer an experiment of this nature.

This experiment has been repeated by me on an extended scale. A large number of electrodes of different materials were prepared and were used to electrolyse blood serum. These are some of the results in tabular form:—

	Anode.	Cathode.
Pt electrodes	Bubbles of gas which rise *en masse* . .	Much gas.
Ag ”	” ” ” ” . .	”
Ni ”	Fair amount of gaseous bubbles . . .	”
Fe ”	Very few bubbles of gas	”
Pb ”	Precipitate forming skin ; a few bubbles .	”
Cu ”	Green precipitate ; a few bubbles . .	”
Zn ”	Firm glove-like precipitate which holds on to the electrode ; no gas at all . . .	”

An examination of the table tells us that the negative poles all give off gas freely, but cause no precipitation.

The positive poles of platinum, silver, and nickel give off gas, but cause no real precipitation.

The positive poles of copper, lead, and zinc cause precipitation, the firmest precipitate being that due to zinc.

With an anode of iron a few bubbles of gas are given off, but no precipitate is formed.

It is evident, then, that surgeons, by using silver or platinum as the active pole, have been unconsciously courting the very danger of gas bubbles which they were endeavouring to avoid. We may draw the conclusion that this danger is nothing but a bogey. There is also no advantage in using iron electrodes, *pace* Dr Stevenson, for no coagulating effect is produced, probably because the protosalts only are produced. Of the heavy metals Pb, Cu, Zn, which produce a precipitate, the latter is to be preferred both because its salts, if absorbed, are less toxic, and also because it forms the largest and firmest precipitate, and that without any gas at all being given off. In an experiment using zinc electrodes the current strength was maintained at 300 m.a. for ten minutes, equivalent to 50 m.a. for one hour, which is about the dose usually given by the Moore-Corradi method; during that time the zinc electrode lost 5 centigrammes in weight and a precipitate weighing 5·35 grammes was formed. This precipitate has been kindly examined for me by Dr Drinkwater and by Miss Isabel Mitchell, and they find that it consists chiefly of a loose compound of albumen and zinc, an albuminate of zinc. By using a coil of zinc wire a firm precipitate binding together the wire spirals is formed. It is difficult to say whether such a precipitate in an aneurysm would, through the deposition of fibrin upon it, lead to the coagulation of the blood and to subsequent organisation of the clot, or whether it would be gradually washed away by the blood stream, or split up by hydrolysis into bodies of smaller molecular weight. The authorities whom I have consulted think that the precipitate would favour the coagulation of the blood. The clinical experience of surgeons and post-mortem examinations would help to settle this question. I venture in conclusion to suggest that surgeons should introduce zinc wire into an aneurysm which they may wish to treat electrolytically rather than the wire recommended by Dr Stewart or other clinical authorities who have not tested the coagulating effects of these metals experimentally in the physical laboratory; for my experiments, which confirm those of Leduc, convince me that the passage of an electric current through an aneurysm in the orthodox Stewart-Moore-Corradi method in no way aids in bringing about the coagulation of the blood. Whatever coagulation is produced is due only to the introduction of the foreign body.

(*Issued separately July* 23, 1908.)

XXVII.—Experiments with Heusler's Magnetic Alloy. By James G. Gray, B.Sc., *Lecturer on Physics in the University of Glasgow. Communicated by* Professor A. Gray, F.R.S.

(MS. received April 6, 1908. Read June 1, 1908.)

Introduction.

In 1903 Heusler made the important discovery that alloys of copper, manganese, and aluminium are strongly magnetic. It was shown in the paper published by Heusler that by properly adjusting the amounts of the component metals present an alloy is obtained the permeability of which is comparable with that of cast iron. Heusler also demonstrated the fact that the properties of this alloy are profoundly modified by thermal treatment.

Since 1903 a number of papers dealing with the subject have appeared in the various scientific journals. In 1905 some experiments were carried out on a sample of the alloy by the author of the present paper, and the results were communicated to the Royal Philosophical Society of Glasgow.* It was shown that the alloy, after quenching from a high temperature, possessed very peculiar magnetic properties. Thus the sample in the quenched condition was practically non-magnetic at room temperature, but at the temperature of liquid air its permeability was much greater than that of the material in the normal, or unquenched, condition, in which variation of temperature produced but little effect. It was therefore decided to construct further samples of the alloy, and a number of rods were accordingly cast at the works of Messrs Stevens & Struthers, the well-known Glasgow brass-founders. Specimens turned from these rods have been the subject of experiment from time to time in the Physical Laboratory of the University of Glasgow, and a great quantity of data have been obtained. The effect of repeated quenching upon the magnetic properties of the alloy has been studied in detail by Mr Alexander D. Ross, until lately Houldsworth Scholar of the University of Glasgow, and some of the more interesting results obtained by him have been communicated to the Royal Society of Edinburgh.†

* *Proceedings of the Royal Philosophical Society of Glasgow*, session 1906–07.

† *Proceedings of the Royal Society of Edinburgh*, vol. xxvii. part ii.

Scope of the Investigation.

The present paper gives an account of some work carried out by the author. The following points were investigated:—

(1) The magnetic properties of the alloy at temperatures lying between 0° C. and 400° C. (about the critical temperature of the material).

(2) The alteration in magnetic properties brought about by heating and cooling the material.

(3) The magnetic properties of the alloy at low temperatures in the quenched condition.

Specimens.

The specimens experimented upon had the following composition:— 25 per cent. manganese, 12·5 per cent. aluminium, a trace of lead, and the remainder copper. It will be seen that the manganese and aluminium were present in atomic proportions, a condition which was shown by Heusler to result in a material possessing maximum permeability for a given amount of dissolved manganese. The specimens were in the form of cylinders, and were tested in the condition which resulted from the dressing operations.

Arrangement of Apparatus.

The magnetometric method was adopted throughout, and the arrangement of the apparatus is shown in Diagram I. The specimen was placed within the magnetising solenoid S. Connected in series with the solenoid was a large circular balancing coil C, placed with its plane perpendicular to, and its centre coincident with, the axis of the solenoid. Also on the axis of the solenoid, and beyond the coil C, was the magnetometer needle, the movements of which could be observed by means of a lamp and scale in the usual way. The coil and solenoid were connected through a reversing key K with a battery B, an ampere-meter A, and a variable resistance R. The stands carrying the magnetometer, solenoid, and coil moved in grooves as shown in the diagram, an arrangement which admitted of the preliminary adjustments being carried out with great convenience and rapidity.

Description of Electric Furnace.

In order to study the behaviour of the material when at temperatures considerably above that of the room, an electric furnace was contrived. A thin porcelain tube T, having an internal diameter somewhat greater than that of the specimen, was wound non-inductively with very fine platinum

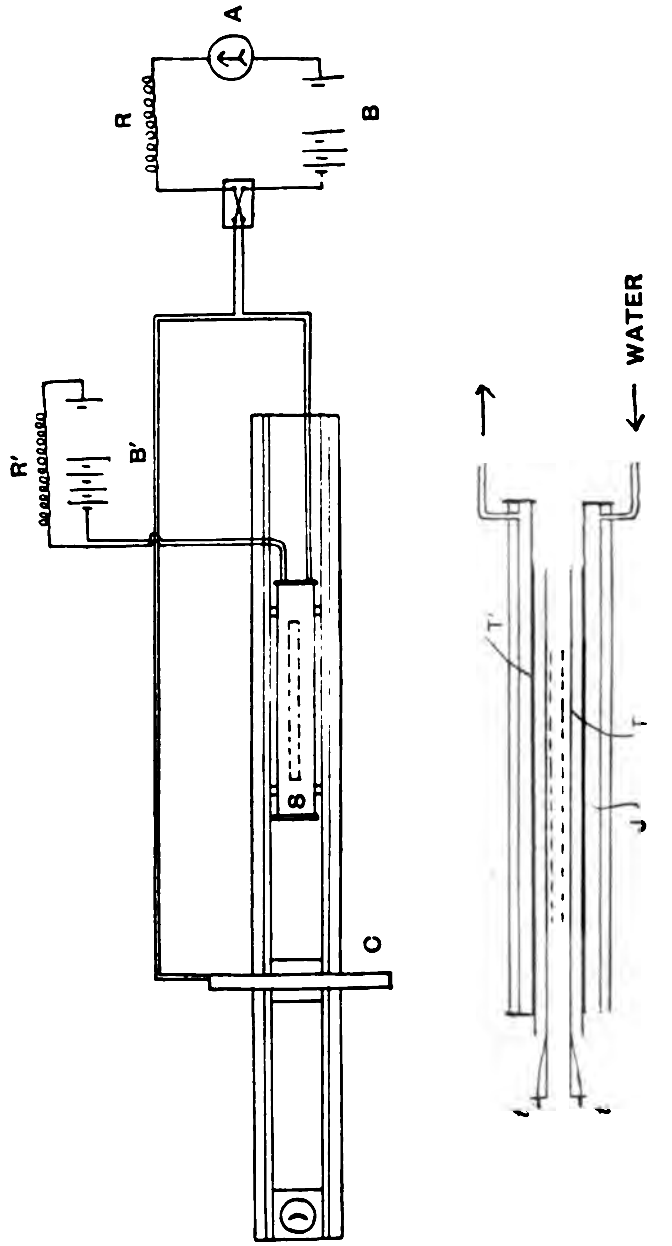


Diagram I.

wire. It was then placed within a tube T′ of hard glass, and the intervening space packed with kaolin clay. The ends of the platinum wire were brought out and connected to a pair of terminals *t*, *t* as shown in the figure. On passing an electric current of suitable strength through the platinum wire it was possible to bring the interior of the porcelain tube, and consequently the specimen, to any desired temperature.

The external diameter of the glass tube being very slightly less than the internal diameter of the magnetising solenoid, the furnace could be slipped as a cartridge within the solenoid.

Between the furnace and the magnetising coil was a water-jacket J, through which a stream of cold water was kept circulating.

The various temperatures employed in the course of the investigation were measured by means of a platinum, platinum-iridium pyrometer. In the experiments contained under (1) and (2) above, the magnetising solenoid, with the furnace in position, was adjusted so that its axis was on the east-and-west line passing through the magnetometer needle. The specimen, with the pyrometer, was then placed in position and the test proceeded with.

Experiments at Low Temperatures.

In carrying out the experiments at low temperatures the furnace was removed, the specimen placed in a glass tube well served with cotton-wool, and the whole slipped within the solenoid. One end of the glass tube was closed; the other end was open, and the tube was bent up to allow of liquid air being poured in.

Preliminary Experiments.

Preliminary experiments were made with a view to testing the adjustment of the apparatus and of ascertaining whether the various specimens varied among themselves. On placing the rods in turn within the coil and carrying out a test, I-H curves were obtained which were practically identical. One of the curves so obtained is shown in Diagram IV., Curve I.

Experiments contained under (1) above.

The experiments (1) were now proceeded with. The specimen was carried through a cycle in the usual way, the temperature being that of the room (15° C.). A current was now passed through the heating coil of the furnace, which resulted in a steady temperature of 120° C. being set up. With the specimen at this temperature a cycle was gone through. This procedure was repeated for various temperatures, and the results obtained are set forth in Diagram II. Curve I. exhibits the magnetic quality of the material at room temperature, Curve II. its magnetic quality

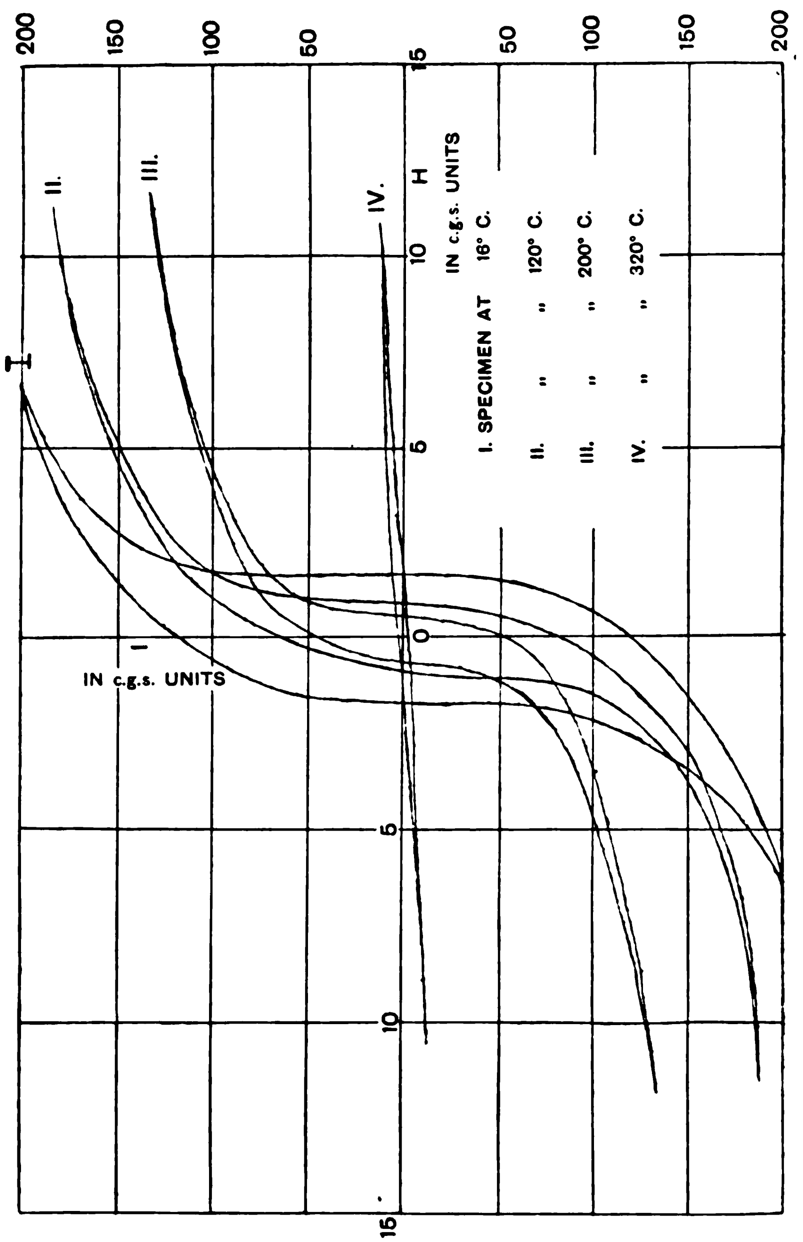


DIAGRAM II.

at 120° C., and so on. It will be seen that the effect of increase of temperature is to diminish continuously the permeability and hysteresis of the material. At 310° C. the specimen became practically non-magnetic.

Diagram III. is of considerable interest. In obtaining the curves there shown the following procedure was gone through:—The specimen was placed within the solenoid and a magnetising field applied. This field was maintained constant throughout. The temperature of the specimen was now gradually increased until the magnetometer deflection had become zero; the specimen was then allowed to cool slowly to room temperature, readings of the magnetometer being taken as the temperature fell. Such curves were first obtained for the Heusler alloy by Mr V. Bruce Hill, and are described by him in the *Physical Review*, vol. xxi., p. 335.

The diagram shows the curves obtained with the specimen subjected to applied fields of amounts 150 c.g.s. units, 75 c.g.s units, and 0·5 c.g.s. units respectively. It will be seen that in the case of the higher fields the permeability of the material is zero at about 500° C., and that the cooling curve in no case coincides with the heating curve. At the higher fields also, as cooling proceeded the susceptibility remained practically zero, until the temperature had fallen to about 350° C. The contrast between the behaviour of the material when heating and when cooling is even more striking when the applied field is weak, as will be seen from the diagram.

Hopkinson* has carried out similar experiments upon alloys composed of nickel and iron, and it is interesting to contrast the results obtained with those exhibited in Diagram III. In Hopkinson's tests the specimens were in the form of rings, and the ballistic method of examining the magnetic quality was employed, the temperature being inferred from the resistance of the secondary coil. A sample containing 4·7 per cent. of nickel and 0·22 per cent. of carbon exhibited magnetic properties, when tested at room temperature, very similar to those of mild steel. On gradually increasing the temperature it was found that the magnetic quality of the material gradually improved until a temperature of about 750° C. was reached, after which it deteriorated rapidly, and the susceptibility became zero at a temperature of about 825° C. On cooling, the susceptibility remained zero until a temperature of about 670° C. had been reached, after which point it increased, until finally at room temperature it was restored to very nearly its initial value.

The behaviour of a nickel-iron containing 25 per cent. of nickel was even more remarkable. This specimen was found to be non-magnetic at room temperature, but on being cooled to about −50° C. it became magnetisable, and remained so as the temperature was gradually increased to about 600° C., at which point the susceptibility became zero. On allowing the specimen

* Hopkinson, *Proceedings of the Royal Society.*, December 12, 1889; January 23, 1890; May 1, 1890.

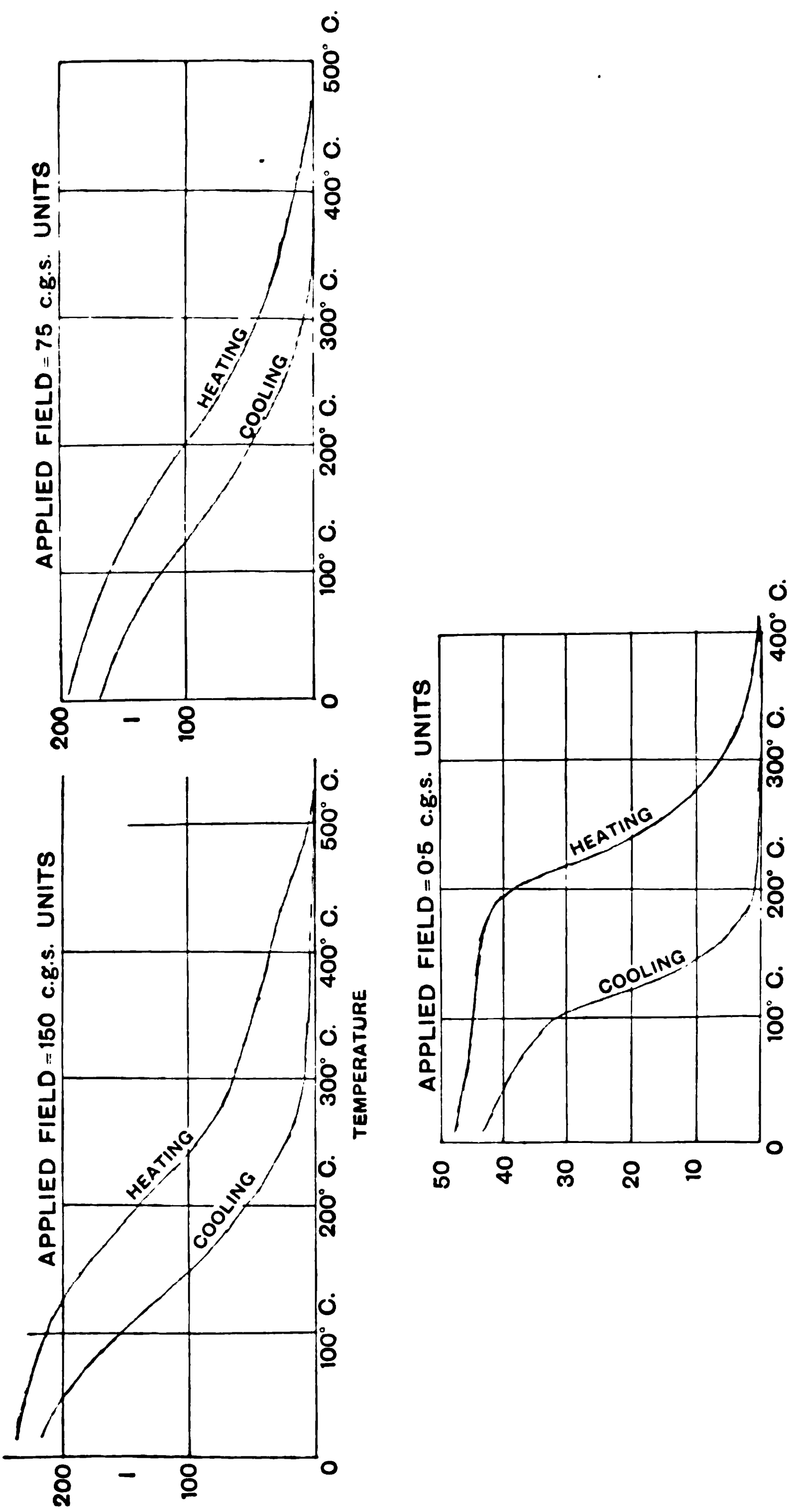

APPLIED FIELD = 75 c.g.s. UNITS
HEATING
COOLING
APPLIED FIELD = 150 c.g.s. UNITS
HEATING
COOLING
TEMPERATURE
APPLIED FIELD = 0·5 c.g.s. UNITS
HEATING
COOLING
0
100° C.
200° C.
300° C.
400° C.
500° C.
200
100
50
40
30
20
10

Diagram III.

to cool, it remained non-magnetic down to the ordinary temperature of the room.

It will be seen from Diagram III. that in the case of the particular Heusler alloys under test the cooling curve never rises to the heating curve.

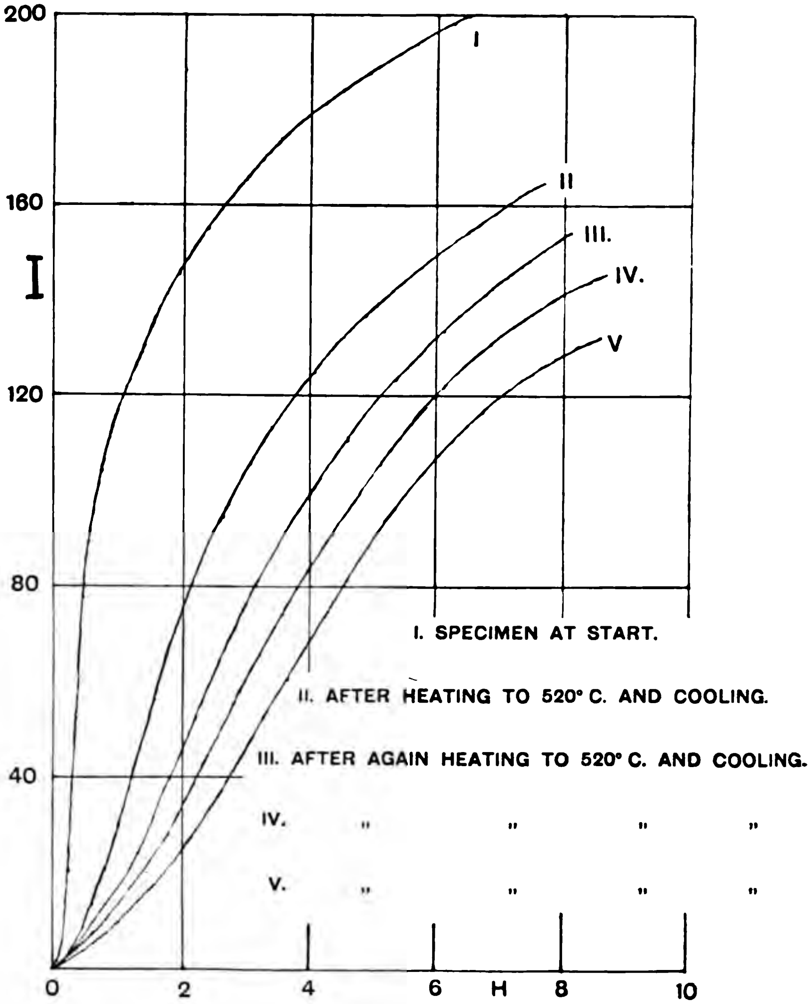
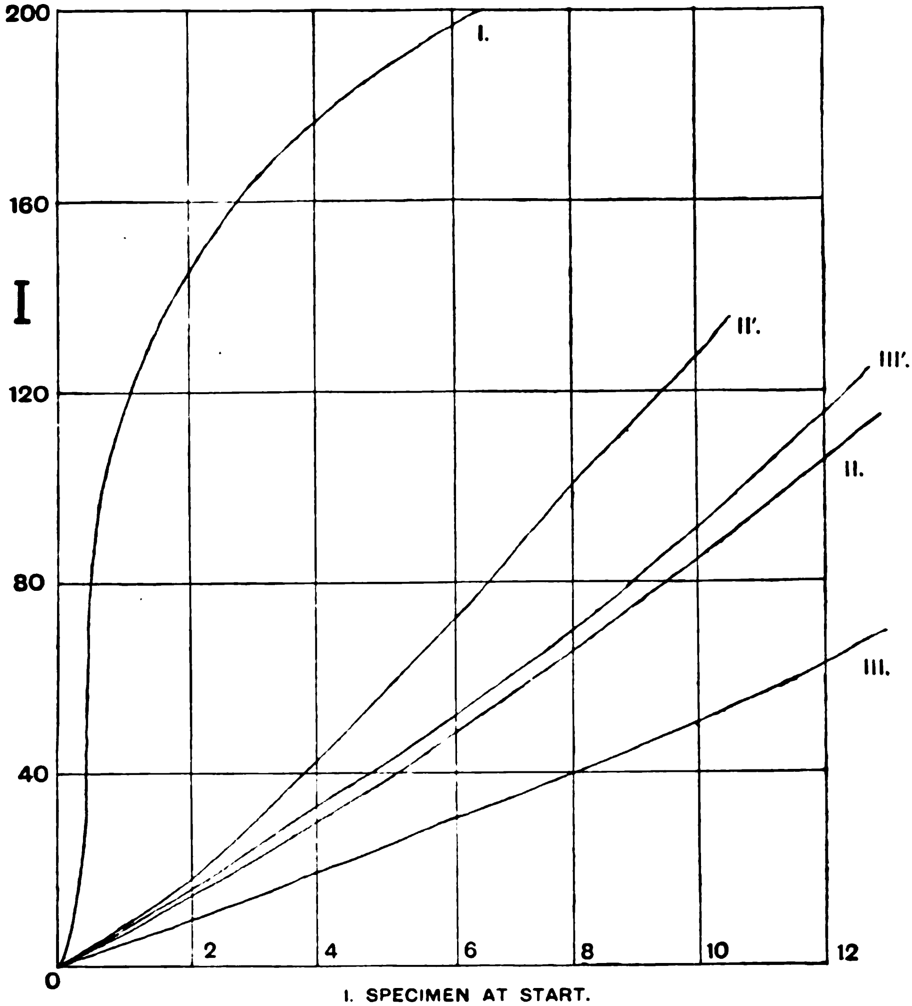


Diagram IV.

Thus heating above the critical temperature results in the magnetic quality being to some extent destroyed. The effect of continued heating and subsequent cooling is shown in Diagram IV. A specimen was placed within the coil and an I–H curve obtained. By means of the furnace the temperature was gradually raised to 520° C. (above the critical temperature); the heating circuit was then broken, the specimen allowed to return to room temperature, and a new I–H curve obtained. This procedure was repeated several times.



I. SPECIMEN AT START.

II. SPECIMEN AFTER QUENCHING AT 750° C.

II'. SPECIMEN IN LIQUID AIR AFTER QUENCHING AT 750° C.

III. SPECIMEN AFTER QUENCHING TWICE AT 750° C

III'. SPECIMEN IN LIQUID AIR AFTER QUENCHING TWICE AT 750° C.

Diagram V.

Curve I. shows the magnetic condition of the specimen prior to heating, Curve II. its condition after one heating, Curve III. its condition after two heatings, and so on. It will be seen that the percentage diminution in susceptibility brought about by the first heating was greatly in excess of that caused by any subsequent heating. Each heating, however, resulted in a diminution of magnetic property, and so far as the experiments went there is no appearance of a steady state having been set up.

The magnetic properties of the Heusler alloys when quenched from a high temperature are very remarkable. Diagram V. shows the results obtained on testing a specimen in both the normal and the quenched conditions. Curve I. was obtained with the specimen in the normal, or unquenched, condition; Curve II. was obtained with the specimen in the quenched condition; Curve II.′ with the specimen completely immersed in liquid air. Curve III. shows the magnetic behaviour of the specimen after two quenchings, and Curve III.′ its behaviour in the doubly quenched condition when at the temperature of liquid air. It will be seen that the effect of the low temperature is to greatly increase the susceptibility of the material. The Heusler alloy in this respect resembles Hopkinson's nickel-iron alloy containing 25 per cent. of nickel. It is interesting to note also, in connection with the comparison of the two materials, that in the Heusler alloy we have a material composed of non-magnetic metals which is strongly magnetic, whereas in the case of the nickel-iron alloy we have an example of a material composed entirely of magnetic metals which is initially non-magnetic at ordinary temperatures.

As on former occasions, the author desires to express his indebtedness to Professor Gray, in whose laboratory the work described above has been carried out.

(*Issued separately July* 23, 1908.)

XXVIII.—The Theory of Hessians in the Historical Order of Development up to 1860. By Thomas Muir, LL.D.

(MS. received February 24, 1908. Read March 16, 1908.)

SPECIAL cases of the determinant

$$\begin{vmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x\partial y} & \frac{\partial^2 u}{\partial x\partial z} & \cdots \\ \frac{\partial^2 u}{\partial y\partial x} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial y\partial z} & \cdots \\ \frac{\partial^2 u}{\partial z\partial x} & \frac{\partial^2 u}{\partial z\partial y} & \frac{\partial^2 u}{\partial z^2} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix},$$

where u is a function of $x, y, z, \ldots$, may well have appeared at a very early date in the history of determinants. The case where $u = ax^2 + 2bxy + cy^2$ may be viewed as traceable to Lagrange (1773), and the case where $u = ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy$ to Gauss (1801); but it is certain that in those cases the elements of the determinants were not looked on as second differential-quotients of u. The general conception first occurred to Hesse in the year 1843.

HESSE, O. (1844, January).

[Ueber die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln. *Crelle's Journal*, xxviii. pp. 68–96: or *Werke*, pp. 89–122.]

In § 15 (p. 83) Hesse passes from the direct subject of his paper to the special case in which the three functions f_1, f_2, f_3 are the first differential-quotients of the homogeneous function of the third degree

$$\Sigma a_{\kappa,\lambda,\mu} x_\kappa x_\lambda x_\mu, \quad \text{or } f \text{ say,}$$

where each of the suffixes κ, λ, μ may be 1 or 2 or 3. The determinant, afterwards called the *Jacobian*, of f_1, f_2, f_3 he says may in that case be styled "the determinant of f." This expression at once recalls that used by Gauss in 1801, namely, "determinant of a form of the second degree," the determinant of $ax^2 + 2bxy + cy^2$, according to Gauss, being $b^2 - ac$, and the determinant of $a_1x^2 + a_2y^2 + a_3z^2 + 2b_1yz + 2b_2zx + 2b_3xy$ being $a_1b_1^2 + a_2b_2^2 + a_3b_3^2 - a_1a_2a_3 - 2b_1b_2b_3$. The two usages, when Hesse's is restricted to the second degree, are not so far apart: for, according to Hesse, the determinants of the same two forms are

$$\begin{vmatrix} 2a & 2b \\ 2b & 2c \end{vmatrix}, \quad \begin{vmatrix} 2a_1 & 2b_3 & 2b_2 \\ 2b_3 & 2a_2 & 2b_1 \\ 2b_2 & 2b_1 & 2a_3 \end{vmatrix}$$

i.e. $$-2^2(b^2-ac), \quad -2^3(a_1b_1^2+a_2b_2^2+\ \ldots\ldots).$$

The elements of Hesse's "determinant of f" being evidently the second differential-quotients of f, those in conjugate places must be equal—that is to say, the determinant is axisymmetric.

The first result enunciated is (p. 85)—*Die Determinante der Determinante einer gegebenen homogenen Function dritten Grades von drei Variabeln ist gleich der Summe der gegebenen Function und ihre Determinante, jede mit einem passenden constanten Factor multiplicirt.* In symbols at a later date this would have been written

$$\mathrm{H}\left\{\mathrm{H}(u_{33})\right\} = cu_{33} + c'\mathrm{H}(u_{33}).$$

Following thereupon is a theorem of like type

$$\mathrm{H}\left\{c_1u_{33} + c_2\mathrm{H}(u_{33})\right\} = c_3u_{33} + c_4\mathrm{H}(u_{33}),$$

and this is used to solve the equation

$$\mathrm{H}(u'_{33}) = u_{33},$$

where u_{33} and u'_{33} stand for ternary cubics, and u'_{33} is the unknown.

The effect of linear transformation, so strikingly brought to the front by Boole three years before, is then (§ 19) entered on, f being no longer a ternary cubic, but any function whatever of $x_1, x_2, \ldots, x_n$, and supposed to be expressed also as a function of the variables $y_1, y_2, \ldots, y_n$ by means of the equations

$$\left.\begin{array}{l} a_1^{(1)}x_1 + a_1^{(2)}x_2 + \ \cdots\cdots\ + a_1^{(n)}x_n = y_1, \\ a_2^{(1)}x_1 + a_2^{(2)}x_2 + \ \cdots\cdots\ + a_2^{(n)}x_n = y_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_n^{(1)}x_1 + a_n^{(2)}x_2 + \ \cdots\cdots\ + a_n^{(n)}x_n = y_n. \end{array}\right\}$$

Denoting the determinant of f when viewed as a function of the x's by ϕ, and when viewed as a function of the y's by ϕ', Hesse affirms that

$$\phi = r^2\phi',$$

where r is the determinant formed from the coefficients of the x's in the transforming equation. His proof is essentially that still followed; that is to say, he recalls that from the multiplication theorem we have

$$\Sigma \pm u_1^{(1)} u_2^{(2)} \ldots u_n^{(n)} = \Sigma \pm a_1^{(1)} a_2^{(2)} \ldots a_n^{(n)} \cdot \Sigma \pm w_1^{(1)} w_2^{(2)} \ldots w_n^{(n)},$$

if $$u_\kappa^{(\lambda)} = a_1^{(\lambda)} w_1^{(\kappa)} + a_2^{(\lambda)} w_2^{(\kappa)} + \cdots + a_n^{(\lambda)} w_n^{(\kappa)},$$

and $$\Sigma \pm u_1^{(1)} u_2^{(2)} \ldots u_n^{(n)} = r^2 \cdot \Sigma \pm v_1^{(1)} v_2^{(2)} \ldots v_n^{(n)},$$

if in addition $$w_\kappa^{(\lambda)} = a_1^{(\lambda)} v_1^{(\kappa)} + a_2^{(\lambda)} v_2^{(\kappa)} + \cdots + a_n^{(\lambda)} v_n^{(\kappa)};$$

and he then merely asserts that application to the case where

$$u_\kappa^{(\lambda)} = \frac{\partial^2 f}{\partial x_\kappa \partial x_\lambda}, \quad w_\lambda^{(\kappa)} = \frac{\partial\left(\frac{\partial f}{\partial x_\kappa}\right)}{\partial y_\lambda}, \quad v_\kappa^{(\lambda)} = \frac{\partial^2 f}{\partial y_\kappa \partial y_\lambda}$$

accomplishes the desired aim. The result, as stated in later phraseology, is that "the Hessian is a covariant." The case where $f = ax^2 + 2bxy + cy^2$ was given by Lagrange in 1773, and the case $f = ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy$ by Gauss in 1801.

The ternary cubic u_{33} is next returned to and shown to be transformable by a linear substitution into the form

$$y_1^3 + y_2^3 + y_3^3 + 6\pi y_1 y_2 y_3$$

and to be such that constants c_1, c_2 are determinable which make

$$c_1 u_{33} + c_2 \mathrm{H}(u_{33})$$

resolvable into linear factors.

Cayley, A. (1845, early).

[Note sur deux formules données par MM. Eisenstein et Hesse. *Crelle's Journal*, xxix. pp. 54–57: or *Collected Math. Papers*, i. pp. 113–116.]

Cayley, who had, like others, been attracted by Boole's epoch-making paper on Linear Transformations, and was about to publish his own first paper on the subject (*Camb. and Dubl. Math. Journ.*, i. pp. 104–122), was naturally interested in that part of Hesse's paper which concerned the "determinant of f." He consequently wrote the note we have now reached, for the purpose of adding to Hesse's results and of extending an identity of Eisenstein's not distantly related to the same subject.

The "équation remarquable" of Hesse's which he starts with he writes in the form

$$\nabla(\mathrm{U} + a\nabla \mathrm{U}) = \mathrm{AU} + \mathrm{B}\nabla \mathrm{U},$$

noting that its author had not given the values of the coefficients A, B, "ce qui parait être très difficile à effectuer." He then announces the analogous theorem: "Soit U une fonction homogène et de l'ordre ν des deux variables x, y, et ∇U la déterminante

$$\frac{\partial^2 U}{\partial x^2}\cdot\frac{\partial^2 U}{\partial y^2} - \left(\frac{\partial^2 U}{\partial x \partial y}\right)^2,$$

l'on a

$$(\nu-2)(\nu-3)\cdot\nabla(U+a\nabla U) = \left\{-\nu(\nu-1)(\nu-3)^2 aJ + \nu(\nu-1)(2\nu-5)^2 a^2 I\right\}U$$
$$+\left\{(\nu-2)(\nu-3)^3 + (\nu-2)(\nu-3)(2\nu-5)a^2 J\right\}\nabla U.$$

En représentant par i, j, k, l, m les coefficients différentiels du quatrième ordre de U, on a

$$I = ikm - il^2 - mj^2 - k^3 + 2jkl,$$
$$J = 4jl - 3k^2 - mi,$$

de manière que I, J sont des fonctions de x, y des ordres $3(\nu-4)$ et $2(\nu-4)$ respectivement." To this he adds the remarkable fact, that if the binary quartic

$$i\xi^4 + 4j\xi^3\eta + 6k\xi^2\eta^2 + 4l\xi\eta^3 + m\eta^4,$$

where i, j, k, l, m are now any quantities independent of ξ, η, be transformed by the substitution

$$\left.\begin{array}{l}\xi = \lambda\xi' + \mu\eta' \\ \eta = \lambda'\xi' + \mu'\eta'\end{array}\right\}$$

and I and J thus become I′ and J′, then

$$I' = (\lambda\mu' - \lambda'\mu)^6 I, \qquad J' = (\lambda\mu' - \lambda'\mu)^4 J.$$

In other words, he makes known for the first time the two "invariants" of a binary quartic, and notes the curious fact that expressions of exactly the same form occur in his equivalent for $\nabla(U_{2,\nu} + a\nabla U_{2,\nu})$.

Another remark is equally suggestive, namely, that simpler results might be reached if U were taken a homogeneous function in x', y' as well as in x, y, and ∇U were defined as

$$\frac{\partial^2 U}{\partial x \partial x'}\cdot\frac{\partial^2 U}{\partial x \partial y'} - \frac{\partial^2 U}{\partial y \partial y'}\cdot\frac{\partial^2 U}{\partial y \partial x'}.$$

For example, U being a quadric in both sets of variables, namely

$$\begin{aligned} U = {} & x_1^2(Ax^2 + 2Bxy + Cy^2) \\ & + 2x_1y_1(A'x^2 + 2B'xy + C'y^2) \\ & + y_1^2(A''x^2 + 2B''xy + C''y^2), \end{aligned}$$

or in later notation

$$U = \begin{array}{ccc|l} x^2 & 2xy & y^2 & \\ \hline A & B & C & y_1^2 \\ A' & B' & C' & 2x_1y_1 \\ A'' & B'' & C'' & x_1^2, \end{array}$$

then we should have *

$$\nabla\nabla U = 2^{10}\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} U.$$

In connection with the first of these results of Cayley's the reader should note that on putting $\nu = 4$ we obtain

$$\nabla(U + a\nabla U) = (-6aJ + 54a^2 I)U + (1 + 3a^2 J)\nabla U,$$

or

$$\nabla(\alpha U + \beta\nabla U) = (-6\alpha\beta J + 54\beta^2 I)U + (\alpha^2 + 3\beta^2 J)\nabla U,$$

and ∴

$$\nabla\nabla U = 54 I \cdot U + 3J \cdot \nabla U.$$

Further, if the particular form of U be

$$ax^2 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

this gives

$$\nabla\nabla U = 12^3(432 I \cdot U - J \cdot \nabla U)$$

where

$$I = ace + 2bcd - ad^2 - eb^2 - c^3,$$

and

$$J = ae + 3c^2 - 4bd.$$

In his famous first paper (February 1845) "On the Theory of Linear Transformations," of which this is merely an offshoot, Cayley states that the invariance of I had been communicated to him by Boole, along with the still more interesting fact that Boole's invariant (*i.e.* the discriminant) is equal to $J^3 - 27I^2$.

Cayley, A. (1847).

[Note sur les hyperdéterminants. *Crelle's Journal*, xxxiv. pp. 148–152: or *Collected Math. Papers*, i. pp. 352–355.]

As already noted, the second section of this short paper concerns what would, a few years later, have been called "the Hessian of the discriminant

$$6abcd + 3b^2c^2 - a^2d^2 - 4ac^3 - 4b^3d$$

of the binary cubic." The result, which is rather inelegantly verified, is that the said Hessian is a numerical multiple of the square of the discriminant.

* Instead of 2^{10} we find in the original 2^3, and in the *Collected Math. Papers* 2^8.

HESSE, O. (1847, August.)

[Ueber Curven dritter Classe und Curven dritter Ordnung. *Crelle's Journal*, xxxviii. pp. 241–256: or *Werke*, pp. 193–210.]

Any homogeneous function of the m^{th} degree in the variables x_1, x_2, x_3 being denoted by u, its first differential-quotients by u_1, u_2, u_3, and its second differential-quotients by u_{11}, u_{12}, . . . there is obtained from Euler

$$\left.\begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= (m-1)u_1 \,, \\ u_{21}x_1 + u_{22}x_2 + u_{23}x_3 &= (m-1)u_2 \,, \\ u_{31}x_1 + u_{32}x_2 + u_{33}x_3 &= (m-1)u_3 \,, \end{aligned}\right\}$$

and thence, on solving,

$$\left.\begin{aligned} \frac{\Delta}{m-1}x_1 &= \mathrm{U}_{11}u_1 + \mathrm{U}_{12}u_2 + \mathrm{U}_{13}u_3 \,, \\ \frac{\Delta}{m-1}x_2 &= \mathrm{U}_{21}u_1 + \mathrm{U}_{22}u_2 + \mathrm{U}_{23}u_3 \,, \\ \frac{\Delta}{m-1}x_3 &= \mathrm{U}_{31}u_1 + \mathrm{U}_{32}u_2 + \mathrm{U}_{33}x_3 \,, \end{aligned}\right\}$$

where, evidently, Δ is used for Hesse's determinant of u, and U_{rs} for the cofactor of u_{rs} in Δ. Using in connection with the latter three equations the multipliers u_1, u_2, u_3 and adding, Hesse derives the interesting result

$$\frac{m}{m-1}u\Delta = \mathrm{U}_{11}u_1^2 + \mathrm{U}_{22}u_2^2 + \mathrm{U}_{33}u_3^2 + 2\mathrm{U}_{23}u_2u_3 + 2\mathrm{U}_{31}u_3u_1 + 2\mathrm{U}_{12}u_1u_2 \,,$$

and this by a process of differentiation leads to six results of the type

$$u_{12}u_1u_3 + u_{13}u_1u_2 - u_{23}u_1^2 - u_{11}u_2u_3 = \frac{m}{m-1}\mathrm{U}_{23}u - \frac{x_2x_3}{(m-1)^2}\Delta \,.$$

The rest of the paper is geometrical.

HESSE, O. (1849, January).

[Transformation einer beliebigen gegebenen homogenen Function 4ten Grades von zwei Variabeln. . . . . *Crelle's Journal*, xli. pp. 243–263: or *Werke*, pp. 223–246.]

A binary quartic u_{24} being the only other homogeneous integral function whose determinant, in Hesse's sense, is of the same degree as the function, there was naturally an inclination to make a study of its properties in the same fashion as had been followed with the ternary cubic. Analogous results are reached, such, for example, as the theorem that *it is possible to determine constants* c_1, c_2 *so that*

$$c_1 u_{24} + c_2 \mathrm{H}(u_{24})$$

may be an exact square. Most of the matter, however, more directly concerns the quartic than its determinant.

Aronhold, S. (1849, July).

[Zur Theorie der homogenen Functionen dritten Grades von drei Variabeln. *Crelle's Journal*, xxxix. pp. 140–159.]

This is an inspiration from, and a striking development of, the latter part of Hesse's paper of the year 1844, and like that paper may be said to concern itself more with the ternary cubic than with the so-called determinant of that function. In regard to the latter, however, there is one very noteworthy result: for, just as Cayley in 1845 established the two invariants I and J of a binary quartic u_{24}, and used them for the expression of $\mathrm{H}\{a\cdot u_{24} + b\cdot \mathrm{H}(u_{24})\}$ in the form $\mathrm{A}\cdot u_{24} + \mathrm{B}\cdot\mathrm{H}(u_{24})$, so Aronhold here announces the two invariants S and T of a ternary cubic, and gives the similar expression for

$$\mathrm{H}\Big\{ a\cdot u_{33} + b\cdot \mathrm{H}(u_{33}) \Big\}.$$

Obtained from this by putting $a=0$ is the result

$$\mathrm{H}\Big\{ \mathrm{H}(u_{33}) \Big\} = 3\mathrm{S}^2\cdot u_{33} - 2\mathrm{T}\cdot\mathrm{H}(u_{33})$$

—the longed-for definite form of Hesse's theorem of the year 1844.

We may also note in passing that the result of eliminating x, y, z from the equations

$$\frac{\partial u_{33}}{\partial x} = 0\,, \quad \frac{\partial u_{33}}{\partial y} = 0\,, \quad \frac{\partial u_{33}}{\partial z} = 0$$

is expressed in terms of S and T, namely,

$$\mathrm{T}^2 - \mathrm{S}^3 = 0\,;$$

or, in later phraseology, that the discriminant of u_{33} is $\mathrm{T}^2 - \mathrm{S}^3$.

Hesse and Jacobi (1849, December).

[Auszug zweier Schreiben des Prof. Hesse an den Herrn Prof. Jacobi und eines Schreibens des Prof. Jacobi an Herrn Prof. Hesse. *Crelle's Journal*, xl. pp. 316–318.]

Hesse having communicated to Jacobi a theorem regarding a homogeneous function of three variables, Jacobi sent back a proof showing that the theorem held in the case of n variables. The function being denoted by u, and being of the m^{th} degree in the variables $x_1, x_2, \ldots, x_n$, Jacobi, like Hesse himself in his paper of 1847 (August), obtains the equations

$$\left.\begin{array}{lcl} x_1u_{11}+x_2u_{21}+ \cdots +x_nu_{n1} &=& (m-1)u_1 \\ x_1u_{12}+x_2u_{22}+ \cdots +x_nu_{n2} &=& (m-1)u_2 \\ \cdots\cdots & & \\ x_1u_{1n}+x_2u_{2n}+ \cdots +x_nu_{nn} &=& (m-1)u_n \end{array}\right\}$$

and thence

$$x_i\Delta = (m-1)\left\{ U_{i1}u_1+U_{i2}u_2+ \cdots +U_{in}u_n \right\} \tag{α}$$

Differentiating both sides of this with respect to x_k, we have

$$\begin{aligned} x_i\frac{\partial\Delta}{\partial x_k} &= (m-1)\left\{ U_{i1}u_{k1}+U_{i2}u_{k2}+ \cdots +U_{in}u_{kn} \right\} \\ &\quad +(m-1)\left\{ u_1\frac{\partial U_{i1}}{\partial x_k}+u_2\frac{\partial U_{i2}}{\partial x_k}+ \cdots +u_n\frac{\partial U_{in}}{\partial x_k} \right\} \\ &= (m-1)\left\{ u_1\frac{\partial U_{i1}}{\partial x_k}+u_2\frac{\partial U_{i2}}{\partial x_k}+ \cdots +u_n\frac{\partial U_{in}}{\partial x_k} \right\} \end{aligned} \tag{β}$$

if k be different from i. A second differentiation, but this time with respect to x_l, gives

$$\begin{aligned} x_i\frac{\partial^2\Delta}{\partial x_l\partial x_k} &= (m-1)\left\{ u_1\frac{\partial^2 U_{i1}}{\partial x_l\partial x_k}+u_2\frac{\partial^2 U_{i2}}{\partial x_l\partial x_k}+ \cdots +u_n\frac{\partial^2 U_{in}}{\partial x_l\partial x_k} \right\} \\ &\quad +(m-1)\left\{ u_{l1}\frac{\partial U_{i1}}{\partial x_k}+u_{l2}\frac{\partial U_{i2}}{\partial x_k}+ \cdots +u_{ln}\frac{\partial U_{in}}{\partial x_k} \right\}. \end{aligned}$$

But on the supposition that l is different from i we have

$$u_{l1}U_{i1}+u_{l2}U_{i2}+ \cdots +u_{ln}U_{in} = 0,$$

and therefore by differentiation with respect to x_k

$$\begin{aligned} & u_{l1}\frac{\partial U_{i1}}{\partial x_k}+u_{l2}\frac{\partial U_{i2}}{\partial x_k}+ \cdots +u_{ln}\frac{\partial U_{in}}{\partial x_k} \\ & +u_{kl1}U_{i1}+u_{kl2}U_{i2}+ \cdots +u_{kln}U_{in} = 0. \end{aligned}$$

Consequently by substitution

$$\begin{aligned} x_i\frac{\partial^2\Delta}{\partial x_l\partial x_k} &= (m-1)\left\{ u_1\frac{\partial^2 U_{i1}}{\partial x_l\partial x_k}+u_2\frac{\partial U_{i2}}{\partial x_l\partial x_k}+ \cdots +u_n\frac{\partial^2 U_{in}}{\partial x_l\partial x_k} \right\} \\ &\quad -(m-1)\left\{ u_{kl1}U_{i1}+u_{kl2}U_{i2}+ \cdots +u_{kln}U_{in} \right\}, \end{aligned} \tag{γ}$$

where l and k are each different from i.

If now special values of $x_1, x_2, \ldots x_n$ make $u_1, u_2, \ldots, u_n$ all vanish, then, Jacobi says, we shall also have

$$\Delta=0, \quad \frac{\partial\Delta}{\partial x_k}=0, \quad U_{rs}=Nx_rx_s$$

for all values of r and s *; and consequently from (γ)

$$x_i\frac{\partial^2\Delta}{\partial x_i\partial x_k} = -(m-1)\left\{ \mathrm{N}x_ix_1\frac{\partial u_{kl}}{\partial x_1} + \mathrm{N}x_ix_2\frac{\partial u_{kl}}{\partial x_2} + \cdots \right\}$$

$$= -(m-1)\mathrm{N}x_i\left\{ x_1\frac{\partial u_{kl}}{\partial x_1} + x_2\frac{\partial u_{kl}}{\partial x_2} + \cdots \right\}$$

$$= -(m-1)(m-2)\mathrm{N}x_iu_{kl},$$

whence

$$\frac{\partial^2\Delta}{\partial x_i\partial x_k} = -(m-1)(m-2)\mathrm{N}\,.\,\frac{\partial^2 u}{\partial x_i\partial x_k}.$$

This result we may formally enunciate as follows:—*If the first differential-quotients of a homogeneous rational integral function all vanish, the elements of the Hessian of the function are proportional to the elements of the Hessian of the Hessian.*

TERQUEM, O. (1851, March).

[Note sur les déterminants. *Nouv. Annales de Math.*, x. pp. 124–131.]

This is an elementary exposition of Hesse's determinant, with simple illustrations from algebraic geometry, the property of "covariance" being made prominent. A curious distinction is made between what are called the "first" and "second" determinants: for example,

$$4ac - b^2$$

is styled "le *premier déterminant* de la fonction hexanôme du second degré à deux variables," and

$$8acf + 4bde - 2ae^2 - 2cd^2 - 2fb^2$$

the "second déterminant de la même fonction rendue homogène et ternaire." The former is, in fact, the determinant of

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2$$

or

$$ax^2 + bxy + cy^2 + dx + ey + f$$

with respect to x, y; and the latter the determinant with respect to x, y, z.

* The first two of these results, which follow from (α) and (β), there is no pressing reason for mentioning: it would have been equally pertinent to note that $u=0$. The third result Jacobi probably obtained (see *Crelle's Journal*, xv. p. 304) by taking in every possible way $n-1$ of the initiatory set of equations and deducing

$$x_1 : x_2 : \ldots\ldots : x_n = \mathrm{U}_{11} : \mathrm{U}_{21} : \ldots\ldots : \mathrm{U}_{n1},$$
$$= \mathrm{U}_{12} : \mathrm{U}_{22} : \ldots\ldots : \mathrm{U}_{n2},$$
$$\ldots\ldots\ldots\ldots$$
$$= \mathrm{U}_{1n} : \mathrm{U}_{2n} : \ldots\ldots : \mathrm{U}_{nn}.$$

This implies that $\mathrm{U}_{rs} : \mathrm{U}_{rs'} = x_s : x_{s'}$

and $\mathrm{U}_{s'r} : \mathrm{U}_{s'r'} = x_r : x_{r'}$;

and from these, by reason of the equality of $\mathrm{U}_{rs'}$ and $\mathrm{U}_{s'r}$ there is got by multiplication

$$\mathrm{U}_{rs} : \mathrm{U}_{s'r'} = x_sx_r : x_{s'}x_{r'}.$$

HESSE, O. (1851, March).

[Ueber die Bedingung, unter welcher eine homogene ganze Function von n unabhängigen Variabeln durch lineare Substitutionen von n andern unabhängigen Variabeln auf eine homogene Function sich zurück führen lässt, die eine Variabel weniger enthält. *Crelle's Journal*, xlii. pp. 117–124: or *Werke*, pp. 289–296].

Hesse here returns to the subject of § 19 of his original paper, calling $\Sigma \pm u_{11}u_{22} \ldots u_{nn}$ or Δ the determinant of u with respect to the variables $x_1, x_2, \ldots x_n$, and $\Sigma \pm u^{11}u^{22} \ldots u^{nn}$ or ∇ the determinant of u with respect to $y_1, y_2, \ldots, y_n$, and proving once more his theorem that

$$\nabla = r^2\Delta .$$

He then supposes that in the result of the transformation y_n does not appear, and says that as this implies that $u^{1n}, u^{2n}, \ldots, u^{nn}$ all vanish, it follows that $\nabla = 0$, and that therefore from the said theorem Δ also must vanish. There is thus obtained the result that, "*Wenn eine homogene ganze Function der* n *unabhängigen Variabeln* $x_1, x_2, \ldots, x_n$, *durch*

$$x_k = a_1^k y_1 + a_2^k y_2 + \cdots + a_n^k y_n$$

in eine Function der Variabeln $y_1, y_2, \ldots, y_n$ *übergeht, in welcher eine dieser Variabeln fehlt, so ist die Determinante dieser Function in Rücksicht auf die Variabeln* $x_1, x_2, \ldots, x_n$, *identisch gleich* 0."

The rest of the paper is occupied with the converse theorem; but as the author himself came to be dissatisfied with his attempt at a proof and returned to the subject seven years later, it need not be entered on here.

SYLVESTER, J. J. (1851, April).

[Sketch of a memoir on elimination, transformation, and canonical forms. *Cambridge and Dub. Math. Journ.*, vi. pp. 186–200: or *Collected Math. Papers*, i. pp. 184–197.]

The expression "determinant of a function" or, more definitely, "determinant of a function in respect to certain variables" occurs repeatedly in Sylvester's writings of the year 1850, the accompanying notation being

$$\square(u);$$

for example, when dealing with ternary quadrics U and V, expressions like

$$\underset{\lambda\mu}{\square}\ \underset{xyz}{\square}(\lambda U + \mu V)$$

are in constant use by him. It is clear, however, that the determinant which he had in mind was not Hesse's, but that which the year following he named the "discriminant." *

The interest of the present paper lies in the fact that, amid much other matter, not only are the said two determinants clearly defined and distinguished, but are shown to be viewable as having a common parentage, being indeed two extreme members of a family group. In the first place, the determinant of any homogeneous integral function is incidentally defined as the resultant of the first partial differential coefficients of the function, when drawing attention to Boole's proposition (1843) that the said determinant "is unaltered by any linear transformation of the variables, except so far as regards the introduction of a power of the modulus of transformation." It is spoken of later in the paper as the "common constant determinant" or the "ordinary determinant" of the function, the word *discriminant* not being proposed until a later date in the same year. In the second place, there is brought into notice in connection with any homogeneous integral function $\phi(x, y, \ldots, z)$ of the n^{th} degree the family of functions

$$\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\ \cdots\ +\zeta\frac{\partial}{\partial z}\right)^r\phi(x, y, \ldots, z),$$

where r has the values $1, 2, \ldots, n$. Corresponding to these there is a family of determinants (*i.e.* discriminants), namely

$$\overline{\left|\right|}_{\xi, \eta, \ldots}\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\ \cdots\ +\zeta\frac{\partial}{\partial z}\right)^r\phi(x, y, \ldots, z),$$

where $r=2, 3, \ldots, n$, the first being according to Sylvester the "Hessian" or "First Boolian" determinant † of ϕ, and the last the "Final Boolian" or "ordinary determinant" of ϕ. The reader is left in the former case to reconcile the new definition with Hesse's own definition, and in the latter case to observe that

$$\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\ \cdots\ +\zeta\frac{\partial}{\partial z}\right)^n\phi(x, y, \ldots, z) = \phi(\xi, \eta, \ldots, \zeta).$$

* See *Philos. Magazine*, ii. (1851) p. 406, and *Cambridge and Dub. Math. Journ.*, vii. (1852) p. 52; or Sylvester's *Collected Math. Papers*, i. pp. 280, 284.

† On p. 194 he says the Hessian of $F(x, y)$ is "the determinant *of the determinant*, in respect to ξ and η, of

$$\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)^2F(x,y)"$$

—an error which is repeated in the *Collected Math. Papers*.

The notation used for the Hessian of ϕ is $H(\phi)$: by "second Hessian" he says he means "Hessian of the Hessian"; by "post-Hessian" the determinant of the function got by taking $r=3$; and similarly for "præter-post-Hessian"!

We may at once remark that much of this nomenclature had a very short life, being supplanted by other coinages made by Sylvester himself. The functions

$$\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\ \cdots\right)^r\phi(x\,,\,y\,,\,\ldots)$$

he soon named the *emanants* of ϕ: and thus the Hessian, post-Hessian, præter-post-Hessian, and other determinants forming the "Hessian (or Boolian) Scale" became known as the discriminants of the emanants of ϕ. To the first member of the scale, however, the word "Hessian" became permanently attached, although Sylvester's mode of defining it as the "discriminant of the quadratic (or second) emanant"* did not spread. It was introduced by Salmon into the first edition of his *Higher Plane Curves* (see p. 72) about a year after Sylvester's first use of it, and met with rapid acceptance.

Salmon, G. (1852).

[A Treatise on the Higher Plane Curves, ... By the Rev. George Salmon, M.A. . . . . xii+316 pp.; Dublin, 1852.]

In sect. ix. (pp. 181–195) Salmon deals with the "General Equation of the Third Degree" on the lines of Aronhold's paper of 1849, and the Hessian naturally comes in for attention. The ternary cubic U which constitutes the non-zero side of the equation he writes in the form

$$\begin{aligned}a_1x^3+b_2y^3+c_3z^3+3a_2x^2y+3b_3y^2z+3c_1z^2x\\ +3a_3x^2z+3b_1y^2x+3c_2z^2y+6dxyz\,,\end{aligned}$$

and gives its Hessian† as

$$\mathrm{a}_1x^3+\mathrm{b}_2y^3+\mathrm{c}_3z^3+3\mathrm{a}_2x^2y+\ \cdots\cdot,$$

* See *Philos. Magazine*, v. p. 122: or *Collected Math. Papers*, i. p. 591.

† It should be noted that this is $\frac{1}{216}$ of the Hessian as defined, that a_1 is expressible as a three-line determinant, and that the performance of the circular substitutions $a_1, b_2, c_3 = b_2, c_3, a_1$, $a_2, b_3, c_1 = b_3, c_1, a_2$, $a_3, b_1, c_2 = b_1, c_2, a_3$, on the expressions for $\mathrm{a}_1, 3\mathrm{a}_2, 3\mathrm{a}_3$ gives us six other of the expressions.

where

$$
\begin{aligned}
\mathrm{a}_1 &= a_1d^2 + b_1a_3^2 + c_1a_2^2 - a_1b_1c_1 - 2da_2a_3 . \\
\mathrm{b}_2 &= b_2d^2 + c_2b_1^2 + a_2b_3^2 - a_2b_2c_2 - 2db_3b_1 . \\
\mathrm{c}_3 &= c_3d^2 + a_3c_2^2 + b_3c_1^2 - a_3b_3c_3 - 2dc_1c_2 . \\
3\mathrm{a}_2 &= c_2a_2^2 + a_2c_1b_1 - 2a_2a_3b_3 - a_2d^2 + 2a_1b_3d + b_2a_3^2 - a_1c_1b_2 - a_1b_1c_2 . \\
3\mathrm{b}_3 &= a_3b_3^2 + b_3a_2c_2 - 2b_3b_1c_1 - b_3d^2 + 2b_2c_1d + c_3b_1^2 - b_2a_2c_3 - b_2c_2a_3 . \\
3\mathrm{c}_1 &= b_1c_1^2 + c_1b_3a_3 - 2c_1c_2a_2 - c_1d^2 + 2c_3a_2d + a_1c_2^2 - c_3b_3a_1 - c_3a_3b_1 . \\
3\mathrm{a}_3 &= b_3a_3^2 + a_3b_1c_1 - 2a_3a_2c_2 - a_3d^2 + 2a_1c_2d + c_3a_2^2 - a_1b_1c_3 - a_1c_1b_3 . \\
3\mathrm{b}_1 &= c_1b_1^2 + b_1c_2a_2 - 2b_1b_3a_3 - b_1d^2 + 2b_2a_3d + a_1b_3^2 - b_2c_2a_1 - b_2a_2c_1 . \\
3\mathrm{c}_2 &= a_2c_2^2 + c_2a_3b_3 - 2c_2c_1b_1 - c_2d^2 + 2c_3b_1d + b_2c_1^2 - c_3a_3b_2 - c_3b_3a_2 . \\
6\mathrm{d} &= -2d^3 + 2d(b_1c_1 + c_2a_2 + a_3b_3) + (a_1b_3c_2 + b_2c_1a_3 + c_3a_2b_1) - a_1b_2c_3 \\
&\qquad - 3(a_2b_3c_1 + a_3b_1c_2) .
\end{aligned}
$$

The invariants S and T are also printed in full, viz.

$$
\begin{aligned}
\mathrm{S} &= d^4 - 2d^2(b_1c_1 + c_2a_2 + a_3b_3) + 3d(a_2b_3c_1 + a_3b_1c_2) - d\cdot a_1b_2c_3 \\
&\quad + d(a_1b_3c_2 + b_2c_1a_3 + c_3a_2b_1) - (b_1c_1\cdot c_2a_2 + c_2a_2\cdot a_3b_3 + a_3b_3\cdot b_1c_1) \\
&\quad + (b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2) - (a_1b_2\cdot c_1c_2 + b_2c_3\cdot a_2a_3 + c_3a_1\cdot b_3b_1) \\
&\quad + (b_3c_3a_2^2 + c_1a_1b_3^2 + a_2b_2c_1^2 + b_2c_2a_3^2 + c_3a_3b_1^2 + a_1b_1c_2^2) , \\
\mathrm{T} &= -8d^6 + 24d^4(b_1c_1 + c_2a_2 + a_3b_3) - \cdot\cdot\cdot\cdot\cdot
\end{aligned}
$$

As these differ from Aronhold's by numerical factors, we are prepared to find corresponding differences in the expressions for the Hessian of the Hessian and for the discriminant, namely,

$$4\mathrm{S}^2\cdot\mathrm{U} \; - \; \mathrm{T}\cdot\mathrm{H(U)} \quad \text{and} \quad \mathrm{T}^2 - 64\mathrm{S}^3$$

respectively.

BRIOSCHI, FR. (1852, August).

[Sur les déterminants des formes quadratiques. *Nouv. Annales de Math.*, xi. pp. 307–311.]

After an introduction of two pages on determinants in general, the determinant of a quadratic form is defined as the determinant whose elements are the second differential-quotients of the form, the editor adding in a footnote the words, "c'est le déterminant *hessien* des Anglais." Starting then from the known fact that if $a_1a_2 - b_1^2 = 0$

$$a_1x_1^2 + a_2x_2^2 + 2b_1x_1x_2 \;=\; \frac{1}{a_1}(a_1x_1 + b_1x_2)^2 ,$$

Brioschi states that similarly, if the determinants of

$$
\begin{gathered}
a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2b_1x_1x_2 + 2b_2x_1x_3 + 2c_1x_2x_3 , \\
a_1x_1^2 + a_2x_2^2 + 2b_1x_1x_2 , \\
a_1x_1^2 + a_3x_3^2 + 2b_2x_1x_3 ,
\end{gathered}
$$

all vanish, the ternary quadric is equal to

$$\frac{1}{a_1}(a_1x_1 + b_1x_2 + b_2x_3)^2 ;$$

and if the determinants of

$$\left.\begin{matrix} a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + 2b_1x_1x_2 + 2b_2x_1x_3 + 2b_3x_1x_4 \\ + 2c_1x_2x_3 + 2c_2x_2x_4 + 2d_1x_3x_4 \end{matrix}\right\} ,$$

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2b_1x_1x_2 + 2b_2x_1x_3 + 2c_1x_2x_3 ,$$

$$a_1x_1^2 + a_2x_2^2 + a_4x_4^2 + 2b_1x_1x_2 + 2b_3x_1x_4 + 2c_2x_2x_4 ,$$

$$a_1x_1^2 + a_2x_2^2 + 2b_1x_1x_2 ,$$

$$a_1x_1^2 + a_3x_3^2 + 2b_2x_1x_3 ,$$

$$a_1x_1^2 + a_4x_4^2 + 2b_3x_1x_4 ,$$

all vanish, the quaternary quadric is equal to

$$\frac{1}{a_1}(a_1x_1 + b_1x_2 + b_2x_3 + b_3x_4)^2 ;$$

and so on generally. An alternative set of conditions is referred to, and is exemplified by the case of the ternary quadric, where the vanishing of $a_1c_1 - b_1b_2$ is substituted for the vanishing of

$$a_1a_2a_3 + 2b_1b_2c_1 - a_1c_1^2 - a_2b_2^2 - a_3b_1^2 ,$$

this latter being equal to

$$\left\{(a_1a_2 - b_1^2)(a_1a_3 - b_2^2) - (a_1c_1 - b_1b_2)^2\right\} \div a_1 .$$

SYLVESTER, J. J. (1853).

[On the conditions necessary and sufficient to be satisfied in order that a function of any number of variables may be linearly equivalent to a function of any less number of variables. *Philos. Magazine*, v. pp. 119–126: or *Collected Math. Papers*, i. pp. 587–594.]

The title at once suggests a connection with Hesse's converse theorem of 1851 (March): the investigation, however, proceeds on totally different lines, and only concerns us because of the doubt thrown on the truth of the said theorem by Sylvester's assertion that the Hessian "is really foreign to the nature" of the question under discussion.

SPOTTISWOODE, W. (1853, August).

[Elementary theorems relating to determinants: second edition, rewritten and much enlarged by the author. *Crelle's Journal*, li. pp. 209–271, 328–381.]

The latter portion (pp. 343–350) of his chapter (§ ix.) "On Functional

Determinants" Spottiswoode devotes to one or two theorems connected with Hessians, and mainly to Hesse's theorem of the year 1849 (December): his proof of the latter, however, is not an improvement on Jacobi's. One of his notations for the Hessian, U being the function and $x, y, z, \ldots$ the variables, is

$$\left\{ \begin{matrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \cdots \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \cdots \end{matrix} \right\} \mathrm{U},$$

suggested, doubtless, by Sylvester's general umbral notation. Further, he uses the word "Hessian" in a geometrical sense, namely, for the locus represented by

$$\mathrm{H(U)} = 0.$$

Brioschi, Fr. (1854).

[Solutions des questions 285, 286. *Nouv. Annales de Math.*, xiii. pp. 402–409.]

The theorems which had been set for proof were geometrical theorems due to Hesse, and as a foundation on which to base them and others Brioschi establishes a general result regarding Hessians. This with only slight departures from the original may be formally enunciated as follows: *If— u be a homogeneous integral function of the* m^{th} *degree in* $r+1$ *variables* $x_0, x_1, x_2, \ldots, x_r$, *the Hessian of which with respect to those variables is* H_{r+1} *and with respect to the variables* $x_1, x_2, \ldots, x_r$ *is* H_r, *then the determinant which is the result of bordering* H_r *by prefixing*

$$0, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_r}$$

as a first row and as a first column is equal to

$$(-1)^{r+1}\frac{m}{m-1}u\mathrm{H}_r + \frac{x_0^2}{(m-1)^2}\mathrm{H}_{r+1}.$$

The bordered Hessian, B say, being equal to

$$\frac{1}{m-1}\left| \begin{matrix} \cdot & u_1 & u_2 & \cdots & u_r \\ (m-1)u_1 & u_{11} & u_{12} & \cdots & u_{1r} \\ (m-1)u_2 & u_{21} & u_{22} & \cdots & u_{2r} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ (m-1)u_r & u_{r1} & u_{r2} & \cdots & u_{rr} \end{matrix} \right|$$

and Euler's theorem regarding the differentiating of homogeneous functions giving

$$\left.\begin{aligned} mu &= x_0u_0 + x_1u_1 + \cdots + x_ru_r, \\ (m-1)u_0 &= x_0u_{00} + x_1u_{01} + \cdots + x_ru_{0r}, \\ (m-1)u_1 &= x_0u_{10} + x_1u_{11} + \cdots + x_ru_{1r}, \\ &\cdots\cdots\cdots\cdots \\ (m-1)u_r &= x_0u_{r0} + x_1u_{r1} + \cdots + x_ru_{rr}, \end{aligned}\right\}$$

there is obtained

$$B = \frac{1}{m-1}\begin{vmatrix} x_0u_0 - mu & u_1 & u_2 & \cdots & u_r \\ x_0u_{10} & u_{11} & u_{12} & \cdots & u_{1r} \\ x_0u_{20} & u_{21} & u_{22} & \cdots & u_{2r} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x_0u_{r0} & u_{r1} & u_{r2} & \cdots & u_{rr} \end{vmatrix}$$

$$= \frac{-m}{m-1}u\begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1r} \\ u_{21} & u_{22} & \cdots & u_{2r} \\ \cdot & \cdot & \cdots & \cdot \\ u_{r1} & u_{r2} & \cdots & u_{rr} \end{vmatrix} + \frac{x_0}{m-1}\begin{vmatrix} u_0 & u_1 & \cdots & u_r \\ u_{10} & u_{11} & \cdots & u_{1r} \\ u_{20} & u_{21} & \cdots & u_{2r} \\ \cdot & \cdot & \cdots & \cdot \\ u_{r0} & u_{r1} & \cdots & u_{rr} \end{vmatrix}.$$

By similar treatment, however, the second determinant on the right of this

$$= \frac{1}{m-1}\begin{vmatrix} (m-1)u_0 & (m-1)u_1 & \cdots & (m-1)u_r \\ u_{10} & u_{11} & \cdots & u_{1r} \\ u_{20} & u_{21} & \cdots & u_{2r} \\ \cdot & \cdot & \cdots & \cdot \\ u_{r0} & u_{r1} & \cdots & u_{rr} \end{vmatrix},$$

$$= \frac{x_0}{m-1}\begin{vmatrix} u_{00} & u_{01} & \cdots & u_{0r} \\ u_{10} & u_{11} & \cdots & u_{1r} \\ \cdot & \cdot & \cdots & \cdot \\ u_{r0} & u_{r1} & \cdots & u_{rr} \end{vmatrix},$$

so that finally we have

$$B = -\frac{m}{m-1}uH_r + \frac{x_0^2}{(m-1)^2}H_{r+1}$$

as desired.

As a corollary it is noted that if H_{r+1} vanishes identically, then

$$(m-1)B + muH_r = 0,$$

i.e.

$$\begin{vmatrix} mu & u_1 & \cdots & u_r \\ (m-1)u_1 & u_{11} & \cdots & u_{1r} \\ (m-1)u_2 & u_{21} & \cdots & u_{2r} \\ \cdot & \cdot & \cdots & \cdot \\ (m-1)u_r & u_{r1} & \cdots & u_{rr} \end{vmatrix} = 0$$

"et l'équation

$$u(x_1, x_2, \ldots, x_r) = 0$$

est elle-même homogène"—a sort of converse of Euler's theorem above referred to.

BRIOSCHI, FR. (1854, March).

[LA TEORICA DEI DETERMINANTI, E LE SUE PRINCIPALI APPLICAZIONI; del Dr Francesco Brioschi: viii+116 pp.; Pavia. Translation into French, by Combescure, ix+216 pp.; Paris, 1856. Translation into German, by Schellbach: vii+102 pp.; Berlin, 1856.]

The last section (§ 11, pp. 106–116) of Brioschi's text-book is headed "Del determinante di Hesse." Opening with the definition of "l'Hessiano," it gives a clear and orderly exposition of a goodly number of the main theorems up till then discovered, with geometrical applications.*

Separated altogether, however, from these is a demonstration (pp. 20, 21) which strictly belongs to this section. Recognising that Hesse's expression (1847, August) for the product of u and its Hessian Δ is in reality obtained by eliminating x_1, x_2, x_3 from four equations, Brioschi performs this elimination openly, with the result:

$$0 = \begin{vmatrix} \frac{m}{m-1}u & u_1 & u_2 & \ldots & u_n \\ u_1 & u_{11} & u_{12} & \ldots & u_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_n & u_{n1} & u_{n2} & \ldots & u_{nn} \end{vmatrix},$$

$$= \frac{m}{m-1}u\Delta + \begin{vmatrix} \cdot & u_1 & u_2 & \ldots & u_n \\ u_1 & u_{11} & u_{12} & \ldots & u_{1n} \\ u_2 & u_{21} & u_{22} & \ldots & u_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_n & u_{n1} & u_{n2} & \ldots & u_{nn} \end{vmatrix}.$$

BELLAVITIS, G. (1857, June).

[Sposizione elementare della teorica dei determinanti. *Veneto, Memorie . . . Istituto,* vii. pp. 67–144.]

Bellavitis (§§ 79, 80) denotes "l'*Hessiano* delle funzione ϕ" by

$$| D_xD_x \quad D_yD_y \quad \ldots\ldots | \phi,$$

* The reason given for the deduction $U_{rs} = Nx_rx_s$, which occurs in his presentation of Jacobi's proof of the year 1849, is disappointing.

calling it also "il determinante delle derivate-seconde." He confines himself to three of the main theorems. Hesse's theorem of 1851 (March) he amplifies, his enunciation being:—*If* u, *a homogeneous integral function of the variables* $x_1, x_2, \ldots, x_n$, *be transformed by means of the substitution*

$$x_k = a_1^{(k)}y_1 + a_2^{(k)}y_2 + \cdots + a_n^k y_n$$

into v, *and one of the new variables, say* y_1, *be absent from* v, *then* (1) *the Hessian* H *of* u *must vanish identically,* (2) *the cofactor of the elements of any row of* H *must be proportional to the coefficients of* y_1 *in the substitution,* (3) *the product of the first differential-quotients of* u *by the said column of coefficients is equal to* 0. The third of these Bellavitis reaches very easily, because generally we have

$$\frac{\partial v}{\partial y_r} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_r} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_r} + \cdots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_r},$$

and therefore when $r = 1$

$$0 = \frac{\partial u}{\partial x_1}a_1^{(1)} + \frac{\partial u}{\partial x_2}a_1^{(2)} + \cdots + \frac{\partial u}{\partial x_n}a_1^{(n)}$$

or

$$0 = u_1 a_1^{(1)} + u_2 a_1^{(2)} + \cdots + u_n a_1^n. \qquad (\pi)$$

As regards the second he notes that on account of the vanishing of H we have in the first place

$$U_{11} : U_{12} : \cdots : U_{1n} = U_{s1} : U_{s2} : \cdots : U_{sn},$$

and in the second place * the set of equations

$$\left.\begin{array}{l} u_1U_{11} + u_2U_{12} + \cdots + u_nU_{1n} = 0, \\ u_1U_{21} + u_2U_{22} + \cdots + u_nU_{2n} = 0, \\ \cdots\cdots\cdots\cdots\cdots \end{array}\right\}$$

from which there is the evident deduction that the said set reduces to a single equation: the identity of this equation with (π) is then assumed.

Hesse's converse theorem he treats with a wise caution, deducing as before from the vanishing of H the existence of a single equation of the form

$$\alpha\frac{\partial u}{\partial x_1} + \beta\frac{\partial u}{\partial x_2} + \cdots = 0,$$

but then adding, "ma rimane da dimostrare che le $\alpha, \beta, \ldots$ sieno quantità costanti."

Lastly, he notes that if there be *two* such equations with constant coefficients, the function is transformable into one with two fewer variables, and all the primary minors of H vanish.

* See (α) in Jacobi's proof of 1849.

BALTZER, R. (1857).

[THEORIE UND ANWENDUNGEN DER DETERMINANTEN, mit Beziehung auf die Originalquellen, dargestellt von Dr Richard Baltzer; vi+129 pp.; Leipzig, 1857. French translation by J. Houel; xii+235 pp.; Paris, 1861.]

In Hesse's converse theorem of 1851 (March) Baltzer (§ 13, 3) wisely substitutes for $\Delta=0$ the condition

$$c_1u_1+c_2u_2+\cdots+c_nu_n = 0$$

(which by a property of Jacobians implies $\Delta=0$), his proof being that the substitution

$$x_k = b_{k1}y_1+b_{k2}y_2+\cdots+b_{k,n-1}y_{n-1}+c_ky_n$$

will then give $\partial u/\partial y_n=0$, for

$$\begin{aligned}\frac{\partial u}{\partial y_n} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial y_n}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial y_n}+\cdots+\frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial y_n}\\ &= u_1c_1+u_2c_2+\cdots+u_nc_n.\end{aligned}$$

In the second place, from the same $n+1$ equations, namely,

$$\left.\begin{aligned}-(m-1)\frac{mu}{m-1}+u_1x_1+\cdots+u_nx_n &= 0\\ -(m-1)u_1+u_{11}x_1+\cdots+u_{n1}x_n &= 0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\\ -(m-1)u_n+u_{1n}x_1+\cdots+u_{nn}x_n &= 0\end{aligned}\right\},$$

he obtains (§ 14, 4)

$$\left.\begin{aligned}-(m-1):x_1:x_2:\ldots:x_n &= \Delta : V_1 : V_2 : \ldots : V_n,\\ &= V_1 : V_{11} : V_{21} : \ldots : V_{n1},\\ &= \cdots\cdots\cdots\cdots\\ &= V_n : V_{1n} : V_{2n} : \ldots : V_{nn}\end{aligned}\right\},$$

where the V's are the cofactors of the corresponding u's in the resultant of the set of equations. The first and $(r+1)^{\text{th}}$ lines imply respectively

$$\begin{aligned}-(m-1):x_r &= \Delta : V_r,\\ \text{and}\qquad -(m-1):x_s &= V_r : V_{sr},\end{aligned}$$

the former of which gives

$$V_r = -\frac{x_r}{m-1}\Delta, \tag{α}$$

and the two together

$$V_{sr} = \frac{x_sx_r}{(m-1)^2}\Delta. \tag{β}$$

The result (α) is essentially the same as the first result reached in Jacobi's proof of 1849 (December)—a proof which Baltzer restates (§ 14, 7, 8) without noting the fact; and (β) is essentially the same as the second of the two results given in Hesse's paper of 1847 (August), it being noted, however, that the case of this where $r=s$ had been established by Hesse in 1844 (see *Crelle's Journal*, xxviii. p. 103, lines 1 and 2).

Hesse, O. (1858).

[Zur Theorie der ganzen homogenen Functionen. *Crelle's Journal*, lvi. pp. 263–269: or *Werke*, pp. 481–488.]

The first part of Hesse's attempted proof of his converse theorem of 1851 (March) was to show that the vanishing of "the determinant of u" led to the establishment of a linear relation connecting the first differential coefficients of u. In this there was an oversight, which Brioschi repeated, but which Bellavitis and Baltzer, from the course followed by them, must have been conscious of. Accordingly, Hesse now returns to the subject, the one object of his six-page paper being "diesen Lehrsatz strenger zu begründen." He first clears the ground a little by setting aside the case where one, and therefore all, of the primary minors of Δ vanish, merely stating that a linear substitution is then possible which will transform u into a function with *two* variables less than before. He then sets himself to supply the want which Bellavitis had drawn attention to, the result being a lengthy (pp. 265–268) and still unconvincing argument.

LIST OF AUTHORS

whose writings are herein dealt with.

(*Issued separately July* 23, 1908.)

XXIX.—On the Origin of the adaxially-curved Leaf-trace in the Filicales. By D. T. Gwynne-Vaughan, M.A., and R. Kidston, LL.D., F.R.S.L. & E.

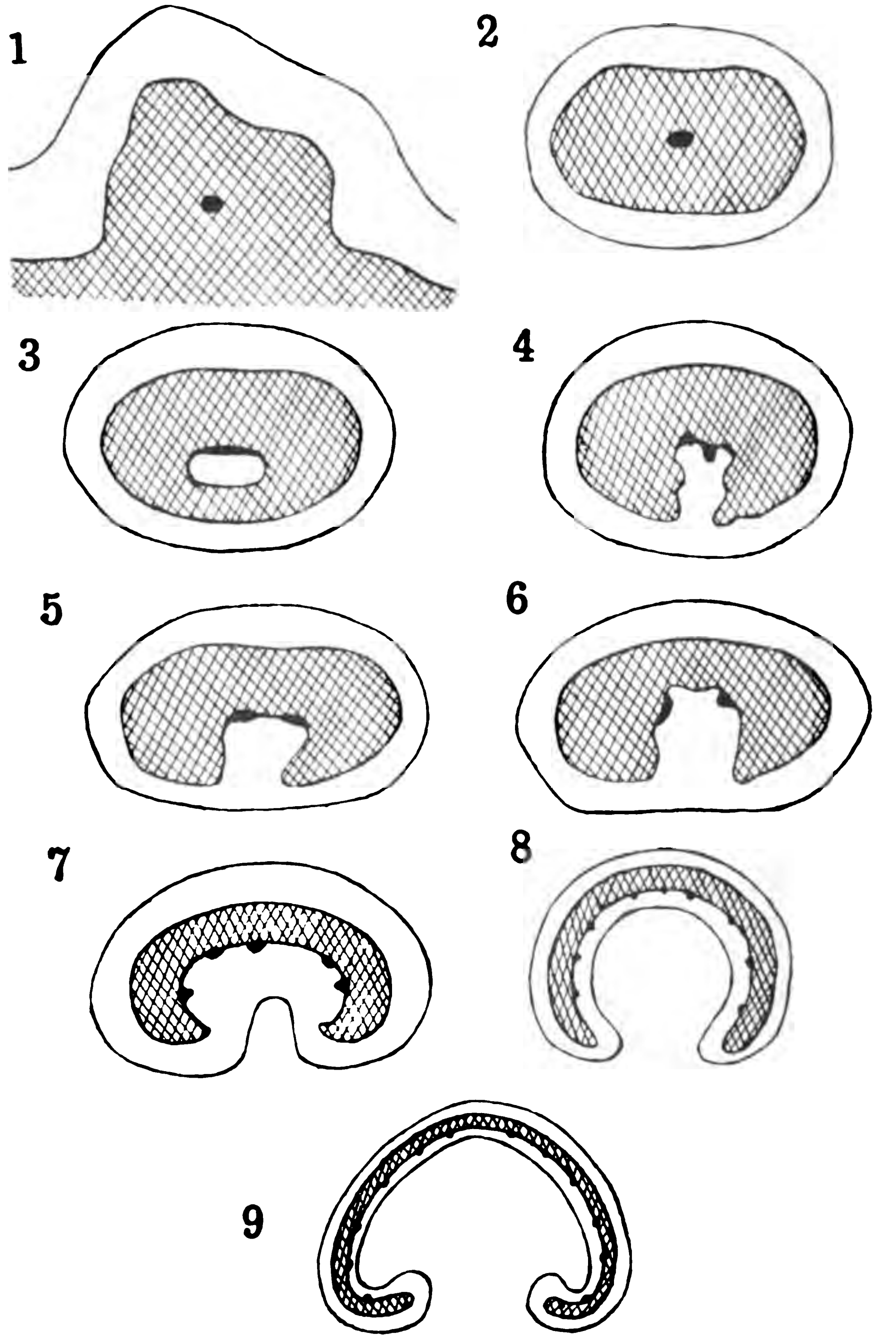
(Read May 4, 1908. MS. received same date.)

WHILE studying the fossil *Osmundaceæ* several interesting points illustrating the origin of the leaf-trace were brought under our notice, which are exhibited with such exceptional clearness in *Thamnopteris Schlechtendalii*, Eichwald, sp.,* that they deserve especial recognition. The full discussion of the subject, however, is reserved until the publication of the next part of our memoir on the fossil *Osmundaceæ*.

The departure of the leaf-trace in *Thamnopteris* is typically protostelic. That is to say, the xylem of the leaf-trace first of all appears as a protuberance on the surface of the xylem of the stem (fig. 1). Later, this separates off without leaving any depression or gap in the stem xylem. Immediately after its departure the leaf-trace xylem is elliptic or oblong in transverse section, with a single mesarch protoxylem group, almost exactly in the middle of the strand (fig. 2). Further out, certain of the centripetal tracheæ situated in front of the protoxylem on the adaxial side of the trace cease to be formed, their place being taken by cells of thin-walled parenchyma (fig. 3). An isolated island of parenchyma is thus produced in the xylem strand just in front of the protoxylem group. This island gradually increases in size in the outer traces, more and more of the tracheæ on the adaxial side being suppressed, until at last it opens out to form an adaxial bay of parenchyma (fig. 4). For some time a few centripetal tracheæ are still to be found in immediate contact with the actual protoxylem elements, but these also eventually die out, and the protoxylems become truly endarch.

As the leaf-trace passes further out through the inner cortex of the stem the xylem becomes tangentially elongated, and the bay of parenchyma progressively wider and more open until the xylem strand assumes the form of a stout crescent with thick, incurved ends. At the same time the protoxylem elements spread out over the concave margin of the bay (fig. 5), and

* Eichwald, *Lethæa Rossica*, vol. i., p. 93, pl. xx., figs. 2 and 5, 1860; Brongniart, *Tableaux des genres des végétaux fossiles*, pp. 35–36, 1849.



Diagrams illustrating the Departure of the Leaf-trace in *Thamnopteris Schlechtendalii*, Eichd., sp.

there divide into two separate groups (fig. 6)—eventually, indeed, into three or more (fig. 7). Usually the protoxylem does not divide until the island has already become a bay, but it may do so before, while the island is still closed in by centripetal xylem. Referring in general to the ontogeny of the leaf-trace of the Ferns, it may be shown that any change in form is first of all initiated by the xylem strand and then followed by the outline of the trace as a whole. It is interesting to note that, while the changes described above are taking place in the xylem of the leaf-trace of *Thamnopteris*, the outline of the whole trace still remains elliptic or oblong. It is not until the sclerotic outer cortex of the stem is reached that a slight depression appears on the adaxial side of the leaf-trace, which then rapidly becomes reniform in transverse section (fig. 7). Once the leaf-trace has actually entered the sclerotic cortex, the xylem strand becomes much narrower and more extended, and the protoxylem groups increase considerably in number. It also gradually becomes more and more curved, first taking the form of a low, rounded arch (fig. 7), the ends of which then slowly approach each other (fig. 8), until at last the characteristic horseshoe-shaped strand with incurved ends is produced (fig. 9). This curvature is followed by the outline of the leaf-trace as a whole—always, however, lagging a little behind that of the xylem.

It is, of course, to be understood that the changes described above are those that take place in the individual leaf-traces of the mature leaves as they pass through the cortex of the stem to the free petioles. Nevertheless it is believed that these stages may be taken as indicating the changes undergone in the ontogeny and phylogeny of the leaf-trace, and that, therefore, they offer useful and reliable suggestions as to the origin and derivation of the adaxially-curved leaf-trace so representative of the Filicales.

The essential points of this theory of the direct or normally oriented horseshoe leaf-trace are as follows. First of all, the xylem strand is derived from a primitively solid, more or less rounded mass with a central mesarch protoxylem. Secondly, this mass became concave on the adaxial side by the substitution of parenchyma for the centripetal elements of the xylem, the protoxylem thereby becoming truly endarch. This view is advanced as an alternative to the suggestion put forward by Mr Tansley,* who derives the directly oriented horseshoe from a Zygopterid type of leaf-trace by the complete reduction of the abaxial arms and the pinnæ that they supplied. Without disputing the possibility that this method could have produced a

* Tansley, A. G., "Lectures on the Evolution of the Filicinean Vascular System," *New Phytologist*, vol. vi., p. 64, 1907.

leaf-trace of the type in question, it seems to us very improbable that it should have anything to do with the case of the *Osmundaceæ*.

In conclusion, we wish to acknowledge our indebtedness to Mons. Th. Tschernyschew, the Director of the Geological Survey of Russia, and Mons. M. Zalessky, for the material of *Thamnopteris* on which our observations were made.

(*Issued separately August* 12, 1908.)

XXX.—On the Theory of the Leaking Microbarograph; and on some Observations made with a Triad of Dines-Shaw Instruments. By Professor Chrystal.

(MS. received June 8, 1908. Read June 15, 1908.)

THE minor fluctuations of the atmospheric pressure are now engaging the close attention of meteorologists, and my own attention has been drawn to them on account of their connection with the oscillations of lake-surfaces generally known since Forel's investigations under the name of seiches.

As these fluctuations often do not exceed a millimetre or two of water, they are not shown by the ordinary self-registering apparatus. In order to record them we may use a specially sensitive form of barograph, such as the Richard statoscope, which is delicate enough to show these small fluctuations, and yet can be brought back to a momentary zero whenever the indicator threatens to go off the scale or beyond the limits of safety. There are, however, two objections to this method of measurement. In the first place, the instrument must either be watched or else provided with a self-acting arrangement for altering the zero. But there is a much more serious and radical objection, which will at once be understood by anyone who has studied a tidal curve showing what are called "secondary tidal oscillations," *i.e.* certain oscillations of much smaller range than the tidal oscillation proper, and of much shorter period, say fifteen to twenty minutes. Fig. 1 is a reproduction of such a curve. At first sight it

FIG I

would appear that the secondary oscillations occur only near high and low water. As a matter of fact, however, they occur throughout the whole day, and are merely masked in the mareogram at the zero of the tide, because the rapid rise of the curve due to the proper tide wipes out the turning-points and even the inflexions due to the minor oscillation. It is necessary, therefore, to devise an instrument which will neglect the barometric variations of larger range and longer period, and record only the minor fluctuations. This is effected by a method which is, in a sense, the converse of that which I described in my memoir on the Investigation of the Seiches of Loch Earn* for damping out the seiches of higher nodality in a limnogram.

This method consists essentially in measuring the difference between the atmospheric pressure and the pressure in a vessel which communicates with the atmosphere by means of a small leak, say through a capillary tube of sufficient length or fineness of bore. If the bore were infinitely fine, the instrument would simply register the atmospheric pressure with all its variations; if the bore were very wide, it would register nothing at all; and by properly adjusting the tube we can arrange so that it only registers the fluctuations of short duration, which pass away before the small flow through the capillary has had time to establish equality of pressure within and without.

THEORY OF THE DINES-SHAW MICROBAROGRAPH.

The instrument, which is represented diagrammatically in fig. 2, consists essentially† of an air-chamber V of considerable capacity (about ·3 cub. ft. = 9440 c.c.), carefully packed to reduce variation of temperature as much as possible. This communicates through a narrow pipe PP with the small air-space under an inverted cylindrical cup, which floats in mercury contained in a cylindrical cup AA. The vertical motions of the cup CC caused by the variations of the pressure outside and inside V are registered by means of a magnifying lever LL, into the details of the construction of which it is unnecessary to enter here.

The air-chamber V communicates with the outer air by means of a capillary tube KK, through which the air leaks slowly outwards or inwards according as the pressure outside is less or greater than the pressure inside.

We shall suppose that the air-space under the floating cup CC is practically negligible in comparison with V, and also that the temperature is constant.

We shall also suppose that the leak is so slow that the cup may be supposed to adjust itself instantaneously to the pressure conditions at any moment; and we shall also neglect the inertia and friction of the movable parts of the apparatus.

* *Trans. Roy. Soc. Edin.*, vol. xliv. p. 373 (1906).
† See *Quart. Journ. Roy. Met. Soc.*, vol. xxxi. p. 39 (1905).

Under these assumptions, the pressure, p, of the inside air is determined solely by its density (*i.e.* by the total mass of the air inside).

Let ϖ denote the pressure of the air outside, measured, like p, in centimetres of mercury; $2a$ the internal, and $2b$ the external, diameters of CC; $2c$ the diameter of AA; u and v the heights above the bottom of AA of the levels of the mercury outside and inside CC; and w the height of any fixed point in CC′, say a point in the bottom inside.

All the lengths are supposed to be measured in centimetres.

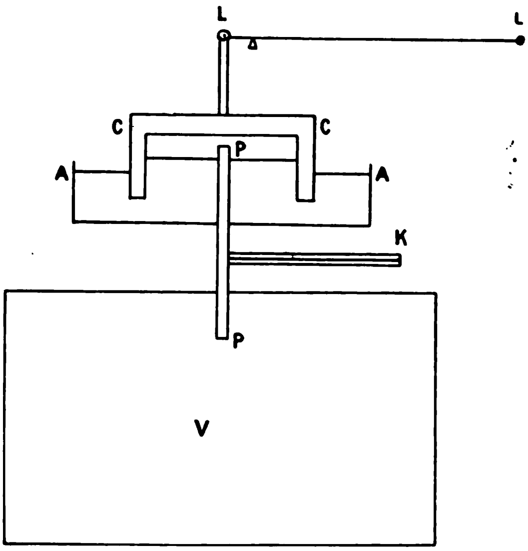


FIG. 2

Then, if we express the condition that the pressures above and below the mercury surface inside CC are equal, we get the equation

$$\varpi - p = v - u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The condition that the whole volume of the mercury remains constant is

$$(c^2 - b^2)u + a^2 v + (b^2 - a^2)w = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The condition for the equilibrium of the floating cup gives

$$b^2 u - a^2 v - (b^2 - a^2)w = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

From (2) and (3) we see that

$$u = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4);$$

$$a^2 v + (b^2 - a^2)w = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (5).$$

The external level of the mercury is therefore unaltered by variations of pressure.

Suppose we start from the condition to which the instrument would settle down after a long period of undisturbed atmospheric pressure ϖ_0. We should then have $p=\varpi_0$, and therefore by (1) $v=u=v_0$ say, and we may now write (1) in the form

$$\varpi - p = v - v_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Also, if w_0 be the value of w in the initial case supposed, we may write (5) in the form

$$a^2v + (b^2 - a^2)w = a^2v_0 + (b^2 - a^2)w_0 ;$$

or

$$v - v_0 = -(b^2/a^2 - 1)(w - w_0) \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Hence, finally,

$$\varpi - p = -(b^2/a^2 - 1)(w - w_0) \quad . \quad . \quad . \quad . \quad . \quad (8);$$

that is, if z be the divergence in centimetres of the index of the instrument from the position of absolute equilibrium for the temperature supposed, and k a constant factor depending on the multiplying power of the registering lever or levers, then

$$\varpi - p = (b^2/a^2 - 1)z/k \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

In other words, the reading is proportional to the difference between the pressures outside and inside the floating cup. We have supposed the excursions of the index so small that the circular motion of the recording pen may be taken without sensible error to be proportional to the vertical motion of the floating cup.

On the Variation of Pressure in a Closed Chamber which Leaks into the Air through a Capillary Tube.*

Let V be the volume of the air-chamber.

v the volume of the air in the chamber at time t when reduced to the pressure $\frac{1}{2}(\varpi + p)$.

R the radius, and L the length, of the capillary tube.

p the pressure of the inside air at time t.

ϖ the pressure of the outside air at time t.

ϖ_0 the common initial pressure outside and inside the chamber, which we shall usually take to be the standard barometric pressure, viz. $981 \times 76 \times 13{\cdot}55 = 1{\cdot}014 \times 10^6$ dynes per sq. cm.

η the coefficient of viscosity of air at 15° C., say $\eta = {\cdot}00018$.

* This part of the theory will apply equally well to any other form of microbarograph in which the reading is proportional to the difference between the atmospheric pressure and the pressure in an air-chamber provided with a leak: for example, to the Richard statoscope, when its air-chamber communicates with the outer air through a capillary tube.

Then, by Poiseuille's Law,

$$\frac{dv}{dt}=\frac{\pi}{8\eta}(\varpi-p)\frac{R^4}{L}=\lambda(\varpi-p) \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where

$$\lambda=\pi R^4/8\eta L .$$

We suppose the temperature kept constant. Hence, since the leak is very slow, we have

$$(p+dp)/p=\left(V+\frac{p+\varpi}{2p}dv\right)/V ;$$

that is,

$$\frac{dv}{dt}=\frac{2V}{p+\varpi}\frac{dp}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Hence the equation which determines the variation of the pressure is

$$\frac{2V}{p+\varpi}\frac{dp}{dt}=\lambda(\varpi-p) ;$$

that is,

$$\frac{dp}{dt}=\frac{\lambda(p+\varpi)}{2V}(\varpi-p) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Now, in the case of the microbarograph, $\varpi-p$ is always small. Hence we may for our purposes replace p and ϖ in the factor $p+\varpi$ by $2\varpi_0$. The equation (3) then becomes

$$\frac{dp}{dt}=\frac{\lambda\varpi_0}{V}(\varpi-p) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3') ;^*$$

or, if

$$1/\mu=\tau=V/\lambda\varpi_0=8\eta LV/\pi\varpi_0 R^4 \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$\frac{dp}{dt}+\mu p=\mu\varpi . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

the solution of which is

$$p=(A+\mu\int_0^t dt\varpi e^{\mu t})e^{-\mu t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where A is an arbitrary constant.

The following special cases are of practical importance:—

Case 1. Suppose air blown into the chamber until the pressure becomes p_0, and this is allowed to leak out again until the barometric pressure falls to ϖ_0. Then, since $p=p_0$ when $t=0$, and $\varpi=\varpi_0$ throughout, we have

$$\begin{aligned} p &= (p_0+\mu\varpi_0\int_0^t dte^{\mu t})e^{-\mu t} \\ &= \{p_0+\varpi_0(e^{\mu t}-1)\}e^{-\mu t} \\ &= \varpi_0+(p_0-\varpi_0)e^{-\mu t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7). \end{aligned}$$

* The exact equation (3) can be reduced to the form $\frac{dy}{dt}+y^2=f(t)$, and could be used instead of the approximate one ; but the calculations are much more complicated.

Hence

$$(p-\varpi_0)/(p_0-\varpi_0)=e^{-\mu t} \quad . \quad . \quad . \quad . \quad . \quad (8).$$

In one of Shaw's tests * it was found that for $t=0$, $k(p_0-\varpi_0)=\frac{3}{2}\times 29$; and for $t=3600$, $k(p-\varpi_0)=\frac{3}{2}\times 9$. Hence

$$e^{3600\mu}=29/9:$$

whence

$$\mu=(\log 29-\log 9)/3600 \log e$$
$$=\cdot 000325;$$
$$\tau=1/\mu=3080^{\text{sec.}}$$

For rough purposes we shall hereafter take $\tau=3000$.

Case 2. Suppose that the air-chamber and the outer air have come to a common pressure ϖ_0, and that the outer air then suddenly rises to the pressure $\varpi_0+\delta\varpi_0$. Then we have

$$p=\{\varpi_0+\mu(\varpi_0+\delta\varpi_0)\int_0^t dte^{\mu t}\}e^{-\mu t}$$
$$=\{\varpi_0+(\varpi_0+\delta\varpi_0)(e^{\mu t}-1)\}e^{-\mu t}$$
$$=\varpi_0+\delta\varpi_0-\delta\varpi_0 e^{-\mu t}.$$

Hence, if y denote $\varpi_0+\delta\varpi_0-p$, to which the microbarograph reading is proportional, we have

$$y=\delta\varpi_0 e^{-\mu t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Case 3. Suppose the pressure in the outer air to begin to rise with a uniform time gradient γ; so that $\varpi=\varpi_0+\gamma t$. We have

$$pe^{\mu t}=\varpi_0+\mu\int_0^t dt(\varpi_0+\gamma t)e^{\mu t}$$
$$=\varpi_0 e^{\mu t}+\gamma\{te^{\mu t}-(e^{\mu t}-1)/\mu\}.$$

Hence

$$p=\varpi_0+\gamma\{(t-1/\mu)+e^{-\mu t}/\mu\} \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Therefore, if we put $\tau=1/\mu$, and $y=\varpi-p$, we get

$$y=\gamma\tau(1-e^{-t/\tau}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11);$$

$$\frac{dy}{dt}=\gamma e^{-t/\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Hence we have the following figure (3),† where the graphs of $y=\varpi-p$ and $y_1=\varpi-\varpi_0$ are OC and OD, the latter being the tangent to the former at the origin, as might be expected.

Thus, at the start, the microbarograph gives the correct gradient; but after a considerable time, if the outside gradient remains constant, the

* *Quart. Journ. Roy. Met. Soc.*, xxxi., p. 43 (1905).

† This figure and the corresponding ones which follow are not drawn to scale, but are diagrammatic merely.

index of the microbarograph comes to rest at a distance $=3000\gamma$ above its zero.

Case 3. As an example of an elevation of pressure with a rounded maximum, take

$$\varpi = \varpi_0 + a(1 - \cos nt),$$

with the condition $a=0$ if $t > 2\pi/n$.

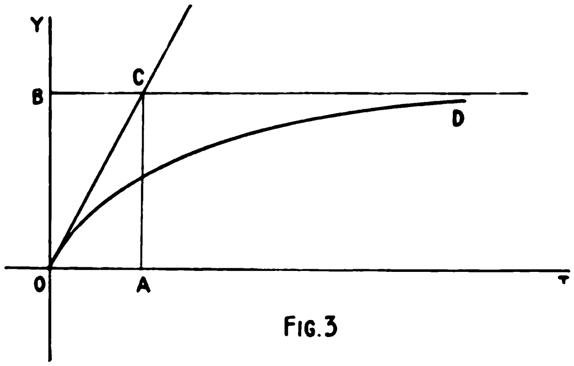


FIG. 3

Then, if $t < 2\pi/n$,

$$pe^{\mu t} = \varpi_0 + \mu \int_0^t dt(\varpi_0 + a - a\cos nt)e^{\mu t}$$

$$= (\varpi_0 + a)e^{\mu t} - \frac{n^2 a}{\mu^2+n^2} - \frac{\mu a}{\mu^2+n^2}\{n \sin nt + \mu \cos nt\}e^{\mu t}.$$

Hence

$$p = \varpi_0 + a - \frac{n^2 a}{\mu^2+n^2}e^{-\mu t} - \frac{\mu a}{\mu^2+n^2}(n \sin nt + \mu \cos nt);$$

$$y = \varpi - p = \frac{n^2 a}{\mu^2+n^2}e^{-\mu t} + \frac{na}{\mu^2+n^2}(\mu \sin nt - n \cos nt);$$

$$y = \frac{na}{\mu^2+n^2}\{\mu \sin nt - n \cos nt + ne^{-\mu t}\}.$$

That is, if $\tan \chi = \mu/n$,

$$y = a \cos \chi\{-\cos(nt+\chi) + \cos \chi\, e^{-\mu t}\} \quad . \quad . \quad . \quad (13).$$

Also, if $t > 2\pi/n$, and $T = 2\pi/n$,

$$pe^{\mu t} = \varpi_0 e^{\mu t} + a(e^{\mu T} - 1) - \frac{\mu a}{\mu^2+n^2}\{\mu e^{\mu T} - \mu\}$$

$$= \varpi_0 e^{\mu t} + \frac{n^2 a}{\mu^2+n^2}(e^{\mu T} - 1);$$

$$p = \varpi_0 + \frac{n^2 a}{\mu^2+n^2}(e^{\mu T} - 1)e^{-\mu t}.$$

Hence

$$y_1 = \varpi - p = -\frac{n^2 a}{\mu^2 + n^2}(e^{\mu T} - 1)e^{-\mu t};$$

or

$$y_1 = -a\cos^2\chi\,(e^{\mu T} - 1)e^{-\mu t} \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Since y_1 is negative, and has no turning-point, y must have (in addition to the quasi-minimum for $t = 0$) both a maximum and a minimum in the interval $0 < t \leqslant T$.

We have, in fact, since $\mu = n\tan\chi$,

$$\frac{dy}{dt} = na\cos\chi\{\sin(nt + \chi) - \sin\chi\, e^{-\mu t}\} \quad . \quad . \quad . \quad (15);$$

and

$$\frac{dy_1}{dt} = na\sin\chi\cos\chi(e^{\mu T} - 1)e^{-\mu t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

By drawing the graphs of $\sin(nt + \chi)$ and $\sin\chi e^{-\mu t}$, it is easy to see that the maxima and minima points above mentioned correspond to values of t given by

$$nt + \chi = \pi - \xi_1 \quad \text{and} \quad nt + \chi = 2\pi + \xi_2,$$

where ξ_1 and ξ_2 are two small positive quantities.

Fig. 4 (not drawn to scale) shows the general character of the graphs of $\varpi - \varpi_0$ and $\varpi - p$, the former being represented by OABC, the latter by ODEF.

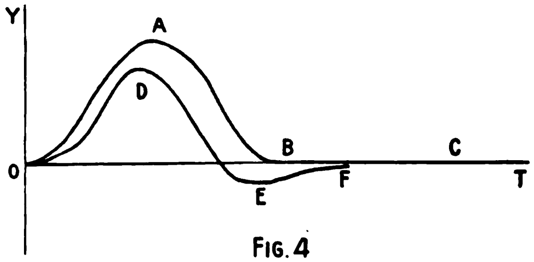


FIG. 4

It is important to notice that a maximum on the microbarograph *precedes* the maximum in the atmospheric pressure which causes it; and, as we shall see presently from a numerical example, this acceleration may be considerable.

A minimum following a maximum on the microbarogram does not necessarily involve a depression following the elevation in the atmospheric pressure.

The equations for calculating ξ_1 and ξ_2, as above defined, are:—

$$f(\xi_1) \equiv \sin\xi_1 - \sin\chi\, e^{-\tan\chi(\pi - \chi - \xi_1)} = 0 \quad . \quad . \quad . \quad . \quad . \quad (17);$$

$$g(\xi_2) \equiv \sin\xi_2 - \sin\chi\, e^{-\tan\chi(2\pi - \chi + \xi_2)} = 0 \quad . \quad . \quad . \quad . \quad . \quad (18).$$

If χ be small, and we may neglect ξ_1^5 or χ^5, then equation (17) may be replaced by

$$\xi_1 = \chi - \pi\chi^2 + \left\{\left(\frac{\pi^2}{2}+\frac{5}{6}\right)\chi^3 + \chi^2\xi_1 + \frac{1}{6}\xi_1^3\right\} - \left\{\left(\frac{\pi^3}{6}+\frac{7\pi}{6}\right)\chi^4 + \pi\chi^3\xi_1\right\},$$

which gives, to the degree of approximation supposed,

$$\xi_1 = \chi - \pi\chi^2 + \left(\frac{\pi^2}{2}+2\right)\chi^3 - \left(\frac{\pi^3}{6}+\frac{8}{3}\pi\right)\chi^4,$$

i.e. $\xi_1 = \chi - 3{\cdot}14159\chi^2 + 6{\cdot}93480\chi^3 - 13{\cdot}54519\chi^4$ . . . (19).

In like manner, under the same assumption, (18) leads to

$$\xi_2 = \chi - 2\pi\chi^2 + \left\{\left(2\pi^2+\frac{5}{6}\right)\chi^3 - \chi^2\xi_2 + \frac{1}{6}\xi_2^3\right\} - \left\{\left(\frac{4}{3}\pi^3+\frac{7}{3}\pi\right)\chi^4 - 2\pi\chi^3\xi_2\right\};$$

$$\xi_2 = \chi - 2\pi\chi^2 + 2\pi^2\chi^3 - \frac{1}{3}(4\pi^3 - 2\pi)\chi^4;$$

i.e. $\xi_2 = \chi - 6{\cdot}28319\chi^2 + 19{\cdot}73921\chi^3 - 37{\cdot}91397\chi^4$ . . . (20).

Numerical Example.

$$\chi = \cdot 1.$$

This gives in degree measures

$$\chi = 5^\circ\ 43'\ 47''.$$

Hence

$$f(\xi_1) = \sin\xi_1 - \cdot 0733646 e^{\cdot 1000336\xi_1} = 0$$

is the equation for ξ_1.

The equation (19) gives for a first approximation

$$\xi_1 = \cdot 1 - \cdot 0314159 + \cdot 0069348 - \cdot 0013545 = \cdot 07416.$$

Calculating more closely, we get the following table:—

ξ_1	$\sin\xi_1$	$e^{\cdot 100034\xi_1}$	$f(\xi_1)$	Δ
·073	·072935	·074187	− ·001252	991
·074	·073932	·074193	− ·000261	987
·075	·074929	·074203	+ ·000726	...

whence

$$\xi_1 = \cdot 074264.$$

Hence, for the time of the maximum we get

$$nt + \cdot 1 = \pi - \cdot 07426;$$
$$t = \pi/n - \cdot 17426/n$$ . . . . . . (21).

Since $\mu/n = \tan\chi = \cdot 10003$, $1/n = 300{\cdot}03$; and the acceleration of the maximum is $\cdot 17426/n = 52^{\text{sec.}}$ In this case $T = 2\pi/n = 1885^{\text{sec.}} = 31{\cdot}43^{\text{min.}}$, which is a commonly occurring period of duration for minor fluctuations of the barometric pressure.

The maximum value of y is given by

$$y_{\text{max.}} = a \cos \chi \{ \cos \xi_1 + \cos \chi e^{-(\pi - \chi)\tan \chi} e^{\xi_1 \tan \chi} \}.$$

Hence

$$y_{\text{max.}} = 1 \cdot 7280 a,$$

as compared with $2a$ in the outer air. If we leave out of account the subsequent fall of the microbarogram, the ratio of damping is ·864.

For the subsequent minimum we have

$$g(\xi_2) \equiv \sin \xi_2 - \sin \chi e^{-(2\pi - \chi)\tan \chi} e^{-\xi_2 \tan \chi} = 0 .$$

A first approximation, given by (20), is

$$\xi_2 = \cdot 1 - \cdot 062832 + \cdot 019739 - \cdot 003791 = \cdot 05311 .$$

Calculating more closely, by means of the equation

$$g(\xi_2) \equiv \sin \xi_2 - \cdot 053685 e^{-\cdot 100034 \xi_2} = 0 ,$$

we have the following table :—

ξ_2	$\sin \xi_2$	$\cdot 053685 e^{-\cdot 100034 \xi_2}$	$g(\xi_2)$	Δ
·052	·051976	·053405	− ·001429	1004
·053	·052975	·053400	− ·000425	1004
·054	·053974	·053395	+ ·000579	...

Hence

$$\xi_2 = \cdot 053423 .$$

For the time corresponding to the minimum we have

$$nt + \cdot 1 = 2\pi + \cdot 05342 ;$$
$$t = 2\pi / n - \cdot 04658 / n \qquad (22).$$

Hence the minimum is accelerated by

$$\cdot 04658 \times 300 \cdot 03 = 14^{\text{sec.}}$$

The minimum value of y is given by

$$y_{\text{min.}} = a \cos \chi \{ -\cos \xi_2 + \cos \chi e^{-(2\pi - \chi)\tan \chi} e^{-\xi_2 \tan \chi} \}.$$

Hence

$$y_{\text{min.}} = - \cdot 00177 a .$$

It follows that the whole range on the microbarogram is 1·7298a, as compared with $2a$ in the outside air. In practice this minimum would, of course, under ordinary circumstances be negligible.

Case 4. To get an idea of the effect of an elevation followed by a depression of barometric pressure, we may suppose

$$\varpi_1 = \varpi_0 + a \sin nt \qquad \text{for } 0 \leqslant t \leqslant 2\pi/n ;$$
$$\varpi_2 = \varpi_0 \qquad \text{,, } 2\pi/n \leqslant t \leqslant \infty .$$

It is easily found that the corresponding values of $\varpi - p$ are given by

$$y_1 = a \cos \chi \{ \sin (nt + \chi) - \sin \chi \, e^{-\mu t} \} . \quad . \quad . \quad . \quad (23)$$
$$y_2 = -a \sin \chi \cos \chi (1 - e^{-\mu T}) e^{\mu t} \quad . \quad . \quad . \quad . \quad (24) :$$

where

$$\tan \chi = \mu / n .$$

The general form of the corresponding graphs of $\varpi - \varpi_0$ and $\varpi - p$ are given in fig. 5 by OABCD and OEHKLM.

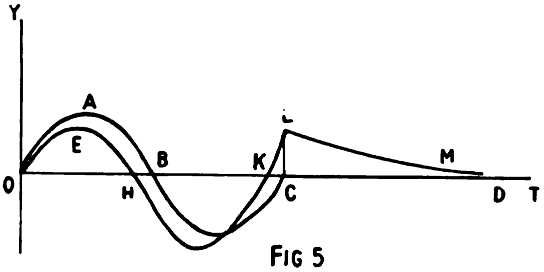


FIG 5

Case 5. As very sharp turning-points often appear on the microbarogram, it will be interesting to consider the effect of a symmetric elevation of barometric pressure whose graph is an isosceles triangle.

We may take $\varpi = \varpi_0 + \gamma t$ when $0 \leqslant t \leqslant \frac{1}{2}\mathrm{T}$; $\varpi = \varpi_0 + \gamma \mathrm{T} - \gamma t$, when $\frac{1}{2}\mathrm{T} \leqslant t \leqslant \mathrm{T}$; $\varpi = \varpi_0$, when $\frac{1}{2}\mathrm{T} < t < +\infty$; so that T is the time of passage of the disturbance.

Then, if $0 \leqslant t \leqslant \frac{1}{2}\mathrm{T}$,

$$p_1 e^{\mu t} = \varpi_0 + \mu \int_0^t dt (\varpi_0 + \gamma t) e^{\mu t} ;$$

whence

$$p_1 = \varpi_0 + \gamma \{ t - 1/\mu + e^{-\mu t}/\mu \} \quad . \quad . \quad . \quad . \quad . \quad (25).$$

If $\frac{1}{2}\mathrm{T} \leqslant t \leqslant \mathrm{T}$,

$$p_2 e^{\mu t} = \varpi_0 + \mu \int_0^{\frac{1}{2}\mathrm{T}} dt (\varpi_0 + \gamma t) e^{\mu t} + \mu \int_{\frac{1}{2}\mathrm{T}}^t dt (\varpi_0 + \gamma \mathrm{T} - \gamma t) e^{\mu t}$$
$$= \varpi_0 e^{\mu t} + \gamma \{ (\mathrm{T} - t + 1/\mu) e^{\mu t} + (1/\mu)(1 - 2e^{-\frac{1}{2}\mu \mathrm{T}}) \} ;$$
$$p_2 = \varpi_0 + \gamma \{ (\mathrm{T} - t + 1/\mu) + (1/\mu)(1 - 2e^{-\frac{1}{2}\mu \mathrm{T}}) e^{-\mu t} \} \quad . \quad . \quad . \quad (26).$$

If $\mathrm{T} \leqslant t \leqslant +\infty$,

$$p_3 e^{\mu t} = \varpi_0 + \mu \int_0^{\frac{1}{2}\mathrm{T}} dt (\varpi_0 + \gamma t) e^{\mu t} + \int_{\frac{1}{2}\mathrm{T}}^{\mathrm{T}} dt (\varpi_0 + \gamma \mathrm{T} - \gamma t) e^{\mu t} + \int_{\mathrm{T}}^t dt \varpi_0 e^{\mu t} ,$$
$$= \varpi_0 e^{\mu t} + (\gamma/\mu)(e^{\frac{1}{2}\mu \mathrm{T}} - 1)^2 ;$$
$$p_3 = \varpi_0 + (\gamma/\mu)(e^{\frac{1}{2}\mu \mathrm{T}} - 1)^2 e^{-\mu t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

From (25), (26), and (27) we get

For $0 < t < \frac{1}{2}T$, $y_1 = \varpi - p_1 = (\gamma/\mu)\{1 - e^{-\mu t}\}$ . . . . . . (28) ;

„ $\frac{1}{2}T < t < T$, $y_2 = \varpi - p_2 = (\gamma/\mu)\{-1 + (2e^{\frac{1}{2}\mu T} - 1)e^{-\mu t}\}$ . . (29) ;

„ $T < t < +\infty$, $y_3 = \varpi - p_3 = -(\gamma/\mu)\{e^{\frac{1}{2}\mu T} - 1\}^2 e^{-\mu t}$ . . . . (30).

Also

$$\frac{dy_1}{dt} = \gamma e^{-\mu t}; \quad \frac{d^2y_1}{dt^2} = -\mu\gamma e^{-\mu t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31);$$

$$\frac{dy_2}{dt} = -\gamma(2e^{\frac{1}{2}\mu T} - 1)e^{-\mu t}; \quad \frac{d^2y_2}{dt^2} = \gamma\mu(2e^{\frac{1}{2}\mu T} - 1)e^{-\mu t} \quad . \quad . \quad (32);$$

$$\frac{dy_3}{dt} = \gamma(e^{\frac{1}{2}\mu T} - 1)^2 e^{-\mu t}; \quad \frac{d^2y_3}{dt^2} = -\gamma\mu(e^{\frac{1}{2}\mu T} - 1)^2 e^{-\mu t} \quad . \quad . \quad (33).$$

The graphs of $\varpi - \varpi_0$ and $\varpi - p$ are OCBK and ODEF in fig. 6.

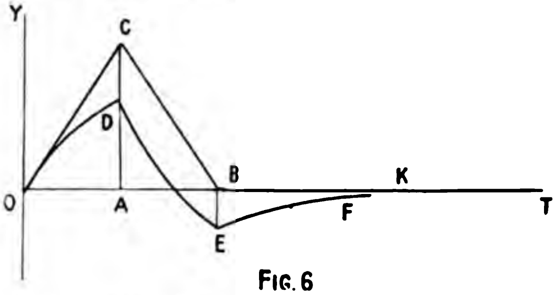


Fig. 6

It should be noticed that here there is no retardation or acceleration of the turning-points on the microbarogram; but there is a pseudo-minimum.

The parts of the microbarogram ODEF are congruent with portions of the exponential curve $y = e^{-\mu x}$; and it should be noticed that the amount of the discontinuity of the gradient is the same at D and E as at C and B respectively.

In the particular case where $T = 30^{\text{min.}} = 1800^{\text{sec.}}$ and $\mu = 1/3000$, we have

$$\tfrac{1}{2}\mu T = \cdot 3, \quad 1 - e^{-\frac{1}{2}\mu T} = \cdot 74082, \quad (1 - e^{-\frac{1}{2}\mu T})^2 = \cdot 06718.$$
$$AC = 900\gamma, \quad AD = 778\gamma, \quad BE = 202\gamma.$$

Hence the range of the microbarogram is 980γ, that is to say, greater than the range of the outside air pressure!

Case 6. Periodic sinusoidal barometric disturbance $\varpi = \varpi_0 + a \sin nt$.

We have merely to drop the term containing $e^{-\mu t}$, which becomes infinitely small after a considerable time has elapsed; and we get

$$y = -a \cos \chi \cos (nt + \chi) \quad . \quad . \quad . \quad . \quad . \quad (34).$$

The acceleration is χ/n, and the ratio of damping $\cos \chi$.

Thus, if $\chi = \cdot 1$, *i.e.* $T = 2\pi/n = 31 \cdot 43^{\text{min.}}$, then $1/n = 300 \cdot 03$; and the acceleration of the maximum (and also of the minimum) is $30^{\text{sec.}}$

Case 7. If the sharp-pointed disturbance of case 5 be repeated periodically, we may determine the nature of the barogram as follows:—

After the variations have settled down to periodicity outside and inside, the pressure at the beginning of each period must be the same as at the end, and there is no acceleration or retardation of the sharp turning-points.

Let us measure the time from the beginning of one of the periods, *i.e.* when the atmospheric pressure is at a minimum. Then we have

$$pe^{\mu t} = \mathrm{A} + \mu \int_0^t dt \varpi e^{\mu t},$$

where A is obviously the inside pressure when $t=0$. Hence, since A must also be the pressure at time T, we must have

$$\mathrm{A}e^{\mu t} = \mathrm{A} + \mu \int_0^{\frac{1}{2}\mathrm{T}} dt(\varpi_0 + \gamma t)e^{\mu t} + \mu \int_{\frac{1}{2}\mathrm{T}}^{\mathrm{T}} dt(\varpi_0 + \gamma \mathrm{T} - \gamma t)e^{\mu t}.$$

This last equation gives

$$(e^{\mu \mathrm{T}} - 1)\mathrm{A} = (e^{\mu \mathrm{T}} - 1)\varpi_0 + (\gamma/\mu)(e^{\frac{1}{2}\mu \mathrm{T}} - 1)^2,$$

that is,

$$\mathrm{A} = \varpi_0 + (\gamma/\mu)(e^{\frac{1}{2}\mu \mathrm{T}} - 1)/(e^{\frac{1}{2}\mu \mathrm{T}} + 1);$$

or

$$\mathrm{A} = \varpi_0 + (\gamma/\mu) \tanh \tfrac{1}{4}\mu \mathrm{T}.$$

Using this value of A, we get for the ascending branch of the microbarogram

$$y_1 = \varpi - p_1 = \frac{\gamma}{\mu}\left\{1 - \left(\frac{2e^{\frac{1}{2}\mu \mathrm{T}}}{e^{\frac{1}{2}\mu \mathrm{T}} + 1}\right)e^{-\mu t}\right\} \quad . \quad . \quad . \quad (35).$$

Similarly, we get for the descending branch

$$y_2 = \varpi - p_2 = \frac{\gamma}{\mu}\left\{\left(\frac{2e^{\mu \mathrm{T}}}{e^{\frac{1}{2}\mu \mathrm{T}} + 1}\right)e^{-\mu t} - 1\right\} \quad . \quad . \quad . \quad (36).$$

From (35) or (36) we get

$$y_{\min.} = -(\gamma/\mu) \tanh \tfrac{1}{4}\mu \mathrm{T} \quad . \quad . \quad . \quad . \quad . \quad (37);$$
$$y_{\max.} = +(\gamma/\mu) \tanh \tfrac{1}{4}\mu \mathrm{T} \quad . \quad . \quad . \quad . \quad (38):$$

and the graphs of $\varpi - \varpi_0$ and $\varpi - p$ are OABCDEF and GHIJKLM in fig. 7. As might have been expected *a priori*, the microbarogram fluctuates about the axis of t, *i.e.* about the line corresponding to $p = \varpi = \varpi_0$.

It should be observed, however, that ϖ_0 is not the mean outside pressure. This last is $\varpi_0 + \frac{1}{4}\gamma \mathrm{T}$: so that the mean pressure outside is higher than the mean pressure inside by $\frac{1}{4}\gamma \mathrm{T}$.

Further,

$$\frac{\text{Inside range}}{\text{Outside range}} = \frac{\tanh \frac{1}{4}\mu T}{\frac{1}{4}\mu T} < 1^* \quad . \quad . \quad . \quad . \quad . \quad (39).$$

In the particular case where $T = 30^{\text{min.}}$ and $\mu = 1/3000$, $\frac{1}{4}\mu T = \cdot 15$; and we find

$$\frac{\text{Inside range}}{\text{Outside range}} = \cdot 9925 :$$

so that the damping is very slight. For $T = 2^{\text{h}}$ the corresponding ratio is ·8951.

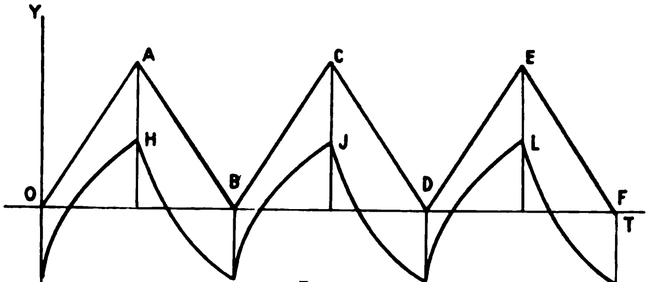


FIG. 7

Case 7. Lastly, let us take the case where the graph of $\varpi - \varpi_0$ is an isosceles trapezium, ODEC in fig. 8. This may be supposed to represent the effect of the passage of a rain-cloud heavily charged with water vapour. By making EC nearly vertical we might represent the effect of a sudden release of pressure by condensation, resulting in a shower.

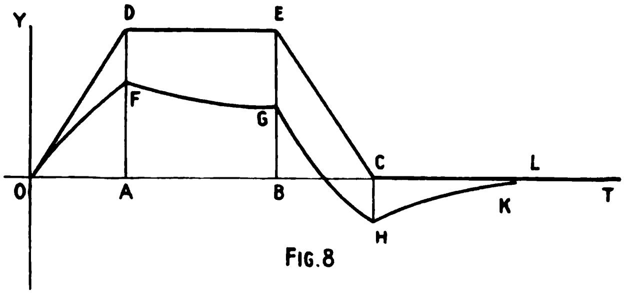


FIG. 8

If the times between O and D, D and E, and E and C be T, U, T respectively, we may represent the atmospheric pressure by

$$\begin{aligned}
\varpi_1 &= \varpi_0 + \gamma t && \text{for } 0 \leqslant t \leqslant T ; \\
\varpi_2 &= \varpi_0 + \gamma T && \text{,, } T \leqslant t \leqslant T + U ; \\
\varpi_3 &= \varpi_0 + \gamma T - \gamma(t - T - U) && \text{,, } T + U \leqslant t \leqslant 2T + U ; \\
\varpi_4 &= \varpi_0 && \text{,, } 2T + U \leqslant t \leqslant \infty .
\end{aligned}$$

* See my *Algebra*, ii., ch. xxix., § 29.

The corresponding values of $\varpi - p$ are then given by

$$\left.\begin{aligned} y_1 &= \frac{\gamma}{\mu}(1 - e^{-\mu t}); \\ y_2 &= \frac{\gamma}{\mu}(e^{\mu T} - 1)e^{-\mu t}; \\ y_3 &= -\frac{\gamma}{\mu}(1 - e^{\mu(T+U-t)} - e^{\mu(T-t)} + e^{-\mu t}); \\ y_4 &= -\frac{\gamma}{\mu}(1 - e^{-\mu T})(1 - e^{-\mu(T+U)})e^{\mu(2T+U-t)} \end{aligned}\right\} \quad . \quad . \quad . \quad (40).$$

The broken curve OFGHK in fig. 8 shows the general form of the microbarogram in this case, the graph of $\varpi - \varpi_0$ being ODECL.

Microbarometric Observations at Loch Earn.

In the autumn of 1905 a series of microbarometric observations were made with the object of throwing some light on the origin of the seiches in Loch Earn. Three Dines-Shaw instruments were placed at the vertices of the triangle, Killin (A), Lochearnhead (B), Ardtrostan (C). If we denote the sides and angles of this triangle by a, b, c; A, B, C, then roughly we have

$$a = 5{\cdot}60, \quad b = 8{\cdot}75, \quad c = 6{\cdot}10 \text{ (miles)}$$
$$A = 39^\circ, \quad B = 98^\circ, \quad C = 43^\circ.$$

The direction BC is about 4° north of east.

In order to interpret a set of observations from three stations only, it is necessary to make some assumption regarding the wave-front or equiphasal line of the atmospheric disturbance. We shall take the simplest possible supposition, and assume the wave-front to be a straight line, and the propagation to be also rectilinear, and uniform with velocity V.

Then, if t_1, t_2, t_3 be the times of passage (say in minutes) of the same phase of a disturbance at A, B, and C, θ the inclination (to northward) of the wave-front to BC, and $\phi = 90^\circ - \theta$ the inclination of the direction of propagation (to southward) to BC, it will readily be seen that we have

$$c \sin(B - \theta)/a \sin\theta = (t_2 - t_1)/(t_3 - t_2).$$

This leads to

$$\cot\theta = \{(t_3 - t_1)\cot B + (t_2 - t_1)\cot C\}/(t_3 - t_2);$$

and, since $\phi = 90^\circ - \theta$, $\cot B = -0{\cdot}14$, $\cot C = 1{\cdot}07$, we have the formula

$$\tan\phi = \{1{\cdot}07(t_2 - t_1) - 0{\cdot}14(t_3 - t_1)\}/(t_3 - t_2) \quad . \quad . \quad . \quad (41).$$

Again, if V be the velocity in miles per hour, we have

$$V = 60a \sin\theta/(t_3 - t_2);$$

that is,

$$V = 336 \sin\theta/(t_3 - t_2) \quad . \quad . \quad . \quad . \quad . \quad (42).$$

On the assumptions we have made, the formulæ (41) and (42) enable us to calculate the direction and velocity of propagation of any disturbance, the same phase of which has been observed at A, B, and C.

It may be of some general interest, and possibly also of some use to others who may investigate the minor fluctuations of the atmospheric pressure with more perfect means in the future, to give a brief account of the Earn observations, and to discuss how far it is possible to deduce definite conclusions from them.

In the first place, it should be mentioned that, of the three instruments used, two came into the possession of the observers of the Lake Survey after they had started work on Loch Earn. There was therefore no opportunity of testing the three instruments together in the laboratory beforehand as regards sensitiveness, equality of leak, regularity of clock, etc.

The time-scale of these instruments is unfortunately very small, viz. ·25 mill. per minute; so that it is barely possible to make determinations of time to a minute on the microbarogram. It should also be mentioned that, since we had no direct telegraphic or telephonic communication between our three observing stations, it was difficult to get frequent and reliable comparisons of the watches of the observers. On the ground of the theoretical calculations in the first part of this paper, and of laboratory comparisons of the three instruments which have recently been made, it may be assumed that the differences of the acceleration of phase in the three microbarograms, whether due to the differences of leakage or to progressive deformation of the disturbance (which undoubtedly takes place in many cases), amounted to less than one minute. Also, our watch comparisons, although they left much to be desired, are accurate to less than a minute. Unfortunately, the clockwork supplied with the instruments did not work in all cases with sufficient regularity, and this, notwithstanding the use of frequent time-marks, introduced a considerable amount of uncertainty into our results.

On account of the necessity of frequent time-comparisons, and also for the purpose of calling the simultaneous attention of observers to any phenomenon of interest the beginning of which may have been noticed at any station, I would strongly advise that any network of microbaric stations should be connected by telephone.

In making time-marks on the microbarogram, it is desirable to avoid handling the instrument. All that is necessary is to shut the door of the room containing it rather briskly at a known time. This will cause the pen to draw a short, sharp line across the trace, and the time can be written opposite these marks when the chart is removed from the drum.

There are two other respects in which the construction of the instrument should be improved. A more refined arrangement should be introduced to secure that the point at which the hook connecting the stem of the floating plunger to the axis of the recording lever touches this axis does not shift arbitrarily; so that the pen shall return exactly to the same place after the instrument has received a jar or the air-pressure has been suddenly altered by slamming the door of the observing room. Also, the steadying spring and all other sources of friction should be removed from the stem of the plunger, or at least be removable, when it is at work.

During the observations the leakage of the Lochearnhead instrument increased, so that it came to register only the more transient or more violent disturbances. Thus on many interesting occasions its readings were unfortunately of no use. For good work it is essential that the microbarograph should be so constructed that it can be rapidly taken to pieces, tested, adjusted, and fitted up again. Considering the great simplicity of its general design, this should not be difficult.

As regards the nature of the phenomena under observation, the following points are to be noted:—

A set of observations from three stations only affords no means of testing the admissibility of our assumptions regarding the rectilinearity of wave-front and path of propagation. The observations themselves seem to indicate that the breadth (in the direction of propagation) of the disturbances in question is not very great, say 10 to 40 miles. Also there were cases in which a disturbance was not observed at all the three stations. It is therefore unlikely that the assumption of a rectilinear wave-front can be more than a very rough approximation.

Again, making all allowance for want of absolute similarity in the instruments, there seems to be no doubt whatever that the distribution of pressure in the disturbance varies as it progresses. This introduces uncertainty in identifying the points on the microbarogram which correspond to the same phase of the disturbance. We are, in fact, face to face with a wave which changes form as it goes along, and therefore has no definite velocity of propagation in the ordinary sense.

All these qualifications must be borne in mind in accepting the following results and conclusions, which make no pretence to be anything more than a mere reconnaissance in a very interesting but as yet almost wholly unknown region of meteorology.

[(Added 27th June 1908.) This warning is the more necessary since one of the referees of this paper, a competent meteorologist, expressed considerable surprise, not to say scepticism, regarding the five cases given in the table on this page, in which the microbaric disturbances come from eastwards.]

Specimens of the Results obtained at Loch Earn.

In the following tables the first column gives the day of the month on which the disturbance occurred; the second, the time, reckoned from midnight, when the maximum or minimum passed Ardtrostan; the third, the direction from which the disturbance came; the fourth, the velocity of propagation in miles per hour.

The letter α prefixed to the date means that the phase that was timed was a maximum, β a minimum.

Day.	Hour.	Direction.	V.	Day.	Hour.	Direction.	V.
Aug.	h. m.			Sept.	h. m.		
α 18	1 53	W.	6·7	α 2	15 11	E. 25° N.	75
β 21	12 17	W. 19° S.	27	α "	15 37	E. 44° N.	19
"	13 53	W. 41° S.	47	β "	21 18	W. 59° S.	21
β "	15 2	W. 52° S.	19	β 3	14 12	W. 25° N.	26
α "	18 17	W. 26° N.	26	α "	16 8	W. 30° S.	48
α "	19 39	W. 36° N.	22	α "	26 18	E. 55° N.	30
β "	21 47	W. 46° S.	13	α 7	8 28	E. 56° N.	19
α "	23 44	W. 49° S.	36	α "	16 34	W. 39° S.	9
α 23	14 7	W. 62° N.	15	α 8	3 39	W. 15° S.	34
β 31	18 36	W. 42° S.	41	α "	17 13	W. 6° N.	27
β "	26 48	W. 62° S.	68	β 9	12 53	W. 63° S.	17
Sept.				α "	13 58	W. 68° S.	22
α 1	20 31	W. 4° S.	21	α 13	3 31	W. 33° S.	45
α 2	10 42	E. 71° N.	46	β "	8 8	W. 44° S.	20

Out of the twenty-seven cases tabulated, the disturbance came from an easterly direction in only five. The average direction for the disturbances coming from westwards is W. 19° S., and for those from eastwards E. 50° N.

The average velocity of propagation is about 30 miles an hour.

In average direction and velocity the minor fluctuations of the atmospheric pressure resemble cyclonic disturbances so closely that the conclusion is naturally suggested that they travel together; but the accuracy and extent of the Loch Earn observations are not such as to justify us in doing more than raising the question.

To give some idea of the general connection of the microbaric disturbances with other phenomena, I append the following notes on three specially interesting days.

Loch Earn, 21st August 1905.

A rainstorm from S.W. began about 8^h 30^m, partially cleared about 9^h, came on heavier about 11^h, and continued with gusts of wind till 18^h 10^m, when the rain stopped and the sky cleared. The total rainfall during the day was 0·88 in.

At Ardtrostan at 9^h the wind was E., velocity 10 (mile/hour); the clouds, 1, were coming from S. by E., and the waves on the lake were coming from E. At 20^h 29^m the wind was S., very light; the clouds were coming from S.W., and the waves on the lake running from W. These surface waves were not high during any part of the day.

During the day the mean velocity of the wind rarely exceeded 10 (mile/hour), and was for the most part less. The extreme velocity in the gusts occasionally reached 15 or 20, and between 17^h and 18^h 15^m the gusts were very frequent, the mean velocity sometimes reaching 15, and the extreme 25.

The standard barometer at Balimeanach fell 0·233 in. from 9^h to 22^h, and the open-air temperature at both these hours was 53°·0 F.

Fig. 9 shows the three microbarograms at Ardtrostan, Killin, and Lochearnhead; part of the anemogram at Ardtrostan; and also the limnogram taken near St Fillans. The direction and velocity of propagation of some of the maxima and minima on the microbarograms are given in the table on p. 454.

Loch Earn, 3rd September 1905.

Westerly Gale.

After six hours or more of calm, the wind began to rise about 2^h 36^m. By 3^h 36^m it had reached a mean velocity of 17 (mile/hour), and it was fluctuating between 15 and 12 just before 7^h. Immediately after 7^h it had shot up to a mean velocity of 25, and between 7^h and 8^h was blowing a gale of 35 mean velocity, the extreme on one occasion being over 52. About 8^h 30^m there was a rather sudden decline, and at 9^h 10^m the mean velocity was 10. Thereafter it rose again, and at 11^h there was a gale of 30. This was maintained, with extreme gusts, occasionally reaching 54, till 15^h, when the gale began to moderate. By 17^h 5^m the velocity had fallen to 3. There was a rise again to 15 at 17^h 35^m, then a fall to 6 at 18^h 35^m, a rise to 13 at 18^h 55^m, a fall to 2 at 19^h 22^m, a rise to 8 at 19^h 30^m, and so on, till practically dead calm was reached about midnight, and continued till 9^h 30^m next morning.

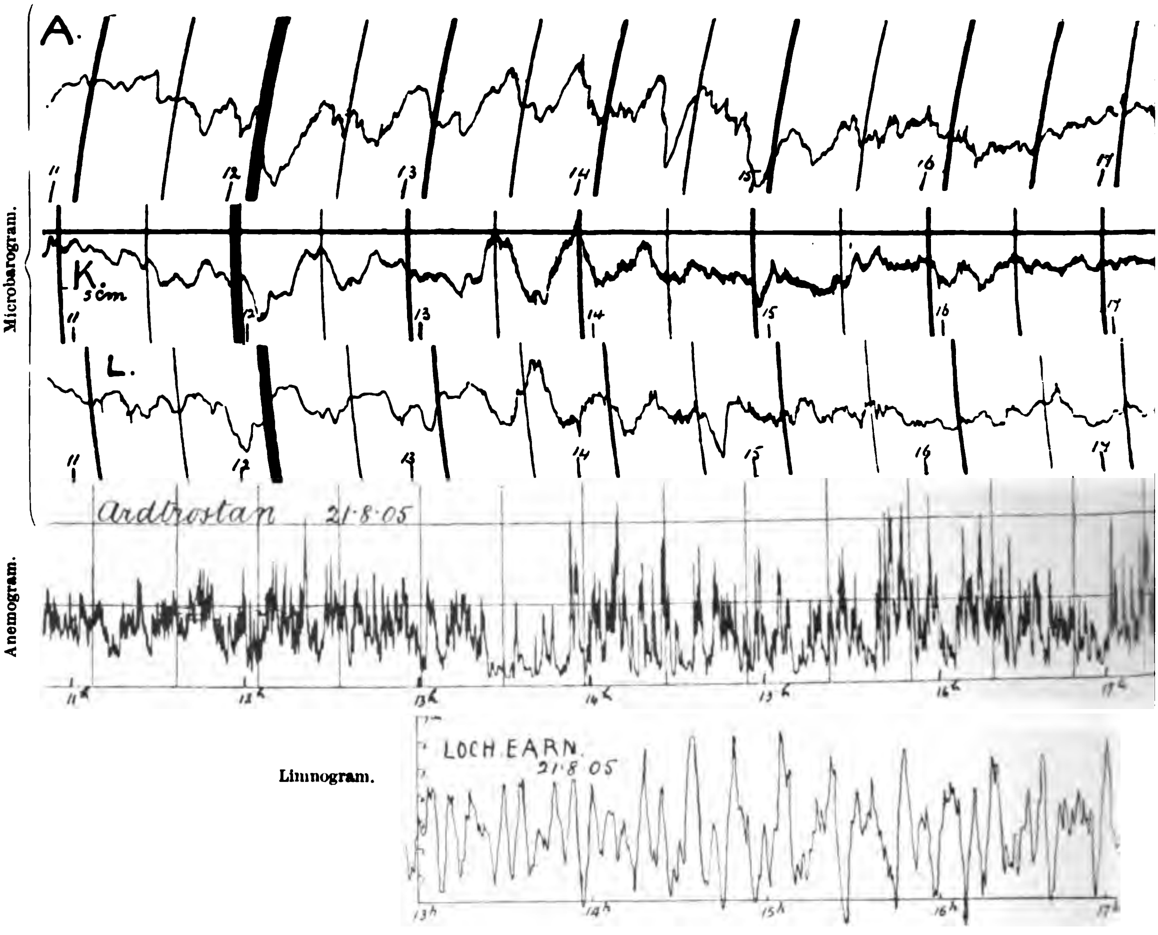


FIG. 9

During the day a number of observations were made with the nephoscope, and the velocity and direction of the cloud-motion deduced from them on the usual assumptions. For easy comparison we place together in the following table the data regarding cloud, wind, and microbaric disturbance. Under "Direction" is given the point of the compass from which the cloud or pressure disturbance came.

3RD AND 4TH SEPTEMBER 1905.

Time.	Cloud.			Wind W.		Microbaric Dist.	
	Cumulus Amount.	Direction.	Velocity.	Mean Velocity.	Extreme Velocity.	Direction.	Velocity.
h. m.							
10 38	9	W. 15° N.	17	25	35	...	...
12 40	3	W. 15° N.	56	35	50	...	...
14 12	...	...	...	...	...	W. 25° N.	26
15 0	2	W. 30° N.	56	34	48	...	...
16 4	3	W. 30° N.	22	20	26	...	...
16 8	...	...	...	...	...	W. 30° S.	48
17 1	3	W. 45° N.	28	6	10	...	...
17 15	3	W. 59° N.	22	5	8	...	...
20 23	7	?	0	3	4	...	...
26 18	...	...	...	...	...	E. 55° N.	30
26 23	...	...	...	1	...	...	...

While there is in this, as in other cases that have been observed, some connection apparent between the directions of the wind and cloud-motion, yet we see that on the fall of the wind and cloud velocity there is a reversion of the current of microbaric disturbances. As might be expected, the wind and cloud velocities go more or less together, but the velocity of propagation of the microbaric disturbances seems to vary independently. The reversion of the last-mentioned observed at 26^h 18^m may point to some discontinuity in the conditions of the upper air.

LOCH EARN, 7TH SEPTEMBER 1905.

Gusty Squalls of Wind and Rain.

At 8^h 28^m occurred one of the most remarkable microbaric disturbances observed during the months of August and September. The microbarograph at Ardtrostan had shown considerable disturbances both of longer and shorter duration since midnight. At 8^h 15^m a sudden rise began, which reached a maximum of $+11·5$ mill. about 8^h 27^m; then there was a fall in about 9^m to a minimum of $-6·5$ mill.; and finally in about 7^m a rise to about the mean previous to the disturbance. The whole disturbance occupied about half an hour, and was accompanied by similar disturbances at Killin and Lochearnhead. The maximum and the minimum at Ardtrostan were sharper than at the other two stations, so that Ardtrostan was either nearer the origin of the disturbance, or nearer its centre if it was a vortex of moderate dimensions. Unfortunately, the pen came to the end of the Lochearnhead chart before the minimum was reached: but the passages of the maximum could be timed, and the direction and velocity of propagation of the disturbance are given in the table on p. 454.

The microbarograms of this disturbance at Ardtrostan and Killin are

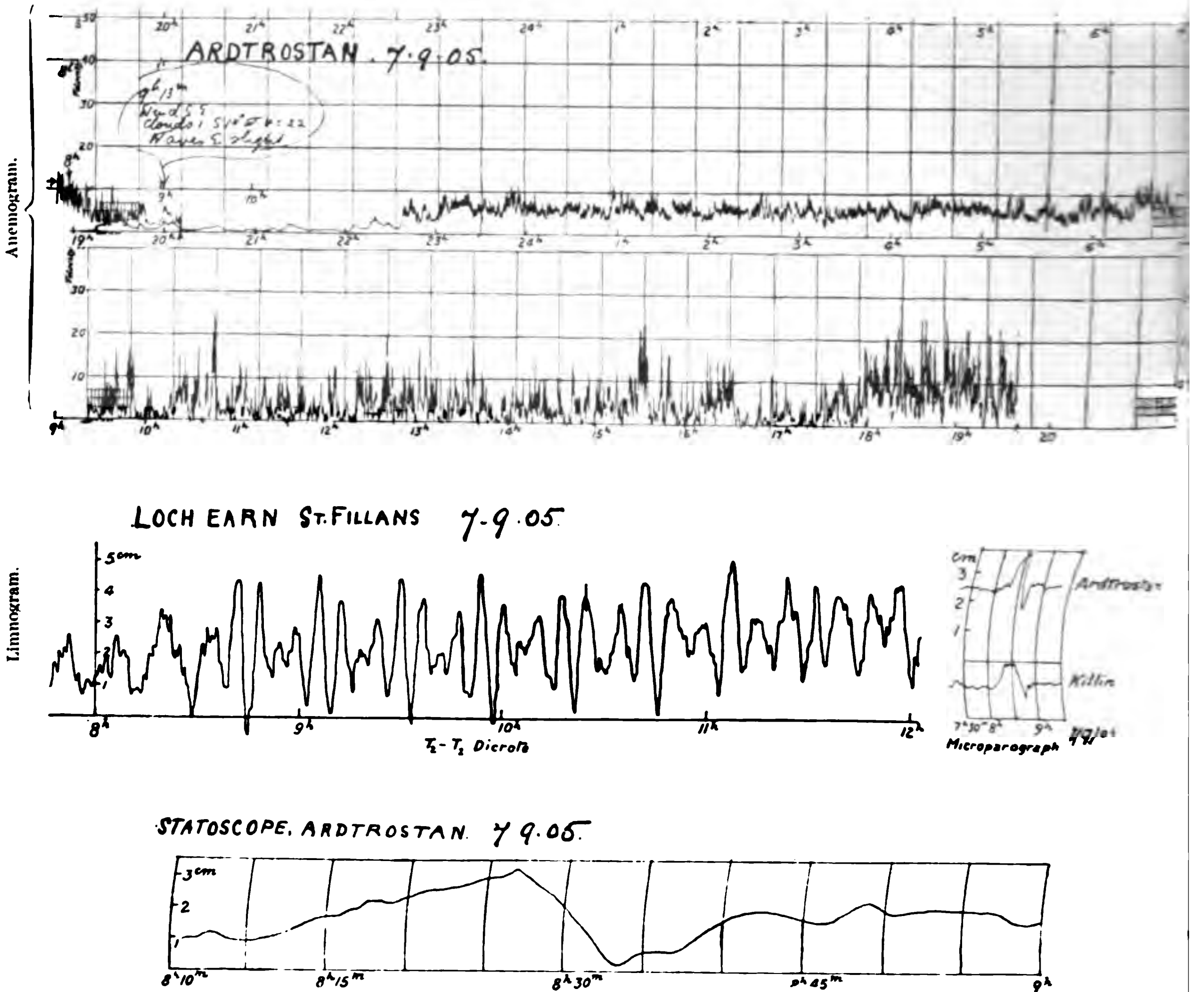


FIG. 10

given in fig. 10, and also a statoscope trace which we had the good fortune to take just at the right time.* In the same figure are shown the

* A comparison of the traces of the microbarograph and statoscope is interesting, as it shows what proportion of the drop in the microbarogram is real and not merely apparent, as suggested by fig. 6 above. The statoscope, of course, shows no false depression, as it was not worked with a leak.

anemogram, and the limnogram showing the alteration of seiche in the lake caused by the microbaric disturbance.

If we can rely on the value of the velocity of propagation given in the table, it is easy to calculate from the breadth of the zigzag on the microbarogram that the breadth of this atmospheric disturbance must have been about 10 miles at Ardtrostan, and 13 at Killin.

The following notes on the weather during the day may be of interest.

The variation of the wind velocity may be seen at a glance from the anemogram reproduced in fig. 10.

8^h 55^m Wind E. 2 (mile/hour). Cloud 1, cumulus towards S. Lake oily calm to S., slight ruffle from S.E. to N. The rain, which had begun about 5^h, now came down in torrents.

9^h 22^m Wind E. to S.E. 1 (mile/hour). Clouds 1, cumulus, drifting from W. 80° S., velocity 22. Rain stopped. Waves on lake slight from E. Wind rose about 9^h 35^m to 13 (mile/hour).

9^h 45^m Wind S.W. Clouds 9, cumulus upper and lower. Drift of lower W. 75° S., velocity 45; drift of upper different, velocity say 22, direction also different.

10^h 0^m Sun came out.

10^h 25^m Wind E. Clouds 5, cumulus, drift from S.W. Waves cross ripples with oily patches.

11^h 20^m Wind E. Clouds 9, cumulus. Waves from E. Shower at St Fillans, passing up the lake towards N.W.

14^h 49^m Wind W. Clouds 9, cumulus, drift from S.S.W. Waves, a moderate swell with small black squalls at times. Shower at Lochearnhead, one of many there during the day.

15^h 27^m Small cloudburst, which lasted two to three minutes. The drops whitened the surface like a shower of hail.

18^h 27^m Wind W.S.W. Clouds 1, cumulus, drift from W. Waves from N.W., a few foam-crested.

At Balimeanach the standard barometer fell from 28·736 in. at 9^h to 28·608 in. at 22^h. The open-air temperatures at the same hours were 56·0° and 54·0°. The total rainfall during the 24 hours was 0·75 in.

During the day the microbarogram was blurred by the characteristic wind vibrations, of which at times over 150 could be counted in an hour. Besides these, there occur irregular undulations on the traces, most marked at Killin, the times between the turning-points of which vary from 4^m to 60^m or more. The traces at the three stations are by no means always similar. Thus, for example, there are two well-marked minima at Killin between 19^h and 20^h, with a maximum between them, the whole range being about 5 mill., and the interval between the minima about 25^m. To this corresponds at Lochearnhead a double-crested maximum, the whole range being about 5 mill. At Ardtrostan there is scarcely anything that can be identified as belonging to the same disturbance. This disturbance cannot, therefore, have extended over the whole of Loch Earn.

One well-marked maximum, which passed Ardtrostan at $16^h\ 34^m$, can be identified at the other two stations. Its direction and velocity are given in the table on p. 454, and it will be seen that it followed more or less the same path as the drift of the lower clouds, and went in a direction opposite to the much more marked microbaric disturbance which occurred earlier in the day.

During the day the conditions in the upper and lower atmosphere were distinctly different; and there must have been at least one well-marked surface of discontinuity.

(*Issued separately August* 13, 1908.)

XXXI.—On the "Negative" Viscosity of Aqueous Solutions. By Dr W. W. Taylor and T. W. Moore, M.A., B.Sc., Carnegie Research Scholar. *Communicated by* Professor Crum Brown.

(Read June 1, 1908. MS. received June 8, 1908.)

The subject of the "negative" viscosity of solutions has of late attracted considerable attention. Since the publication of the paper by one of us with Ranken * in the *Transactions of the Royal Society of Edinburgh*, there have appeared a paper by Jones and Veazey,† and two by Getman.‡ §

Since the latter of the papers by Getman contains a comprehensive review of the older literature on the subject, it is unnecessary for us to do more than indicate those points which are of fundamental importance. First noticed by Hübner in 1873, "negative" viscosity was observed by Sprung in the course of his extensive researches to occur in solutions of several potassium and ammonium salts, while corresponding sodium salts do not cause it. Later investigations added to the salts which are known to give rise to it; mention may be made of hydrobromic acid, in which it occurs at low temperatures but not at higher temperatures, whilst it does not occur at all in the case of hydrochloric acid.‖

Euler's explanation of the phenomenon, based upon Nernst and Drude's theory of electrostriction, was put forward at a time when no exception was known to the rule that "negative" viscosity only occurs in aqueous solutions of a limited number of electrolytes. It became untenable so soon as Wagner and Mühlenbein ¶ found that solutions of certain organic substances in organic liquids, *e.g.*, cyanobenzen in alcohol, also possessed "negative" viscosity.

Rüdorf ** had just previously obtained values for the viscosity of carbamide solutions in water at 25° C., which were smaller than the viscosity of the solvent, but Fawsitt,†† working with a very pure sample of

* *Trans. Roy. Soc. Edin.*, xlv., part ii., p. 397, 1906.
† *Amer. Chem. Journ.*, xxxvii. p. 405, 1907.
‡ *Journal de Chemie Physique*, v. p. 344, 1907.
§ *Jour. Amer. Chem. Soc.*, xxx. p. 721, May 1908.
‖ Taylor and Ranken, *Proc. Roy. Soc. Edin.*, xxv. p. 231, 1904.
¶ *Zeit. f. physikal. Chem.*, xlvi. p. 872, 1903.
** *Zeit. f. physikal. Chem.*, xliii. p. 257, 1903.
†† *Proc. Roy. Soc. Edin.*, xxv. p. 52, 1904.

carbamide, was unable to confirm Rüdorf's results. In our previous paper it was conclusively shown that aqueous solutions of carbamide do not exhibit "negative" viscosity at 25° C., but that at low temperatures (+8° C.) a dilute solution is less viscous than water. The suggestion, first put forward by Arrhenius, was revived by Getman, that the anion and the undissociated molecule have the same influence on the viscosity of the solvent, but that the cation has the opposite effect, the tendency of the cation being to diminish the viscosity, that of the others to increase it. Jones and Veazey extended this idea, starting from the observation that the salts which diminish the viscosity of water are salts of metals with large atomic volumes. Several instances are adduced in support of this view; thus potassium chloride and potassium nitrate solutions are less viscous than water, while many other potassium salts are not. In the latter, the increase in viscosity due to the anion is supposed to more than counterbalance the decrease due to the potassium ion. Potassium has the largest atomic volume of all the metals, with the exception of rubidium and cæsium, and, as anticipated by the hypothesis, the chlorides of those metals diminish the viscosity of water to a greater extent than does potassium chloride. In connection with potassium chloride, it is further pointed out that of all the elementary anions, chlorine has the largest atomic volume, with the sole exception of bromine, which is slightly larger: this unusually large anion is supposed to act in the same direction as the cation, or, at least, not to counteract its effect. Now, potassium bromide solutions actually do possess "negative" viscosity, and to a greater extent than equivalent solutions of potassium chloride; but potassium iodide, the atomic volume of iodine being practically the same as chlorine, causes a still greater decrease in the viscosity of water than either the chloride or bromide.

It would thus appear that, taken alone, the hypothesis of Jones and Veazey cannot account for the occurrence of "negative" viscosity even in the case of aqueous solutions of electrolytes. This conclusion is confirmed by a consideration of other solutions which have "negative" viscosity.

The cases of potassium ferrocyanide and potassium ferricyanide are perhaps the most convincing. The anions $Fe(CN)_6''''$, and $Fe(CN)_6'''$, differ mainly in this, that the one has four negative charges while the other has only three, and their effect on the viscosity of water can scarcely be attributed to differences in volume; this being so, the ferrocyanide with four $K^{\cdot}$ should diminish the viscosity of water to a greater extent than the three $K^{\cdot}$ of an equi-*molecular* solution of the ferricyanide. But the facts are quite the reverse, as the following data show:—

TEMPERATURE 25° C.

Mols. per litre.	Absolute Viscosity. $K_3Fe(CN)_6$.	Absolute Viscosity. $K_4Fe(CN)_6$.
·5	·00988	·01120
·25	·00932	·00996
·125	·00910	·00944

The differences are similar at 15° C. At 1·6° C. the viscosity of the ferricyanide solutions was negative. These differences cannot be explained by differences in degree of ionisation of ferrocyanide and ferricyanide, for freezing-point determinations made some time ago, but not yet published, prove that the degree of ionisation of the two salts is about the same at the same molecular concentrations.

All the more complete investigations on the viscosity of solutions bear out the conclusion that, in general, concentration-viscosity curves pass through a minimum, although, owing to limitations of solubility, it may not always be possible to obtain a minimum at all temperatures. It is not, however, at all impossible that the size of ions may have a direct influence on the absolute value of the viscosity, and thus be an important factor in determining whether this minimum value for the solutions is less than the viscosity of the pure solvent at the same temperature. In order definitely to test this point, we have determined the viscosity of aqueous solutions of certain organic salts whose cations must have widely different ionic volumes. The five salts chosen were tetra-methyl ammonium iodide, tetra-ethyl ammonium chloride and bromide, tetra-propyl ammonium chloride and iodide. These substances were used on account of their stability, and because they are typical electrolytes in every way comparable with salts of the alkalies or of ammonium. They were purchased from Kahlbaum, with the exception of tetra-propyl ammonium chloride, which was made from the corresponding iodide by treatment with freshly precipitated silver oxide, and exact neutralisation of the filtrate with hydrochloric acid. As some of the salts are deliquescent, all the solutions were made up approximately by weighing, and the actual concentrations found by titration with standard silver nitrate.

In order to have the same ratio of molecules of solvent to one molecule of solute, all the concentrations are expressed in gm. molecules of solute to 1000 gm. of water.

The densities were determined by means of an Ostwald-Sprengel pycnometer. The viscosities were determined by the Ostwald-Poiseuille transpiration method, the apparatus and mode of working being the same as that employed in our previous work, and fully described in the paper by

Taylor and Ranken.* The only change made was that sodium hydroxide in aqueous alcohol, followed by concentrated nitric acid, was frequently employed, instead of chromic acid in sulphuric acid, to clean out the viscosity tubes. The transpiration times of pure water for each tube were determined just before the determination of the transpiration times for each solution; this is an absolutely necessary precaution. It is unnecessary to reproduce tests which were made to check the accuracy of the determination of the viscosity. It may be taken to be the same as mentioned in our previous paper.

Experimental Results.

In the following tables—

m is the concentration of the solution in mols. per 1000 gm. of water.

d is the density of the solution referred to water at 4° C.

η/η_0 is the ratio of the viscosity of the solution to that of the solvent at the given temperature, as directly determined by experiment.

η is the viscosity of the solutions in absolute units, and is obtained from the ratio given in the third column by multiplication with η_0 as determined by Thorpe and Rodger † (η_0 at 25° C. = ·00891, at 35° C. = ·00720).

Tetra-Methyl Ammonium Iodide.

25.° C.

m	d	η/η_0	η
·0685	1·0022	1·0042	·00894
·1300	1·0064	1·0060	·00895
·2550	1·0149	1·0124	·00898

35°.

m	d	η/η_0	η
·1106	1·0019	1·0036	·00723
·1527	1·0050	1·0076	·00726
·1774	1·0065	1·0127	·00729

* *Loc. cit.*

† *Phil. Trans.*, 185, p. 449, 1894.

Tetra-Ethyl Ammonium Chloride.

25° C.

m	d	η/η_0	η
·2914	·99713	1·0987	·00979
·5893	·99755	1·2187	·01086
·7878	·99815	1·3155	·01173
1·148	·99935	1·5255	·01359

35° C.

m	d	η/η_0	η
·2935	·99401	1·1076	·00798
·4747	·99426	1·1760	·00846
·7509	·99467	1·3000	·00936
1·0922	·99570	1·4571	·01049

Tetra-Ethyl Ammonium Bromide.

25° C.

m	d	η/η_0	η
·2819	1·0071	1·0995	·00980
·5082	1·0149	1·1895	·01059
·7568	1·0227	1·2930	·01152
1·1363	1·0342	1·4737	·01314

35° C.

m	d	η/η_0	η
·3309	1·0054	1·1100	·00799
·4524	1·0084	1·1455	·00825
·7218	1·0180	1·2640	·00910
1·1367	1·0305	1·4470	·01042

Tetra-Propyl Ammonium Chloride.

25° C.

m	d	η/η_0	η
·2733	·99520	1·2405	·01105
·2771	·99529	1·2450	·01109
·6177	·99433	1·6235	·01447
·7884	·99425	1·8250	·01626

35° C.

m	d	η/η_0	η
·1772	·99266	1·1350	·00818
·3324	·99161	1·2600	·00907
·7884	·99013	1·8030	·01298

Tetra-Propyl Ammonium Iodide.

25° C.

m	d	η/η_0	η
·2665	1·0143	1·2151	·01083
·5314	1·0307	1·493	·01331

35° C.

m	d	η/η_0	η
·1855	1·0052	1·1240	·00809
·2970	1·0141	1·2430	·00895
·4182	1·0205	1·3332	·00960

Discussion of Results.

From the results contained in the above tables, and shown graphically in figs. 1 and 2, it is at once apparent that the viscosity increases from

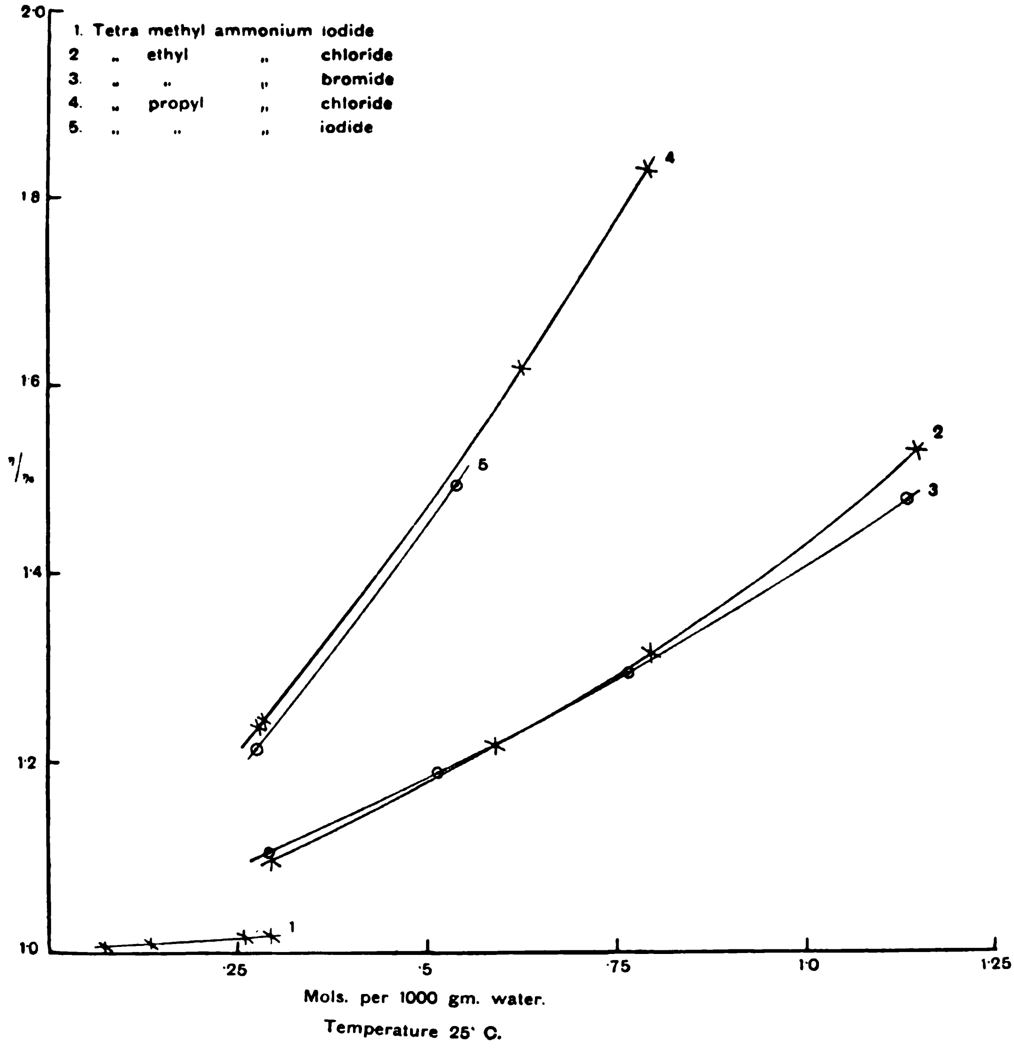


Fig. 1.

tetra-methyl ammonium to tetra-ethyl ammonium, and is largest for tetra-propyl ammonium salt; *i.e.* the salt whose cation has the largest volume has the largest viscosity, while the one with the least cation has the smallest viscosity.

In no case is there the slightest approach to "negative" viscosity, and the curves in figs. 1 and 2 do not give any indication that it would occur at any other concentration.

It should be explained that the small range of solutions of tetra-methyl ammonium iodide examined was due to its slight solubility in water.

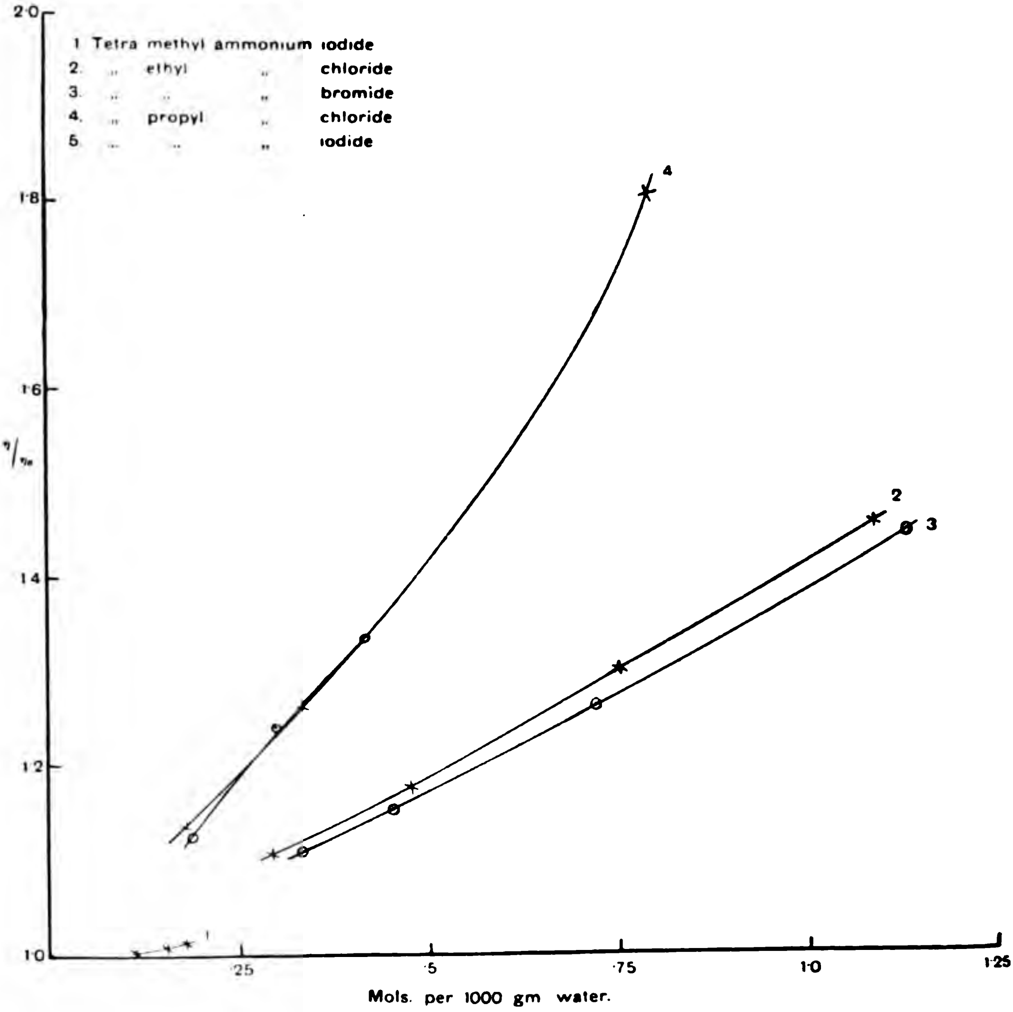


Fig. 2.

From the present point of view, however, it was not worth while to investigate solutions of the chloride which is much more soluble.

It is remarkable how slight the influence of the anion is. The particular salts used were chosen so as to afford some information on this point, without impairing their value as tests for the influence of the volume of the cations. In particular, the values for tetra-ethyl ammonium chloride and bromide lie very close together, much more so than the viscosities of hydrochloric acid and hydrobromic acid, of the potassium salts, or of the ammonium salts. From this it would appear that the viscosity of salt solutions cannot be regarded as simply an additive property.

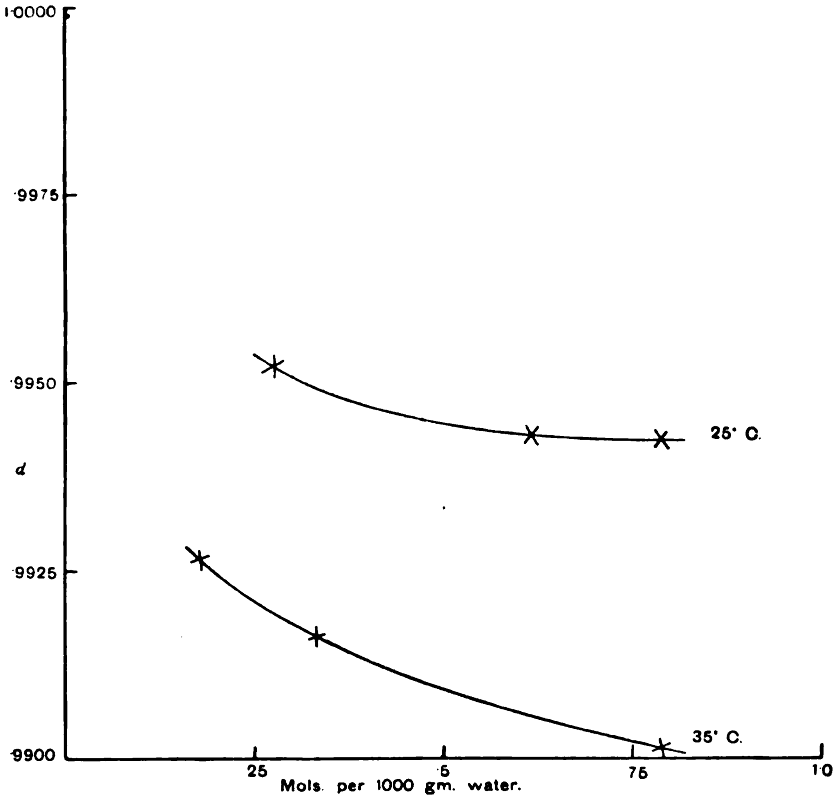


Fig. 3.

The data are insufficient to enable one to make an effective comparison of the effect on η of the change from methyl to ethyl to propyl, but the following approximate figures are of some interest. They are for ·5 molar solutions at 25° C.; the value for ammonium iodide is taken from Getman's tables, for tetra-methyl ammonium iodide by extrapolation, and for tetra-propyl ammonium iodide and tetra-ethyl ammonium bromide by interpolation from the curves in fig. 1.

·5 m.	NH_4I	η = ·00858	diff. ·00062
	$(CH_3)_4NI$	·00920	·00132
	$(C_2H_5)_4NBr$	·01052	·00242
	$(C_3H_7)_4NI$	·01294	

It must be remembered that these figures are not strictly comparable, for the ammonium iodide concentration is in mols. per litre, the others in mols. per kilogram of solvent; also the difference in viscosity between tetra-ethyl ammonium bromide and iodide is no doubt very slight, and would reduce the difference between it and the methyl compound.

In conclusion, attention is directed to a very curious density relation which has been found to exist in the case of the tetra-propyl ammonium chloride solutions. On reference to the tables it will be noticed that the density of the more concentrated solutions is less than that of the more dilute ones; the difference is quite marked, and holds at all the concentrations examined, and at both temperatures, viz. 25° C. and 35° C. (see fig. 3). The decrease of density with increase of concentration is more marked at the higher temperature. Tetra-propyl ammonium iodide does not show this peculiarity, nor do any of the other substances under the conditions of examination. But it will be noticed that the density of the tetra-ethyl ammonium chloride increases very slightly with increasing concentration, and more especially at 35° C. In comparison with ammonium chloride the differences are very striking,

25° C.

Concentration.	Density (Water at 4° = 1).		
	NH_4Cl.	$(C_2H_5)_4NCl$.	$(C_3H_7)_4NCl$.
·2733	...	...	·9952
·2914	...	·9971	...
·4517	1·0071	...	...
·5893	...	·9976	...
·6177	...	...	·99433
·7884	...	...	·99425
·9185	1·0138	...	...
1·148	...	·9994	...
1·402	1·0204	...	...

Here, too, the values for ammonium chloride are taken from Getman's tables, but the concentrations have been recalculated into the same units as the others. It will also be noticed that the actual densities of the solutions of the substituted ammonium salts are very small, being even less dense than water (d 25°/4° = 0·99707) in the case of the tetra-propyl ammonium chloride. Whether the two are directly connected or not, the abnormally low densities of these solutions are in marked contrast with their viscosities, which are abnormally large. It is intended to fully investigate the density relations of these substances through as wide a range of concentration and temperature as may be found practicable; other properties of the solutions which may be related to the volume of the solution will be taken into consideration. Work has already been commenced in this direction.

We desire to express our thanks to the Carnegie Trust for a grant in aid of the expenses, both for apparatus and materials used in connection with this research.

Chemistry Laboratory,
University of Edinburgh.

(*Issued separately* August 13, 1908.)

XXXII.—On the Effects on the Metabolism of Chloroform administered by Different Channels. By D. Noël Paton, M.D.

(From the Physiological Laboratory of the University of Glasgow, and the Laboratory of the Royal College of Physicians, Edinburgh.)

(Read June 15, 1908. MS. received June 18, 1908.)

Preliminary.

Death under the administration of chloroform as an anæsthetic is an accident which, although fortunately rare, is well known to all; but the occurrence of a late poisoning, sometimes manifesting itself several days after the administration of the drug, is less well recognised. This too is rare, but its occurrence indicates the very profound effect which such a drug may exercise upon living tissues.

A very large amount of work has been done upon the immediate influence of chloroform upon the nervous system and upon the circulatory and respiratory mechanisms, but few observations on its more lasting effects upon the general metabolism have been recorded.

A protoplasm poison so powerful as chloroform might be expected to modify profoundly the chemical changes in the body. Upon the cells of the liver, after removal from the body, I found in 1893 that chloroform has a very marked action, accelerating the disintegrative necrobiotic changes (*Phil. Trans. Roy. Soc.*, vol. 185, p. 248), and at the same time accelerating the conversion of glycogen to glucose. That, during life, it has a marked effect upon the metabolism of protein, has been shown by Salkowski (*Virchow's Arch.*, Bd. 115, p. 339) and by Strassmann (*Virchow's Arch.*, 1889), both of whom found that its administration caused an increased excretion of nitrogen in fasting animals. Kast and Mester (*Ztsch. f. klin. Med.*, Bd. xviii., p. 469, 1891) also got similar results. Vidal (*C.R. de la Soc. biol.*, 1896, p. 474) records a series of observations on the excretion of nitrogen in the human subject before and after operations of different degrees of severity, in which chloroform had been administered, and he finds that, irrespective of the character of the operation, there is a rise in the excretion of nitrogen, when compared with its ingestion. He also investigated the question on fasting rabbits, and found that in these animals the excretion of nitrogen, after the separation of the albumin, is markedly increased by the simple inhalation of the drug.

Thompson (*Brit. Med. Jour.*, vol. i., 1906, p. 608) describes the result of a study of the immediate effect of the administration of chloroform and of ether upon the urinary secretion. His experiments extended over two to four hours, and he states that with chloroform there is first a reduction in the volume and in the concentration of the urine secreted, followed by a rise in the amount. He finds that exudation of leucocytes and hyaline casts is apt to occur, probably due to the partial vascular stasis in the kidney, and that albuminuria occasionally is developed. Sometimes an increase in the reducing substances in the urine was observed.

The effect of chloroform upon the gaseous exchanges in respiration does not seem to have been studied experimentally. During the anæsthesia, Oliver and Garrett (quoted in Richet's *Dict. de Phys.*, vol. iii., p. 624) found an increase of carbon dioxide expired, and a decrease of oxygen absorbed, apparently as a result of the respiratory deficiency.

Present Investigation.

In 1900, when studying, with Dr Eason, the effect of various drugs upon the elaboration of urea in the liver, I intended to investigate the effect of chloroform, anticipating that its action would be similar to that of alcohol and sulphonal (*Jour. of Phys.*, vol. xxvi., p. 166, 1901). Other work interfered with the prosecution of these investigations. My attention was again directed to the subject by the appearance of a paper by Stiles and Stuart M'Donald on "Delayed Chloroform Poisoning" in August 1904 (*Scot. Med. and Surg. Jour.*, 1904, p. 97). In this an excellent *résumé* of the previous work on the subject is given.

The object of the present series of experiments is to investigate the effect of chloroform on the protein metabolism, to ascertain whether chloroform in any way influences the elaboration of urea in the liver, and whether its action is the same when administered by the respiratory passages, by the mouth, and hypodermically.

I.—General Plan of Investigation.

In this investigation the effect of the drug upon the chemical changes in the living matter of the body is deduced from the changes in the excretion of the products of disintegration of proteins in the urine: any *quantitative* change in the protein metabolised being indicated by changes in the total amount of nitrogen excreted per diem, while *qualitative* changes are indicated by alterations in the distribution of nitrogen and of sulphur in the different nitrogen- and sulphur-containing constituents of the urine. The animal was kept upon a fixed diet, and the average normal amount of these different constituents was determined for some days. Then the drug was administered, and their amounts were again investigated for a longer or shorter period, till any disturbance had passed off and the normal condition was again restored.

Since an interference with the usual course of protein metabolism is generally indicated by a change in the relative amount of nitrogen in urea and in ammonia, in the earlier experiments these constituents were alone specially investigated, but in the later experiments other constituents were also dealt with.

II.—Method of Experiment.

Female dogs of about 15 to 20 kilos. were used for the experiments. Catheterisation was not used in collecting the urine. This was done by keeping the dog in a galvanised iron cage with a sloping bottom, under which the urine, after passing through a rough filter of glass-wool, was collected in a glass vessel containing a little dilute hydrochloric acid to prevent decomposition. The dog was fed on porridge and milk, which produced firm fæces, which were at once removed from the cage, and with which the urine was never contaminated. (For a comparison of the analysis of urine collected thus and by catheterisation, see *Jour. of Physiol.*, xxv., p. 445, 1900.) The diet generally consisted of:—

Oatmeal = 120 grm.
Water = 500 c.c.

made into porridge by boiling for twenty minutes.

Milk = 500 c.c.

When nitrogenous equilibrium was established, the chloroform was administered:—

A. In the first series of experiments, by the respiratory passages, with the addition of 10 per cent. alcohol, following Schäfer's recommendation for decreasing the toxic effect upon the heart (*Trans. Roy. Soc. Edin.*, vol. xli., p. 311, 1904).

B. In the second set of experiments, by the stomach, dissolved in oil.

C. In the third set of experiments, by hypodermic injection.

In some of the earlier experiments the dog fasted on the day on which the chloroform was administered, the effect of a one-day fast being investigated a few days before the chloroform was given.

III.—Methods of Analysis.

The urine was collected at 10 each morning. It was measured, the reaction and specific gravity taken, and either the whole or a part diluted to a convenient volume. When a deposit was present, it was collected in a Y-glass and examined microscopically.

The Total Nitrogen was determined by Kjeldahl's method.

The "Urea" Nitrogen was determined by Bohland's or by Mörner and Sjöqvist's method, or by both. A useful comparison of these two methods is thus afforded. (See *Jour. of Physiol.*, vol. xxxiii., p. 6, 1905.)

The Ammonia Nitrogen was determined by Schloessing's method.

The Creatinin Nitrogen, when investigated, was determined by Folin's method.

The Uric Acid Nitrogen in the early experiments was determined by Hopkins' method, and in the later experiments (XI. and XII.) by Folin's method.

The Oxidised and Unoxidised Sulphur were determined by the method described in *Jour. of Physiol.*, vol. xxiv., p. 335, 1899.

Most of the determinations of nitrogen were carried out under my supervision by Mr Alfred Patterson, Chemical Attendant in the Royal College of Physicians Laboratory. The sulphur determinations were made by Miss Jean Robertson, Chemical Assistant in the Laboratory.

IV.—Results.

Series I.—Chloroform by the Respiratory Passages.

The chloroform was administered upon cotton wadding placed in a mask. Since several determinations have now been recorded of the chloroform in the blood during full anæsthesia, showing that it amounts to something not far short of 50 mg. per 100 c.c. (Gréhant and Quinquand, *C.R.*, 1883; Nicloux, *C.R. Soc. biol.*, t. lx., p. 244; Buckmaster and Gardner, *Proc. Roy. Soc.*, vol. lxxix., 1907), it was not considered necessary to dose the amount given. Generally in two hours about 80 c.c. were used.

Experiment I.—Retriever Bitch. Weight, 16 kilos.

Table I.

Day.	Urine in c.c.	Sp. G.	Total Nitrogen.	Urea Nitrogen.		NH_3 Nitrogen.	
				Bohl.*	M. & S.†		
1	880	1013	6·22	5·32	...	0·579	
2	910	1013	4·96	...	...	0·403	
3	620	1013	4·12	...	...	0·299	
4	690	1014	4·06	3·39	...	0·274	
5	220	1025	2·77	2·41	2·27	0·126	Fast.
6	640	1010	3·95	3·33	3·11	0·252	
7	570	1013	3·30	2·86	2·66	0·224	
8	770	1018	6·94	5·63	5·68	0·512	
9	690	1015	5·48	4·68	4·56	0·397	Fast. Chloroform 1 hour.
10	770	1014	4·87	4·26	4·01	0·229	
11	690	1012	4·09	3·50	3·33	0·319	

Weight, 15 kilos.

No protein and no reduction of Fehling's solution.

* Bohl. = Bohland's phospho-tungstic acid method.

† M. & S. = Mörner and Sjöqvist's method.

TABLE II.

Day.	Percentage of Total Nitrogen. Urea. Bohl.	Percentage of Total Nitrogen. Urea. M. & S.	NH$_3$.	Not in these.	
1	86	...	9·3	...	
2	...	...	8·5	...	
3	...	...	7·2	...	
4	86	...	6·7	...	
5	87	82	4·5	13·5	Fast.
6	84	78	6·2	15·8	
7	85	81	6·7	12·3	
8	81	82	7·3	10·7	
9	88	84	7·3	8·7	Fast. Chloroform 1 hour.
10	87	83	4·6	12·4	
11	85	81	7·8	11·2	

EXPERIMENT II.—Retriever Bitch. Weight, 16 kilos.

TABLE III.

Day.	Urine in c.c.	Sp. G.	Total Nitrogen.	Urea Nitrogen. M. & S.	NH$_3$ Nitrogen.	
1	700	1013	4·54	3·56	0·476	
2	Lost	...	...	...	...	
3	720	1012	3·70	...	0·411	Fast. Chloroform 2 hours in morning and 1 hour in afternoon.
4	300	1025	4·20	3·43	0·304	
5	260	1040	9·41	8·09	0·420	
6	680	1016	6·66	5·46	0·515	
7	570	1017	5·71	4·45	0·420	
8	480	1015	4·28	3·23	0·268	
9	770	1012	4·93	3·89	0·341	
10	680	...	...	...	...	

Weight, 16·6 kilos.

On the second day after chloroform there was a distinct trace of protein in the urine—by Esbach's tube 0·04 per cent. On the third day this increased to 0·1 per cent. It disappeared on the sixth day. On the third day there was a copious deposit of granular cells like renal epithelium, but no tube-casts. These cells were present till the sixth day. The urine was very dark for three days. No reduction of Fehling's solution.

TABLE IV.

Day.	Percentage of Total Nitrogen.		Not in these.	
	Urea. M. & S.	NH_3.		
1	78	10·4	11·6	
2	...	...	...	
3	...	11·1	...	
4	83	7·2	9·8	Fast. Chloroform 3 hours.
5	86	4·4	9·6	
6	82	7·7	10·3	
7	79	7·5	13·5	
8	79	6·4	14·6	
9	79	6·9	14·1	
10	...	...	...	

EXPERIMENT III.—Retriever Bitch. Weight, 14·7 kilos.

TABLE V.

Day.	Urine in c.c.	Sp. G.	Total Nitrogen.	Urea Nitrogen. Bohl.	NH_3 Nitrogen.	
1	450	1018	4·56	...	0·515	
2	560	1013	4·00	3·14	0·434	
3	520	1014	3·78	2·87	0·403	
4	320	1025	3·84	3·28	0·218	Fast. Chloroform 2 hours.
5	220	1032	5·49	4·76	0·330	
6	570	1015	3·98	3·44	0·344	
7	550	1012	3·08	2·83	0·232	
8	570	1014	3·08	2·49	0·145	
9	560	1015	...	...	...	
10	710	1013	2·58	1·90	0·254	
11	630	1015	...	...	...	

Dog weighed 13·6 kilos.

A trace of protein appeared on the fourth day after chloroform, and was more marked on the sixth day. On the fourth day there was a deposit of granular cells, but no tube-casts. The urine gave a very slight reduction of Fehling's solution on the day after chloroform.

TABLE VI.

Day.	Percentage of Total Nitrogen. Urea. Bohl.	NH$_3$.	Not in these.	
1	...	11·3	11·7	
2	78	10·3	11·7	
3	75	10·6	14·4	
4	85	5·6	9·4	Fast. Chloroform 2 hours.
5	86	6·0	8·0	
6	86	8·5	5·5	
7	91	7·5	} 8·5	
8	80	4·6		
9	...	...	...	
10	75	9·6	5·6	

EXPERIMENT IV.—Retriever Bitch. Weight, 14·7 kilos.

TABLE VII.

Day.	Urine in c.c.	Sp. G.	Total Nitrogen.	Urea Nitrogen. Bohl.	Urea Nitrogen. M. & S.	NH$_3$ Nitrogen.	
1	630	1015	5·38	4·51	4·51	0·375	
2	310	1031	5·24	4·37	4·34	0·350	Chloroform 2 hours.
3	380	1028	8·99	7·59	7·70	0·616	
4	570	1020	6·99	5·74	5·63	0·444	
5	735	1013	5·77	4·68	4·73	0·406	
6	950	1013	6·27	5·12	5·12	0·434	
7	520	1015	4·22	3·47	3·50	0·246	
8	630	1015	4·79	3·95	3·89	0·330	
9	975	1011	...	...	...	...	

Dog weighed 12 kilos.

No protein. No reduction of Fehling's solution.

TABLE VIII.

Day.	Percentage of Total Nitrogen.			Not in these.	
	Urea.		NH_3.		
	Bohl.	M. & S.			
1	83·5	83	6·9	10·1	
2	83·3	83	6·6	10·4	Chloroform 2 hours.
3	83	84	6·9	9·1	
4	82	80·5	6·3	13·2	
5	81	82	7·0	11·0	
6	81·6	81·6	6·9	11·5	
7	82	83	5·8	11·2	
8	80	81	6·8	12·8	

EXPERIMENT V.—Retriever Bitch. Weight, 12·8 kilos.

A weighed quantity of plasmon was added to the diet.

TABLE IX.

Day.	Urine in c.c.	Sp. G.	Total Nitrogen.	Urea Nitrogen.		
				Bohl.	M. & S.	
1	620	1022	9·04	7·34	7·34	
2	50	1016	0·89	0·71	...	Fast.
3	410	1031	9·18	7·70	7·81	
4	740	1015	7·73	6·78	6·64	
5	140	1047	4·57	4·10	4·08	Fast. Ether for 2 hours.
6	660	1022	9·72	8·60	8·48	
7	620	1022	9·52	8·32	8·26	
8	300	1037	8·15	7·36	7·25	Fast. Chloroform for 2 hours.
9	450	1031	10·9	9·80	9·83	
10	560	1020	7·84	7·08	6·97	
11	530	1023	10·11	9·16	9·04	
12	430	1025	8·57	7·59	7·39	

Dog weighed 10 kilos.

Trace of protein on day after ether. Marked reduction of Fehling's solution on day of chloroform. Trace of protein on day after chloroform.

Table X.

Day.	Percentage of Total Nitrogen in Urea.		
	Bohl.	M. & S.	
1	81	81	
2	79	...	Fast.
3	83	85	
4	87	86	
5	89	89	Fast. Ether for 2 hours.
6	88	87	
7	87	87	
8	90	89	Fast. Chloroform for 2 hours.
9	90	90	
10	90	89	
11	90	89	
12	87	86	

Table XI.

Day.	Total Sulphur.	Sulphur Oxidised.	Oxidised per cent. of Total Sulphur.	
1	...	...	...	
2	...	...	...	
3	1·34	0·941	70	
4	1·16	0·803	69	
5	0·80	0·547	68	Ether.
6	1·50	1·11	74	
7	2·17	1·37	63	
8	...	...	...	Chloroform.
9	1·83	1·20	66	
10	1·72	...	...	
11	2·03	1·38	68	
12	1·39	...	...	

EXPERIMENT VI.—Setter Bitch. Weight, 15·1 kilos.

Food—oatmeal, 100 grm.; water, 500 c.c.; milk, 500 c.c.

TABLE XII.

Day.	Urine in c.c.	Sp. G.	Total Nitrogen.	Urea Nitrogen. M. & S.	NH_3 Nitrogen.	Creatinin Nitrogen.	
1	970	1012	5·68	...	...	0·15	
2	860	1013	5·94	...	...	0·15	
3	160	1032	2·74	...	...	0·04	
4	870	1012	5·18	4·34	0·24	0·13	
5	420	1020	4·41	3·74	0·22	0·10	Chloroform for 3 hours.
6	490	1025	8·99	7·64	0·54	0·13	
7	660	1014	7·84	6·40	0·46	0·12	
8	620	1016	6·50	5·40	0·34	0·13	
9	550	1016	5·33	4·41	0·38	0·10	
10	575	1010	3·54	3·04	0·21	0·06	
11	1100	1015	8·61	7·44	0·60	0·13	
12	225	1025	3·54	...	...	0·06	

Dog weighed 12·6 kilos.

On the day after chloroform the urine was dark in colour, and remained so for four days. No proteins; no reduction of Fehling's solution; no bile-pigments.

TABLE XIII.

Day.	Percentage of Total Nitrogen.		Creatinin.	Nitrogen not in these.	
	Urea. M. & S.	NH_3.			
1	...	...	2·6	...	
2	...	...	2·5	...	
3	...	...	1·4	...	
4	83	4·6	2·0	10·4	
5	85	5·0	2·4	7·6	Chloroform for 3 hours.
6	85	6·0	1·4	7·6	
7	81	5·9	1·5	10·9	
8	83	5·2	1·9	10·0	
9	82	7·0	1·97	9·0	
10	85	6·0	1·9	7·0	
11	86	7·0	1·4	5·6	
12	...	...	1·6	...	

TABLE XIV.

Day.	Total Sulphur.	Sulphur Oxidised.	Oxidised per cent. of Total Sulphur.	
1	0·89	0·53	66	
2	1·17	0·72	61	
3	0·56	0·27	49 } 63	
4	0·95	0·74	78 } 63	
5	1·05	0·76	72	Chloroform.
6	1·13	0·76	67	
7	1·19	0·81	68	
8	1·15	0·83	72	
9	1·03	0·61	60	
10	0·61	0·40	66	
11	1·94	1·60	82 } 62	
12	0·81	0·34	42 } 62	

The results of these experiments may be considered under the following heads:—

1. *Amount of Water excreted.*—The general result shows that chloroform tends to diminish the secretion of urine on the day on which it is administered (see Thompson, *loc. cit.*).

2. *Total Nitrogen.*—In these experiments the increased excretion of nitrogen, *i.e.* the increased protein katabolism, already recorded by Strassmann, Vidal, and others (*loc. cit.*), is shown in Experiments II., IV., and VI., but not in I., III., and V. (see Table I.).

TABLE XV.

Total Nitrogen.

		Before.	After.
Experiment	I.	Average of 4 days, 4·24	Average of 3 days, 4·81
"	II.	" 2 " 4·06	" " 7·26
"	III.	" 3 " 4·11	" " 4·11
"	IV.	" 1 " 5·38	" " 7·60
"	V.	" 2 " 9·62	" " 9·60
"	VI.	" 4 " 4·91	" " 7·77

3. *Urea Nitrogen.*—Without entering upon a discussion of the question of how far the nitrogen determined by Bohland's and by Mörner's and Sjöqvist's methods actually represents the nitrogen elaborated into urea (see *Jour. of Phys.*, xxxii., 1, 1905), these experiments show that, in the absence of amino acids, both give a fair approximation to the true result, and that both methods are of value for comparative results.

The results of these experiments may be tabulated:—

TABLE XVI.

Urea Nitrogen.

	Bohland's.			Mörner's.		
Exp.	Before.	After.	Per cent. Rise.	Before.	After.	Per cent. Rise.
1	84 85	88 87	4	78 81	84 83	. 6
2	...	...	...	78	86	10
3	78 75	86 86	10	74 73·5	86 87	10
4	83·5	84 82	0	83	84 80·5	0
5	88 87	90 90	3	87 87	90 89	3
6	...	...	...	83	85 81	·01

The effect of the administration of chloroform to the extent to maintain full anæsthesia for periods of from one to three hours in the dog is generally, although not always, to cause a rise in the proportion of nitrogen in the form of urea.

4. *Ammonia Nitrogen.*—In Experiments I., II., and III., there was a distinct fall in the proportion of nitrogen in ammonia; in IV. and VI., where the rise in urea nitrogen was absent, there was no change. In V. it was not determined.

5. *Creatinin Nitrogen.*—In the one experiment in which this was investigated (Experiment VI.) there was unfortunately no change in the distribution of the "urea" nitrogen. No very marked change in the amount of creatinin nitrogen could be determined, but it roughly follows the excretion of total nitrogen. There is no distinct fall in the proportion of creatinin nitrogen.

6. *The Purin and Uric Acid Nitrogen* were not investigated in this series.

7. *The Nitrogen not in Urea and Ammonia* showed a fall in Experiments I., II., and III., the experiments in which the urea nitrogen rose, and no marked change in IV. and VI.

8. *Sulphur.*—This was determined only in Experiments V. and VI., where the disturbance in the proportion of urea nitrogen was trivial or absent. The total sulphur ran fairly parallel with the total nitrogen, although the usual divergences were manifest.

The Oxidised Sulphur, both as regards total amount and its proportion to the total sulphur, showed no variation ascribable to the influence of chloroform. A curious tendency for a day of low proportionate excretion to be followed or preceded by a day of high excretion was observed.

In these experiments protein was present in the urine once (Experiment II.) in appreciable amount, and twice (Experiments III. and IV.) in mere traces. In Experiment II. there was a deposit of fatty cells resembling those of the kidney, but not forming tube-casts; and a few similar cells were present in Experiment III.

No reduction of Fehling's solution was obtained except in Experiment III., when a very slight reduction was noted on the day after the administration of chloroform. The reduction was so slight that no attempt was made to determine if it was due to sugar.

In Experiment V. the administration of ether for two hours, although it was followed by a slight albuminuria, caused no marked change in the protein metabolism.

Series II.—Chloroform by the Stomach.

In the next two experiments 20 to 25 c.c. of chloroform dissolved in 100 c.c. of olive oil were administered by the stomach-tube, and as sickness was to be expected, food was withheld on the day of administration, sometimes entirely, sometimes merely till late in the evening. The dog was on both occasions somewhat excited, but showed no signs of anæsthesia.

EXPERIMENT VII.—Setter Bitch. Weight, 18 kilos.

TABLE XVII.

Day.	Urine in c.c.	Sp. G.	Total Nitrogen.	Urea Nitrogen. M. & S.	Uric Acid Nitrogen.	Creatinin Nitrogen.	NH_3 Nitrogen.
1	570	1016	} 5·99	5·28	0·079	...	} 0·128
2	910	1014	} 5·91	5·28	0·091	0·124	} 0·128
*3	410	1016	3·42	2·83	0·049	0·079	0·123
†4	360	1036	6·42	5·28	0·018	0·126	0·44
5	945	1018	12·56	10·24	0·189	0·236	0·96
6	675	1016	6·01	4·46	0·135	0·145	0·527
7	360	1019	4·54	3·50	0·032	0·149	0·316
8	830	1014	6·31	5·20	0·074	0·150	0·594
9	360	1013	3·29	2·70	0·050	0·044	0·134
10	780	1012	5·38	4·54	0·085	0·079	0·260
11	560	1017	5·29	4·40	...	...	...
12	920	1014	...	...	...	...	...

Dog weighed 15 kilos.

On third day dog fasted, and 20 c.c. chloroform in 75 c.c. olive oil given. Dog vomited.*

On fourth day dog fasted, and 20 c.c. chlorofo
given at 1 p.m. Dog took food at night.†

Next day urine dark. No protein. No reductio
Two days later a trace of protein and a large deposi
day no protein reaction.

TABLE XVIII.

Day.	Percentage of Total Nitrogen.			
	M. & S.	Uric Acid.	Creatinin.	]
1	89	1·4	...	
2	89	1·4	...	
*3	82	1·4	2·0	
†4	82	2·8	1·9	
5	82	1·5	1·8	
6	74	2·2 } 1·5	2·4	
7	77	0·7 }	2·3	
8	82	1·1	2·3	
9	84	1·5	1·3	
10	83	1·5	1·4	
11	...	...	...	
12	...	...	...	

TABLE XIX.

Sulphur.

Day.	Total.	Oxidised.	Oxidi. cen Total S
1	0·46	0·30	6
2	1·19	0·79	6
*3	0·81	0·48	5
†4	1·20	0·70	5
5	2·40	1·36	5
6	1·20	0·81	6
7	0·74	0·50	6
8	1·17	0·65	5
9	0·44	0·25	5
10	0·81	0·54	6
11	0·92	0·60	6
12	...	...	.

Experiment VIII.—Same Dog. Weight, 15·4 kilos.

Table XX.

Day.	Urine in c.c.	Sp. G.	Total Nitrogen.	Urea Nitrogen. M. & S.	Uric Acid Nitrogen.	Creatinin Nitrogen.	NH_3 Nitrogen.
1	1080	1011	{ 6·06	{ 5·01	0·05	0·119	{ 0·400
2	565	1012	{ 6·06	{ 5·01	0·05	0·119	{ 0·400
*3	370	1015	3·19	2·77	0·007	0·085	0·249
†4	440	1036	10·30	8·46	0·026	0·149	0·638
5	425	1020	5·46	4·17	0·017	0·100	0·557
6	470	1022	5·21	4·12	0·101	0·14	0·386
7	970	1012	5·77	4·34	0·174	0·12	0·408
8	1020	1011	4·33	3·39	0·091	0·086	0·413
9	670	1013	4·73	3·84	0·080	0·12	0·492
10	490	1011	3·02	2·43	0·039	0·069	0·199
11	655	1012	3·42	2·88	0·052	...	0·182

Dog weighed 14 kilos.

On third day, 25 c.c. chloroform in oil at 1 p.m. Dog vomited.*

On fourth day, 25 c.c. chloroform in oil at 11 a.m. Took food at night.†

On day after urine dark. Gmelin's reaction. Trace of protein. No reduction of Fehling's solution. Deposit of fatty renal cells. Gmelin's test positive for three days. Protein disappeared on second day, but fatty cells and tube-casts present. Five days after chloroform the urine seemed normal.

Table XXI.

Day.	Percentage of Total Nitrogen.				Nitrogen not in these.	
	M. & S.	Uric Acid.	Creatinin.	NH_3.		
1	82	0·8	1·9	5·4	9·9	
2	82	0·8	1·9	5·4	9·9	
3	83	0·2	2·6	7·8	6·4	Chloroform.
4	82	0·2	1·4	6·2	10·2	"
5	76	0·3	1·8	10·2	11·7	
6	71	1·9	2·6	7·4	17·1	
7	75	3·0	2·0	7·0	13·0	
8	78	2·1	2·0	9·5	8·4	
9	81	1·7	2·5	10·4	4·4	
10	80	1·0	2·3	6·6	10·1	
11	82	1·5	...	5·3	...	

1. *Total Nitrogen.*—These experiments show the very marked effects of chloroform administered by the stomach upon the protein katabolism. In both, a very markedly increased excretion of total nitrogen followed the administration of the substance.

2. *Urea Nitrogen.*—A very marked fall in the proportion of urea nitrogen, as determined by Mörner and Sjöqvist's method, is manifest in each experiment, reaching its maximum on the third day. In Experiment VII. it amounts to 17 per cent., and in Experiment VIII. to 13 per cent. It is to be regretted that the conformity of the results obtained by the two methods of determining the urea nitrogen in the previous series of experiments decided me to employ one only in the present set.

3. *Ammonia Nitrogen.*—A rise in the proportion of ammonia nitrogen accompanies the fall in urea nitrogen, but the maximum of the one does not always occur on the same day as the minimum of the other (figs. 1 and 2, p. 493). In both experiments the maximum proportion of ammonia nitrogen is reached two or three days after the minimum proportion of urea nitrogen.

4. *Creatinin Nitrogen.*—The total excretion of creatinin nitrogen in these experiments follows fairly closely the excretion of total nitrogen. Hence the *proportionate* excretion does not undergo very marked variations, but in both experiments the rise in creatinin nitrogen is not proportionate to the rise in total nitrogen on the day after the administration, and hence the proportion falls somewhat.

5. *Uric Acid Nitrogen.*—In both experiments there is an increase in the total amount of uric acid nitrogen excreted, but this occurs earlier, on the day following the chloroform, in Experiment VII., and later, three or four days after, in Experiment VIII. In Experiment VII. the proportionate increase appears on the day after the administration, and in Experiment VIII. not until the third day and after.

6. *The Nitrogen not in Urea and Ammonia*, and the nitrogen not in urea, ammonia, creatinin, and uric acid, show a rise after the administration of chloroform.

7. *Sulphur.*—The excretion of sulphur in Experiment VII. runs parallel with the excretion of total nitrogen. The proportion of oxidised sulphur is decreased after the administration of chloroform, but is not lowest on the day of the lowest proportion of urea nitrogen.

In both experiments, proteins, granular cells, and tube-casts were present in the urine. There was no reduction of Fehling's solution. In Experiment VIII. bile-pigments appeared in the urine, and the animal was slightly jaundiced.

Series III.—Hypodermic Administration.

Two experiments were performed upon dogs of 18 and 15 kilos., 5 c.c. of chloroform being administered in Experiment IX. on two successive days, and in Experiment X. twice on the same day. The animals showed no sign of intoxication, but in Experiment X. the dog took only half its food on the day of administration and the next day, and the milk only on the following day.

Experiment IX.—Same Dog. Weight, 18 kilos.

Table XXII.

Day.	Urine in c.c.	Sp. G.	Total Nitrogen.	Urea Nitrogen. M. & S.	Uric Acid Nitrogen.	Creatinin Nitrogen.	NH_3 Nitrogen.
1	335	1028	5·26	4·62	...	...	...
2	640	1018	6·92	6·36	...	0·22	...
3	745	1015	6·55	5·68	·007	0·25	0·238
*4	910	1012	6·24	5·24	·000	...	0·439
†5	165	1026	} 6·03	} 4·98	} ·034	} 0·24	} 0·369
6	690	1022	} 6·03	} 4·98	} ·034	} 0·24	} 0·369
7	610	1017	5·71	4·68	·048	0·13	0·277
8	740	1014	3·64	2·60	·059	0·15	0·305
9	515	1016	5·29	4·23	·046	0·20	0·420
10	680	1017	6·38	5·40	·040	0·24	0·313
11	580	1015	4·68	4·06	·029	0·10	0·288
12	775	1011	4·16	3·50	·023	0·14	...
13	500	1010	2·83	...	...	0·09	...
14	730	1017	5·72	...	...	0·15	...

Dog weighed 15·8 kilos.

On fourth day, 5 c.c. chloroform.*

On fifth day, 5 c.c. chloroform.†

From second day of chloroform to third day after, slight reduction of Fehling's solution on cooling.

From next day to fourth after, protein present; on third day, deposit of fatty cells.

TABLE XXIII.

Day.	Percentage of Total Nitrogen.				Nitrogen not in these.
	M. & S.	Uric Acid.	Creatinin.	NH_3.	
1	87	...	...	...	...
2	91	...	3·2	...	...
3	86	0·1	3·3	3·6	6·5
*4	84	0·0	...	6·8	...
†5	82	0·5	3·9	6·1	7·5
6	82	0·5	3·9	6·1	7·5
7	81	0·8	2·3	4·8	11·0
8	72	1·6	4·2	8·3	13·9
9	80	0·8	3·7	7·9	7·6
10	84	0·6	3·9	4·9	6·6
11	87	0·6	2·1	6·8	3·5
12	84	0·5	3·3	...	...
13	...	...	2·6	...	...
14	...	...	...	...	...

TABLE XXIV.

Sulphur.

Day.	Total.	Oxidised.	Oxidised per cent. of Total Sulphur.
1	0·782	0·624	79
2	0·927	0·734	79
3	0·940	0·741	79
*4	0·940	0·628	66
†5	0·940	0·628	66
6	1·08	0·721	66
7	0·968	0·583	60
8	0·940	0·618	66
9	0·707	0·466	66
10	0·796	0·549	68
11	0·755	0·556	73
12	...	...	...
13	...	...	...
14	...	...	...

EXPERIMENT X.—Collie Bitch. Weight, 15·4 kilos.

TABLE XXV.

Day.	Urine in c.c.	Sp. G.	Total Nitrogen.	Urea Nitrogen. M. & S.	Uric Acid Nitrogen.	Creatinin Nitrogen.	NH_3 Nitrogen.
1	400	1012	2·26	1·84	...	0·08	0·076
2	880	1016	3·98	3·28	...	0·16	0·347
3	460	1010	2·20	1·39	...	0·09	0·140
9	530	1007	1·51	1·38	0·00	0·05	0·066
10	640	1010	3·88	2·45	0·00	0·10	0·238
*11	770	1012	5·56	...	...	0·18	0·409
12	275	1017	4·65	3·91	0·018	0·11	0·608
13	550	1016	4·65	3·91	0·018	0·11	...
14	...	...	...	...	...	...	...
15	610	1017	6·36	5·24	0·073	0·20	0·456
16	730	1010	3·75	2·96	0·051	0·11	0·419
17	550	1010	3·61	2·11	0·022	0·08	0·148
18	700	1010	4·21	3·43	0·028	0·12	0·329

Dog weighed 14·7 kilos.

On eleventh day, 5 c.c. chloroform at 11, and again at 5.*

From the day of chloroform to fourth day after only part of food was taken.

On second day after, trace of protein, present till eighteenth day; on fourth day, 0·07 per cent. by Esbach's method.

Slight reduction of Fehling's solution from second to fourth day after.

TABLE XXVI.

Day.	Percentage of Total Nitrogen.				Nitrogen not in these.
	M. & S.	Uric Acid.	Creatinin.	NH_3.	
1	83	...	3·9	3·3	...
2		...	4·0	8·6	...
3		...	4·0	6·3	...
9	84	0·0	3·3	4·4	7
10		0·0	3·2	7·1	
*11	...	...	3·2	7·4	...
12	83	0·4	2·4	6·5	8
13	...	0·4	...	...	...
14	...	...	...	...	...
15	82	1·1	3·1	7·1	7
16	79	1·3	2·9	11·3	5·5
17	81	0·6	3·0	5·8	9·6
18	81	0·6	2·8	7·9	7·7

1. *Total Nitrogen.*—These two experiments show no definite change in the total nitrogen excreted, although in the second experiment a possible rise is masked by the fact that the dog did not take all its food.

2. *Urea Nitrogen.*—In Experiment IX. there is a distinct fall in the proportion of urea nitrogen during three days after the administration of chloroform, reaching its maximum, 18 per cent., on the third day. In Experiment X. the fall is slight, less than 5 per cent., and is delayed till the fifth day.

3. *Ammonia Nitrogen.*—In Experiment IX., on the third day, the day of the most marked fall in urea nitrogen, there is a rise in the proportion of ammonia nitrogen, and the same thing occurs in Experiment X. on the fifth day.

4. *Creatinin Nitrogen.*—In neither experiment is there any marked change in the proportion of creatinin nitrogen. The total amount excreted varies fairly directly with the excretion of total nitrogen.

5. *Uric Acid Nitrogen.*—In Experiment IX. the *proportion* of uric acid nitrogen is markedly raised, especially on the day of the lowest excretion of urea nitrogen. In Experiment X. the change is not evident. In both experiments there is a distinct increase in the *total excretion* of uric acid nitrogen; in the first, upon the third day after the administration of chloroform, and in the second, upon the first and second days after.

6. *Nitrogen not in these Combinations.*—In the first experiment there is a distinct rise upon the second and third day after the chloroform. In the second experiment a very doubtful rise occurs upon the sixth day.

7. *Sulphur.*—The sulphur was estimated in the first experiment only. The proportion of oxidised sulphur shows a marked decrease, beginning on the day after the chloroform, and reaching its maximum on the third day after—the day of the greatest fall in urea nitrogen.

In both these experiments proteins were present in the urine, and in the first fatty cells resembling renal epithelium were seen. They were unfortunately not examined for in the second experiment. In neither experiment was there any marked reduction of Fehling's solution.

V.—Consideration of Results.

These experiments show that in the dog chloroform, *when given by the respiratory passages*, for two or three hours has either no effect upon the nitrogenous metabolism or increases the protein disintegration, as already demonstrated by previous investigators. The effect upon the nitrogen in urea was either to leave it unaffected or *to increase its proportion*, while, at the same time, the nitrogen in ammonia was either unchanged or *decreased* when the urea was increased. These results suggest that the hepatic metabolism, by which ammonia compounds are changed to urea, is stimulated—an effect very similar to that produced by the administration of a protein diet (*Jour. of Phys.*, vol. xxv., p. 443, 1900)—and that this stimulation, in these experiments at least, is not followed by any after-depression.

The toxic action of the drug upon the kidneys is manifested by the presence of proteins and of renal epithelium in the urine, most markedly in Experiment II., where the animal was anæsthetised for two hours, and again for one hour.

When chloroform is given *by the stomach* the effect is very different. The protein disintegration, as indicated by the excretion of nitrogen, is markedly increased, and this increase is best marked on the day of, or the day after, the administration of the drug. At this time the disturbance in the proportion of nitrogen in the various compounds is not very pronounced, and the presence of protein in the urine is not so distinctly marked as it afterwards becomes. There is apparently a period of simple increase of the protein disintegration.

After two or three days there occurs a very marked fall in the proportion of nitrogen in urea, with a rise in the proportion of nitrogen in ammonia—a rise which, however, is not inversely proportionate to the fall in urea nitrogen, and which is somewhat delayed (figs. 1 and 2). Along with this there is a rise in the proportion of uric acid nitrogen, which apparently may precede (Experiment VII.) or succeed (Experiment VIII.) the most marked fall in the urea nitrogen. The proportion of nitrogen not in these analysed compounds increases concomitantly with the fall in the urea, and previously to the rise in the ammonia. In both the experiments the rise was a very marked one, corresponding to something like 6 per cent. of the total nitrogen. Since the urea nitrogen was determined by the method of Mörner and Sjöqvist, the amino acids were precipitated, and not estimated with the urea, and hence almost certainly this increase in the undetermined nitrogen is due to the appearance of amino acids. It is to be regretted that, since the previous examination of the urine after the administration of chloroform by the respiratory passages had shown so close a correspondence between the results of the methods of Bohland and of Mörner and Sjöqvist, the former method was abandoned in the present series of analyses, for a divergence between these would have more definitely proved that the undetermined nitrogen is really in amino acids. That the rise in the proportion of undetermined nitrogen was not due to the appearance of protein is indicated, first, by the fact that in Experiment VII. the rise occurred before the appearance of the protein, and, second, by

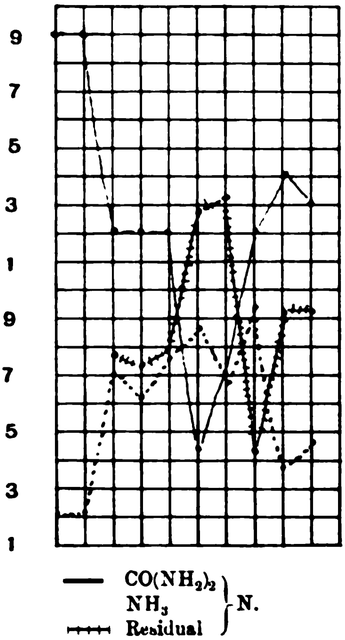


FIG. 1.—Experiment VII.

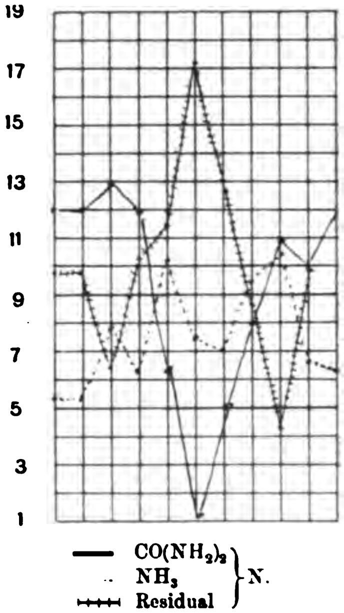


FIG. 2.—Experiment VIII.

the fact that the amount of protein was insufficient to account for the nitrogen thus excreted.

The proportion of unoxidised sulphur to the total sulphur (Experiment VII.) shows, like the nitrogen in urea, a distinct fall, curiously interrupted by a rise on the two days on which the proportion of urea nitrogen was lowest. I am unable to explain this.

The whole series of changes is such as would be produced by a direct toxic action on the liver, similar to that produced by the administration of alcohol and sulphonal (*Jour. of Phys.*, xxvi., p. 166, 1901). The effect is not immediate, but is nevertheless the result of the poisoning of the cells. This is clearly shown by the histological changes produced in the liver cells by giving chloroform by the mouth, already described by Doyon and now being investigated by Dr Clark. The effect upon the kidney, as indicated by the presence of protein and the appearance of renal epithelium in the urine, is much more marked when chloroform is given by the mouth than when administered by the respiratory passages. Zweifel (*Berl. klin. Wochensch.*, 1874, p. 245) and Pohl (*Arch. f. exp. Path. u. Pharmac.*, Bd. xxviii., p. 251) found traces of chloroform in the urine, and Nicloux (*Jour. de Pharm. et Chim.*, 24 [6], p. 64) has found that the amount after administration by the respiratory passages is very small. But the fact that protein appeared in the urine in certain of my experiments in which it was given in this way without exerting a true toxic action on the liver, and the very frequent occurrence of transient albuminuria described by many surgeons, seems to indicate that its effect on the kidney is direct.

The question naturally arises of how far these differences in the effect of chloroform given by the respiratory passages and by the mouth are merely a result of difference of dose. The amount of chloroform in the blood necessary to produce anæsthesia has now been investigated by several observers—Tissot (*C.R. Soc. biol.*, 1906, p. 198), Nicloux (*C.R. Soc. biol.*, t. lx., p. 144), Buckmaster (*Proc. Roy. Soc.*, vol. lxxix., 1907)—and the general result is to indicate that, to produce the state, something like 30 to 50 mg. per 100 c.c. must be present in the blood of the dog or cat. After the administration is stopped, the amount rapidly falls, and the elimination is practically complete in about three hours (see succeeding paper).

So far no determinations of the amount of chloroform in the blood after it has been administered by the mouth have been recorded, but a series of experiments carried out by Miss D. Lindsay upon the blood of rabbits after the administration of chloroform in oil, nearly equivalent to that given by me to the dogs experimented upon, shows that the percentage amount never rises so high as during administration by the lungs, and that the drug persists in the blood for a much longer period—about six hours.

The greater effect upon the metabolism of chloroform when administered by the mouth is not the result of a larger dose, but must be due rather to the direct influence upon the liver, and to the more prolonged action.

In rabbits, I have found that the administration of 1 c.c. by the mouth, which leads to the appearance of at most between 20 and 30 mg. per 100 of $CHCl_3$ in the blood, is infinitely more toxic than a prolonged administration by the lungs. Of 12 rabbits, no less than 11 showed albuminuria, while 7 died, and 2 which appeared to be dying were killed.

How far the fatal result is due to the injury to the kidneys, how far to the toxic action on the liver, and how far to the more general injury to the tissues produced, we have no data to enable us to determine. The appearance of acetone and diacetic acid in the urine shown by various investigators is probably a mere expression of this disturbed metabolism, whereby the β oxybutyric acid set free does not undergo its normal katabolism, but is converted into acetone. It is probably secondary to the hepatic change, and is an accompaniment, not the cause, of the death of the animal.

A study of the effects of *hypodermic* administration of the drug throws some light upon its mode of action. With a dose in two portions of 5 c.c.

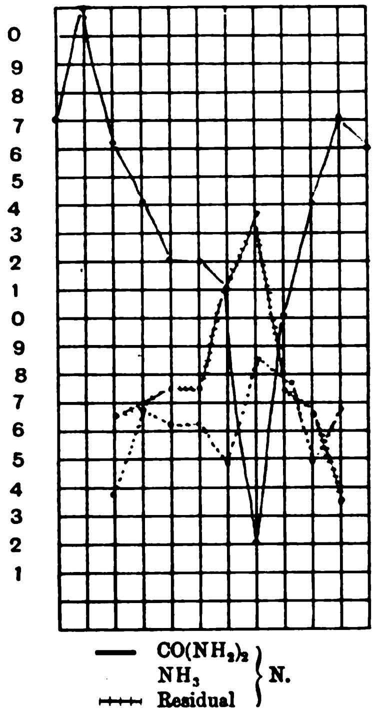


FIG. 3.—Experiment IX.

there was no definite change in the excretion of nitrogen; but in Experiment IX. there was a sharp fall in the urea nitrogen, with a corresponding rise in the amino acid nitrogen, somewhat preceding a rise in the ammonia nitrogen (fig. 3). In Experiment X. the changes were less marked, but in the same direction.

Given hypodermically, chloroform, then, acted in the same manner as when given by the mouth, but to a lesser degree. How far this lesser degree is due to the smaller dose administered it is difficult to say. Miss Lindsay's observations on rabbits seem to show that the administration of 1 c.c. of chloroform by the mouth leads to the appearance of a higher percentage of chloroform in the arterial blood than does the administration of the same amount hypodermically; and hence dogs which got two doses of 25 c.c. of chloroform by the mouth may have had a higher proportion of the drug in the circulating blood than those which had two doses of 5 c.c. hypodermically. The respect in which the hypodermic administration resembles the administration by the mouth is in the persistence of the drug.

The hypodermic administration of chloroform to rabbits appears to be almost as fatal as the administration by the mouth, and the appearance of proteins and tube-casts in the urine is almost as frequent. Dr Clark also finds that the changes in the liver are as marked as when the drug is given by the stomach.

It would seem that the acute transitory action of chloroform administered by the respiratory passages leads to a stimulation of hepatic metabolism, while the less acute and more sustained action leads to injury without previous stimulation.

Although in these experiments the administration of chloroform by the respiratory passages failed to cause the grave disturbances, metabolic and structural, which were caused by its administration by other channels, the results of such administration by other channels seem to show how, in the event of the chloroform given by the lungs not being eliminated with the usual rapidity, serious injuries might be effected, which would lead to the symptoms of late chloroform poisoning.

Moore and Roaf (*B.M.J.*, vol. ii., 1906, p. 721) have shown how chloroform is fixed to the proteins of the blood, and Nicloux (*C.R. Soc. biol.*, t. lx., pp. 206, 248, 1906; t. lxii., p. 1153, 1907) has also shown the way in which it is anchored to certain tissues. Sherrington and Sowton have proved that the action of chloroform upon the heart depends upon its tension in the circulating fluid, and that it is greater in saline solutions than in the protein-containing blood where it becomes fixed.

These observations, taken in conjunction with our results, seem to show that chloroform, when given by the respiratory passages, is rapidly taken up and first dissolved in the blood, and in this condition acts upon the nerve-centres, the excess being rapidly eliminated; while, when given by the stomach or hypodermically, the assumption is slow, more stable compounds are formed, the elimination is consequently delayed, and the drug has thus more time to produce a slow toxic effect upon the protoplasm of the tissues.

The onset of late chloroform poisoning after anæsthesia would thus be due to delayed elimination brought about by unusually firm fixation, or by respiratory deficiency.

(*Issued separately* August 15, 1908.)

XXXIII.—On the Rate of Elimination of Chloroform when Administered by Different Channels.

By Dorothy E. Lindsay and D. Noël Paton, M.D. (*From the Physiological Laboratory of the University of Glasgow.*)

(Read June 15, 1908. MS. received June 18, 1908.)

One of us (D. N. P.) has shown in the preceding paper that chloroform, when administered to dogs by the respiratory passages, may produce a purely stimulating action upon hepatic metabolism; whereas when it is given by the mouth, and to a less extent when it is given hypodermically, it exercises a distinctly toxic action, decreasing the activity of hepatic metabolism and leading to degenerative changes in the liver cells.

In attempting to find an explanation of these phenomena, it was necessary to know whether any difference existed in the distribution of the drug throughout the system after it had been given by these channels.

To elucidate this, the following series of observations were undertaken upon rabbits.

D. Noël Paton is responsible for the carrying out of the experiments, and Dorothy E. Lindsay is responsible for the determination of the amount of chloroform in the blood.

Nicloux's * method for the chloroform in the blood was used throughout the investigation (*Bul. de la Soc. Chimique de Paris*, 3rd series, 1–33, p. 321, 1906). The blood was collected in alcohol (95 c.c.'s alcohol, 5 c.c.'s of 5 per cent. tartaric acid solution) and distilled, using Nicloux's apparatus. The distillate was transferred to a flask and boiled with a 10 per cent. solution of alcoholic potash for half an hour with a reflux condenser. After neutralisation with nitric acid, the chlorine was estimated by titration with silver nitrate, using potassium chromate as indicator. Several duplicate estimations were made with Volhardt's method, and the results were found to agree very closely, so that the other method was used as a rule.

* We are indebted to Monsieur Nicloux for sending us his apparatus.

The following table gives results obtained when working with known quantities of chloroform:—

Amount $CHCl_3$ taken.	Amount $CHCl_3$ found.	Percentage.
11·9 mgrs.	11·8 mgrs.	99·1
12 „	11·9 „	99·1
12 „	11·85 „	98·75
49·8 „	49 „	98·3

Two or three times during the course of the investigation blood was taken from animals to which no chloroform had been given, and in each of these cases negative results were obtained.

Chloroform by the Respiratory Passages.

The taking up and elimination of chloroform from the blood when given by the respiratory passages has been investigated by several observers.

Gréhant and Quinquand (*C. R.*, 1883, p. 753) found that to produce anæsthesia in dogs the chloroform in the blood must amount to something like 50 mgrs. per 100 c.cm. These observations were confirmed by Nicloux (*C. R. Soc. biol.*, T. lx. p. 144), who further showed that after the administration was stopped the elimination went on, at first rapidly and then more slowly, and that even after three hours 7·5 mgrs. might be present.

Buckmaster and Gardiner (*Proc. Roy. Soc.*, vol. lxxix., 1907), working upon cats, found that the amount of chloroform was about 20 or 30 mgrs. per 100 c.c.'s when anæsthesia, as indicated by the corneal reflex, was produced. They also showed that the chloroform is first rapidly and then more slowly taken up by the blood until a state of equilibrium between assumption and elimination is established, so that continued administration does not necessarily further increase the percentage of chloroform in the blood.

Brodie and Widdows (*Brit. Med. Journ.*, vol. xi., 1906, p. 79) have recorded somewhat similar results.

Tissot (*C. R. Soc. biol.*, T. lx., 1906, p. 198) shows that the proportion of chloroform in the blood at the onset of anæsthesia depends upon the rate at which it is given. If the animal is anæsthetised, in three or four minutes the amount may be as much as 60 or 70 mgr. per 100 c.c., but if the chloroform is given drop by drop and the anæsthesia slowly produced, the amount may be only 34 or 35 mgr. or even less per 100 c.cm. As he shows, the anæsthesia is determined by the amount fixed in the brain rather than by the amount in the blood.

Since these observations had been made on dogs and cats, and since rabbits were the most convenient animals for the study of the question when the chloroform was given by the stomach and hypodermically, it was considered desirable to reinvestigate the subject on these animals.

The rabbits were chloroformed by placing a towel saturated with chloroform on the face till full anæsthesia was produced. They were then either killed immediately or allowed to live for a varying period. Death was produced by a blow behind the head, and the blood, about 20 c.c., was collected by cutting the carotid and receiving the blood in a measured volume 100 c.c.'s of the alcohol and tartaric acid, used by Nicloux. The following results were obtained:—

Kept under for	No.	Weight.	Time after Adminis-tration.	Amt. of $CHCl_3$ in Blood.	Remarks.
10 mins.	XXII.	1660 grms.	0	77·3 and 78	Respiration stopped.
5 ,,	XXIII.	1500 ,,	0	61	Respiration and heart stopped, cyanosis.
3 ,,	LXII.	800 ,,	0	30·8	Full anæsthesia.
	LXIII.	...	0	46·4	,, ,,
5 ,,	LXIV.	800 ,,	0	58·	,, ,,
	XXIX.	1820 ,,	25 minutes	25·9	Conjunct reflexes restored, others not. Drowsy, difficult to rouse.
15 ,,	XXVIII.	1600 ,,	1 hour	16·6	
10 ,,	XLII.	1150 ,,	1 ,,	15·4	Drowsy and staggery.
,, ,,	XLIII.	1200 ,,	1 ,,	13·2	Normal.
,, ,,	XXXIV.	1050 ,,	1 ,,	3·3	Wild rabbit, complete recovery.
,, ,,	LIX.	...	2 hours	18·	Kept in cage.
,, ,,	LXVII.	1400 ,,	2 ,,	19·4	Kept in cage; still sluggish.
,, ,,	LX.	2510 ,,	2 ,,	4·5	Left free; lively.
15 ,,	LVII.	500 ,,	2 ,,	4·7	
10 ,,	LXV.	...	2¾ ,,	21·4	Kept in cage.
,, ,,	LV.	2250 ,,	3 ,,	7·4	
,, ,,	LIV.	1700 ,,	3¼ ,,	10·	Breathing stopped during anæsthesia.
15 ,,	LVIII.	450 ,,	4 ,,	0·	

When our results were tabulated, the very varying amounts of chloroform found in the blood at the end of two and three hours puzzled us considerably, but upon investigation we found that in the more recent experiments, in which the proportion of chloroform was so high, the rabbit had been placed in a very small cage in the research-room without any possibility of moving about. To test if this might explain the results obtained, two rabbits were chloroformed alongside one another by the same operator and to the same extent for ten minutes. One was then put in the small cage, and the other in a large stall where it moved about freely. Both were killed at the end of two hours. In the first, 19·4 mg. of chloroform per 100 c.cm. of blood were found; in the second, only 4·5 mg. (LXVII. and LX.). This observation suggests the importance of adequate respiration in the post-anæsthetic condition.

Chloroform by the Stomach.

The rabbits were anæsthetised with ether, and a catheter was passed down the œsophagus and a measured quantity (10 c.c. = 1 c.c. chloroform), of a 1 in 10 mixture of chloroform in olive oil was run into the stomach. The animals were killed at varying periods after the administration of the drug.

No.	Weight.	Time after Administration.	Amount of $CHCl_3$ in Blood.	Remarks.
IV.	2100 grms.	35 minutes.	11·3 and 11·8	
XL.	1300 „	3 hours.	32·5	Very staggery.
XXXVII.	1100 „	4¼ „	16·	Recovered immediately.
XXX.	1620 „	4½ „	25·	
XLI.	950 „	5½ „	19·3	2¼ hours after, staggery. 5½ hours after, recovered completely.
XLVI.	1900 „	6¼ „	14·9	
LIII.	1550 „	6½ „	4·	

It appeared to us that very possibly with the mode of administration an accumulation of chloroform might take place in the liver. Two or three observations were made upon the amount of chloroform in that organ:—

No.	Time.	Liver.	Blood.
XL.	3 hours	36·2	32·5
XXX.	4½ „	21·3	25

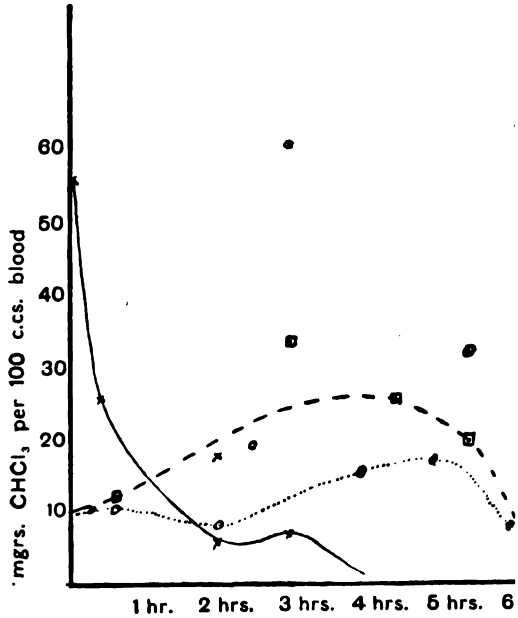
Tissot (*C. R. Soc. biol.*, 1906, p. 205) and Nicloux (*C. R. Soc. biol.*, 1906, p. 208) have found that when chloroform is administered by the respiratory passages the percentage in the liver is far below that in the blood, and this conclusion we can confirm from our own observations. In one rabbit, chloroformed for ten minutes and killed a few minutes later, when just recovering from the anæsthetic, the blood contained 31·4 mg. per cent. and the liver only 15·2 mg. per 100 cm. Hence our results help to explain the more marked action of chloroform on the liver when administered by the mouth.

CHLOROFORM HYPODERMICALLY

1 c.c. of chloroform was injected under the s animals were killed at varying periods after.

No.	Weight.	Time after Administration.	Amount of $CHCl_3$ in Blood.	
II.	2000 grms.	10 minutes.	10 and 11	
I.	2000 „	20 „	9·4 and 9·3	
III.	2300 „	30 „	10·4 and 12	
X.	1550 „	1 hour.	10·6 and 12·9	
XIV.	2150 „	2 hours.	7·69 and 7·7	
XVI.	2140 „	2½ „	18·7	Abs a
XXV.	1250 „	3 „	60·8	Res
XIX.	2000 „	4 „	12·5	
XXXVIII.	1400 „	4 „	15·8	
LXI.	1800 „	4 „	22·2	Dro
XXXIII.		5 „	18·6	
XLV	2020 „	5 „	16 and 16·6	
XXXIX.	1000 „	5½ „	31·6	Sha
L.	2250 „	6 „	8·5	

These results, leaving out from the curves the ex be formulated in the figure :—



Chloroform administered by respiratory passages ——
„ „ hypodermically .....
„ „ by stomach —

In experiment LXI. the chloroform in the liver was also determined, and 27·7 mg. per 100 grm. was found to be present, showing a fixation of chloroform to that organ under hypodermic as well as under gastric administration.

Conclusions.

These observations show that, in the rabbit, chloroform when given by the respiratory passages is rapidly taken up by the blood, and that anæsthesia is produced when the amount reaches about 30 to 40 mg. per 100 c.cm.

It is then rapidly eliminated, so that, in normal cases, by the end of two hours it has almost entirely disappeared from the blood. This elimination, however, is subject to marked variation, and at the end of two hours there may still remain in the blood no less than 20 mg. of the drug, and even after 3¼ hours 10 mgrs. may remain.

These experiments seem to explain the occurrence of late chloroform-poisoning in a certain number of cases where the drug remaining in the body has acted on the tissues for a prolonged period.

When given by the mouth, chloroform is slowly absorbed, the percentage amount reaching its maximum between four and five hours after administration, and then slowly disappearing. The amount taken up by the blood after the administration of about 1 c.c. per 1000 grm. is rarely sufficient to produce anæsthesia—in only one case did it reach 32 mgrs., and in that case the animal was not anæsthetised, but merely staggery. In all probability, the fixation of the drug by the proteins and corpuscles prevented its full action on the nerve centres.

When chloroform is given hypodermically, absorption is generally more rapid than when it is given by the mouth, and the percentage in the blood reaches its maximum in about four or five hours. The amount present is rarely sufficient to produce anæsthesia, but in one case it was produced when only 18·7 mgrs. were present. In one case when the respiration and heart stopped, no less than 60·9 mgrs. were found in the blood at the end of three hours; in another, at the end of 5½ hours 31·6 mgrs. were present, and yet no anæsthesia, but merely staggering, was observed.

These observations show that the more marked action of chloroform on the metabolism when it is administered by the mouth or hypodermically is probably to be explained by the more prolonged action of the drug upon the protoplasm. The more marked action of the drug when administered by the mouth, as compared with the action of the same dose hypodermically administered, is probably due to the more direct action upon the liver cells, and possibly to its less rapid elimination, indicated by its more marked accumulation (see figure).

(*Issued separately August* 21, 1908.)

XXXIV.—Algebra after Hamilton, or Multenions. By Alex. M'Aulay, M.A., Professor of Mathematics and Physics, University of Tasmania, Hobart. *Communicated by* Professor C. G. Knott.

(First MS. received December 1906. Read March 4, 1907. Supplement received from Tasmania with revision of first proof, June 1908.)

Summary.

Ever since I have learnt something of the meanings of Grassmann's Ausdehnungslehre, and have at the same time learnt to regard the beauties of that system with something akin to awe, I have been persuaded that on the lines of Quaternion Algebra there is to be built a system very much like the Ausdehnungslehre, but an improvement thereon. Of course it will be matter for differing opinions whether what I call Multenions is really an improvement on the Ausdehnungslehre. I here record my own personal opinion that it is.

I do not suppose that anybody will maintain that a multitude of different kinds of multiplication within the bounds of one method can be regarded as anything but a blemish,—a blemish that may be justified by necessity and utility. The Ausdehnungslehre seems to me to have this blemish, and Multenions not to have it. Whether along with the absence of the blemish there becomes present an additional difficulty of manipulation is questionable. I have not found it so, but this may be due to the fact that I have so long been in the habit of thinking through the quaternion machinery.

I think these general remarks are all that can be usefully set down as a summary. The paper itself appears to be too condensed to admit of a true summary other than a mere table of contents.

This paper is to be regarded as a preliminary outline of what is necessarily a large subject, which demands time and labour for due development. It is based on the work of Hamilton, Grassmann, and their followers. Their treatises will, however, be referred to but scantily. Chapter iv. of Octonions, which I propose to regard as virtually a part of the present paper, contains copious references to the source, Ausdehnungslehre, from which it was practically derived.

List of some of the terms used below:—Fictor, fictit, fictorplex, continent fictorplex. Multenion, multit, multiplex, continent multiplex. Fictorlinity, fictorcolinity, multilinity. Replacement, retroplacement, rigid replacement, unireplacement. Complement, conjugate, reversate and reciprocal of a multenion. "Sequence" is used to prevent confusion with "order" in the technical sense. The conventions as to notation are as far as possible those of Quaternions as adopted by Hamilton.

[*April* 1908].—The lack, in Tasmania, of all mathematical literature bearing on Matrices, has made it impossible to check to what extent many of the results are novel or simply in different guise.

1. **Introductory illustration.**—I was recently led back to the present work by facing a problem of mere laborious calculation in the method of least squares, and this method will probably serve as an indication of the kind of inquiry in which Multenions may be expected to be of practical utility.

In the method, a supposed set of values of the unknowns is contemplated in various ways:—(1) The true unknowns are often symbolised; (2) a particular set called the most probable set is mainly considered; (3) a quite arbitrary set is sometimes treated of. In all these three ways we look upon a set as a whole, as a sort of individual belonging to a community of individuals. In Multenions we use a single symbol, a, to denote such a set, and call a a fictor. If the unknowns are denoted by $x, y, \ldots$ we may say that $a=(x, y, \ldots)$. The particular value of a for which $x=1$ and $y=z=\ldots=0$ is called a fictit (denoted by ι_1); and the value for which x is any scalar and $y=z=\ldots=0$ is said to be x times the fictit ι_1 just mentioned. More generally, any fictor $a=(x, y, z, \ldots)$ is put as the sum $x\iota_1+y\iota_2+\ldots$, where $\iota_1=(1, 0, 0, \ldots)$, $\iota_2=(0, 1, 0, \ldots)$, etc.; or

$$(x, y, z, \ldots)=x(1, 0, 0 \ldots)+y(0, 1, 0, \ldots)+\text{etc.}$$

Multenions is based on *fictits.*

2. **The Laws, Symbols, and Parts of Multenions.**—The laws differ from Grassmann's (Ausdehnungslehre) in some important respects. The product of two units (fictits or multits) is never zero, so that fictit products are not "combinatorial." There is but one mathematical meaning of product. Grassmann's various products are "parts" of our one product. Hence we speak of combinatorial *parts*, but not of combinatorial *products.*

Before enunciating what we are to consider as fundamental laws, it is well to state that in § 3, below, an extension of these laws will be considered. The extension is of the nature of an alternative or permissible simplifying restriction, and will, in the subsequent part of the paper, be accepted or rejected as we see fit.

[The references to the odd number N in the fundamental laws might be omitted without any serious alteration of what follows in the present paper. We almost invariably suppose our symbols confined to a given "multiplex" of order n less than N. The reason for introducing N is to make quaternions a particular symmetrical case of our subject, multenions, namely, when N = 3. We shall generally, however, in our references to quaternions make the mathematically equivalent assumption that quaternions is a particular *unsymmetrical* case of our present subject obtained by putting $n=2$. We thus suppose quaternions based on *two* fictits, i, j; the third (vector but not fictor), k, being defined as the *fictor product*, ij.] [*April* 1908.—For final views regarding (1) and (2) see end of Supplement.]

Fundamental Laws of Multenions.

(1) There are given an odd number (N) of primitive units (here called **fictits**), $\iota_1, \iota_2 \ldots$, besides the unit **scalar** 1. [Sometimes, below, the scalar 1 is classed as a multit (that is a fictit product) and sometimes as a unit apart; I think the context will always serve to prevent inconvenience which might result from the ambiguity.] These are such that all the laws of ordinary algebra except the commutative law for multiplication apply to the expression (here called a **multenion**)

$$\Sigma(x_0 + x_1\iota_1 + x_2\iota_2 + \ldots\ldots)(x'_0 + x'_1\iota_1 + x'_2\iota_2 + \ldots\ldots)\ldots\ldots,$$

where Σ has its usual significance, and every x is a scalar.

(2) Scalars are commutative not only with each other but with fictits. [It at once follows, and will always be assumed without proof or reference, that scalars are commutative with multenions.]

(3) $\iota_1^2, \iota_2^2, \ldots\ldots$, and also the product $\iota_1\iota_2 \ldots\ldots \iota_N$ of all the fictits, are all *scalars differing from zero.*

(4) In a product $\iota\,\iota'\iota'' \ldots\ldots$ (here called a multit) of fictits, if two adjacent *different* fictits be interchanged the product changes in sign, but otherwise remains unchanged. [It is by reason of the word "different" that a multit is *not* a combinatorial product of its constituent fictits.]

(1), (2), (3), (4), and not what follows, are regarded as our fundamental laws.

The multenions $q_1, q_2 \ldots\ldots$ are said to be (1) **independent** or (2) not independent according as (1) Σxq is only zero ($x_1, x_2 \ldots\ldots$ being scalars) when every x is zero, or (2) $\Sigma xq=0$ for some values of $x_1, x_2 \ldots\ldots$ not all zero.

An expression of the form $\Sigma x\iota$ is called a **fictor**. If $a_1, a_2, \ldots\ldots a_n$ are n given independent fictors, all fictors of the form Σya are said to form a **fictorplex** of order n, and it is called the fictorplex $a_1\, a_2 \ldots\ldots a_n$. [As this is liable to be confused with the **fictor-product** $a_1\, a_2 \ldots\ldots a_n$, which is a multenion, it is safest always to use the word "fictorplex" when $a_1\, a_2 \ldots\ldots a_n$ is used in the latter sense.] The fictorplex $\iota_1\, \iota_2 \ldots\ldots \iota$ of all the fictits is called the **continent fictorplex**; and it is best to understand by "fictorplex," unless the context obviously implies the contrary, always a fictorplex of lower order than N.

If $a_1, a_2, \ldots . a_n$ are n given independent fictors, all multenions of the form

$$\Sigma(x_0 + x_1 a_1 + x_2 a_2 + \ldots .)(x'_0 + x'_1 a_1 + x'_2 a_2 + \ldots .) \ldots .$$

are said to form a **multiplex** of order n, and it is called the multiplex $a_1 a_2 \ldots . a_n$. The multiplex $\iota_1 \iota_2 \ldots . \iota_N$ of all the fictits is called the **continent multiplex.**

[First N.B.—A fictorplex is a complex of given fictors; but a multiplex is not defined as a complex of given multenions, as the last phrase would naturally mean all multenions of the form $x_1 q_1 + x_2 q_2 + \ldots .$ where $q_1, q_2 \ldots .$ are the given independent multenions. Of course a multiplex *is* a complex of multenions, but it is not *any* such complex. Thus the continent multiplex $\iota_1 \iota_2 \ldots . \iota_N$ is a complex of all the 2^{N-1} independent multits; and the complex $1, \iota_1, \iota_2$ is not a multiplex, whereas the complex $1, \iota_1, \iota_2, \iota_1 \iota_2$ is a multiplex.

Second N.B.—The importance of distinguishing between the continent fictorplex (or multiplex) and fictorplexes (or multiplexes) of lower orders arises from the statement in law (3) that $\iota_1 \iota_2 \ldots . \iota_N$ is a scalar, and the consequential statement in law (1) that N is odd. There are but 2^{N-1} independent multits in the continent multiplex, whereas there are 2^n independent multits in any other multiplex of order n.]

A product $\iota \iota' \iota'' \ldots . = \upsilon$ of fictits is called a **multit**, and unless the context implies the contrary it is to be understood that all the fictits $\iota, \iota', \ldots .$ are different. [All like pairs can be got rid of by transpositions and use of the relations $\iota^2 =$ scalar, $\iota'^2 =$ scalar, $\ldots .$]

The **order** of a multit is the number of constituent fictits in it (when they are all different as just prescribed). The part $S_a q$ of a multenion q depending on the a^{th} order multits, is called the a^{th} **order part**, or a^{th} part of q. The zeroth part $S_0 q$, being a scalar, will be called the **scalar** part, and will be denoted by Sq.

The following important theorem is intimately connected with the fundamental laws. *If* $\lambda_1, \lambda_2 \ldots . \lambda_n$ *be* n *multenions such that* $\lambda_1^2, \lambda_2^2 \ldots . \lambda_n^2$ *are all scalars differing from zero, and each pair is anti-commutative* (*that is*, $\lambda_1 \lambda_2 = -\lambda_2 \lambda_1$, *etc.*); *then if* n *is even the* 2^n *multiplicative combinations are independent; and if* n *is odd they are independent unless the product* $\lambda_1 \lambda_2 \ldots . \lambda_n$ *is a scalar, and if it is,* 2^{n-1} *of the combinations* (*which may be taken either as the odd formed combinations or as the even formed*) *are independent.* [An odd formed multiplicative combination means a product of an odd number of different λ's.]

If $\Sigma x \upsilon = 0$ where υ is a multiplicative combination, put the relation $\Sigma x \upsilon = 0$ in the form $q = r$; where every term of q is commutative and every term of r is anti-commutative with λ_1; that is, in every one of the odd formed combinations, and in one of the even formed. Thus

$$\lambda_1 q = q\lambda_1 = \lambda_1 r = r\lambda_1, \ \lambda_1 r = -r\lambda_1.$$

Hence

$$\lambda_1 q = \lambda_1 r = 0.$$

Multiplying by λ_1,

$$q = r = 0.$$

Hence in the supposed relation λ_1 occurs in each of else in each of the odd formed combinations; and sir Let $\lambda_1, \lambda_2 \ldots\ldots$ occur in the even formed, and λ'_1, formed combinations. The supposed relation now bec

$$f_0(\lambda_1, \lambda_2, \ldots\ldots) = f_1(\lambda'_1, \lambda'_2, \ldots\ldots).$$

Since $\lambda_1, \lambda_2, \ldots\ldots$ each occur in *every* even formed (

$$f_0 = x\,\lambda_1\,\lambda_2 \ldots\ldots \lambda_{2a},$$

and similarly

$$f_1 = y\,\lambda'_1\,\lambda'_2 \ldots\ldots \lambda'_{2b+1},$$

where x and y are scalars (that is, f_0 and f_1 each conta the relation $f_0 = f_1$, multiplying by $\lambda_1\,\lambda_2 \ldots\ldots \lambda_{2a}$ we ge

scalar = product of odd number of λ_1,)

The scalar is not zero because the square of the prod the λ's must be present in the product, because if λ would be anti-commutative with the product, that is, v scalar. Hence, when n is even no such relation can h

If n is even, then, the 2^n multiplicative combinatic

If n is odd, similar reasoning shows that th combinations of $\lambda_1, \lambda_2, \ldots\ldots \lambda_{n-1}$ are independent; ar when $\lambda_1\,\lambda_2 \ldots\ldots \lambda_n = \varpi$ is a scalar, every odd formed even formed by multiplication by ϖ, the odd forn independent, and again the even are independent.

[In the above $\lambda_1, \lambda_2, \ldots\ldots$ are not fictits in genera fictors, but multenions. Thus υ is not a multit in ger again confine υ to meaning a multit.]

If, as a particular case, we put

$$\lambda_1 = \iota_1, \ \lambda_2 = \iota_2, \ \ldots\ldots$$

we have for the *continent* multiplex $\iota_1\,\iota_2 \ldots\ldots \iota_N$:—

(1) N *must* be odd, since $\iota_1\,\iota_2 \ldots\ldots \iota_N$ = scalar.

(2) The continent multiplex is a complex of 2^{N-1} in(

which may be taken as (*a*) the odd formed multits, or (*b*) the even formed multits, or (*c*) the multits not containing some one assigned fictit, such as ι_1 or ι_N.

(3) We may take as a definition that $S_a q = S_{N-a} q$.

(4) The $\frac{1}{2}(N+1)$ odd parts of q are independent, and the $\frac{1}{2}(N+1)$ even parts are independent.

For the multiplex $\iota_1 \iota_2 \ldots \iota_n$ where $n < N$ we have:—

(1) If $\varpi = \iota_1 \iota_2 \ldots \iota_n$, when n is odd ϖ like a scalar is commutative with every multenion of the multiplex; when n is even ϖ behaves like a fictor in commutation with multenions, and in that ϖ^2 is a scalar differing from zero. [When n is odd, ϖ, unlike a scalar, is independent of 1, and whereas x is commutative with S_a, ϖ is not; when n is even, multiplication by ϖ changes the orders of the parts of q in a quite different way from multiplication by a fictor. Thus $\varpi S_a q$ is of the $(n-a)^{\text{th}}$ order, whereas $a S_a q$ (a a fictor) contains parts of orders $a \pm 1$ only.]

(2) The multiplex is a complex of 2^n independent multenions, which may be taken as the 2^n multits.

(3) The $n+1$ parts of q are independent.

Whether n is equal to or less than N, there are ${}_nC_a$ multits of order a.

Returning for a moment to the general theorem above, note that $\lambda_1, \lambda_2 \ldots \lambda_n$ were not confined to a given multiplex such as that of $\iota_1 \iota_2 \ldots \iota_n$. In considering the rigid replacement below they will be so confined. Does this confinement produce any noteworthy effects? None that I have been able to find. For instance, if n is odd it may still be true that $\lambda_1 \lambda_2 \ldots \lambda_n$ is a scalar even when $\iota_1 \iota_2 \ldots \iota_n$ is not (*i.e.* $n \neq N$), as we see by putting

$$\lambda_1 = \iota_1, \ \lambda_2 = \iota_2, \ \ldots \ \lambda_{n-1} = \iota_{n-1}, \ \lambda_n = \iota_1 \iota_2 \ldots \iota_{n-1}$$

(or, more generally, if we put $\lambda_n = \lambda_1 \lambda_2 \ldots \lambda_{n-1}$).

The reader will perceive the bearing of this negative result when he studies the unfinished investigation of the rigid replacement below.

The proofs of the following important statements are simple and are left to the reader.

$$v^2 = \text{scalar} \quad \ldots \quad (1)$$

$vv' = \text{scalar}$ if, and only if, $v' = \pm v$. In other words, Svv' is or is not zero according as v and v' are not or are indentical; and when they are identical, $Svv' = vv'$.

Putting, then, $q = \Sigma x v$ we get by operating by $Sv^{-1}(\ )$.

$$x = Sv^{-1} q, \quad \text{or}$$
$$q = \Sigma v S v^{-1} q = \Sigma v^{-1} S v q \quad \ldots \quad (2)$$

Similarly,

$$\alpha = \Sigma\iota S\iota^{-1}\alpha = \Sigma\iota^{-1}S\iota\alpha \quad . \quad . \quad (3)$$
$$Sqq' = \Sigma xx'v^2 \text{ when } q = \Sigma xv,\ q' = \Sigma x'v \quad . \quad . \quad (4)$$
$$Sqq' = Sq'q \text{ always} \quad . \quad . \quad (5)$$
$$Sqq' \ldots q^{(n)} = Sq'q'' \ldots q^{(n)}q \text{ always} \quad . \quad . \quad (6)$$
$$S\alpha\alpha' = \Sigma xx'\iota^2 \text{ when } \alpha = \Sigma x\iota,\ \alpha' = \Sigma x'\iota \quad . \quad . \quad (7)$$
$$S\alpha\alpha' = S\alpha'\alpha \text{ always} \quad . \quad . \quad (8)$$

(8) is, of course, a particular case of (5), and (5) a particular case of (6).

If r and ρ are an *arbitrary* multenion and fictor respectively; then, if p, q are multenions and α, β fictors,

$$p = q \text{ when } Spr = Sqr \quad . \quad . \quad (9)$$
$$\alpha = \beta \text{ when } S\alpha\rho = S\beta\rho \quad . \quad . \quad (10)$$

It is convenient to define here the **conjugate** Kq of q, the **reversate** Qq of q, and an associate Pq of q (for which I have not thought it necessary to provide a name). If v is a multit of order a, Qv is defined as a multit with the same fictits written in the reverse sequence; Kv is defined as v^{-1}; and Pv is defined as $(-)^a v$. Qq, Kq, and Pq are then defined by

$$Qq = \Sigma xQv,\ Kq = \Sigma xKv,\ Pq = \Sigma xPv \quad \text{when} \quad q = \Sigma xv \quad . \quad . \quad (11)$$

The following may now be proved:—

$$Q^2 = 1,\ K^2 = 1,\ P^2 = 1 \quad . \quad . \quad (12)$$
$$QK = KQ,\ QP = PQ,\ KP = PK \quad . \quad . \quad (13)$$
$$P(qr) = PqPr \quad . \quad . \quad (14)$$
$$Q(qr) = QrQq,\ K(qr) = KrKq \quad . \quad . \quad (15)$$

K and Q are called **retroplacements** on account of the sequence change of (15), P is called a **proplacement**, and all three are called replacements. They are further called **uni-replacements** on account of (12). It will be seen that all these three symbols are commutative [(13)]. Replacements, in general, will be considered later. These three are by far the most important. QP and KP are easily shown to be uni-retroplacements and QK (like $P^2 = Q^2 = K^2 = 1$) a uni-proplacement; that is, a product of uni-replacements is itself a uni-proplacement or uni-retroplacement respectively according as the number of retroplacements in the product is even or odd.

The following equations are true (and the similar equations also where P, Q, K are replaced by any product of powers thereof):—

$$S.qPr = S.rPq, \quad S.qKr = S.rKq, \quad S.qQr = S.rQq \quad . \quad . \quad (16)$$

Also

$$S.qKq' = \Sigma xx' = S.q'Kq \text{ when } q = \Sigma xv,\ q' = \Sigma x'v \quad . \quad . \quad (17)$$

and in particular

$$S.qKq = \Sigma x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Also

$$K(qKq) = qKq, \quad Q(qQq) = qQq \quad . \quad . \quad . \quad . \quad (19)$$

and similarly for uni-retroplacements in general, but not for uni-proplacements.

P, Q, and K and their powers and products are all commutative with S_a, that is,

$$K(S_a q) = S_a(Kq), \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Also

$$PS_a q = (-)^a S_a q \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Thus $\frac{1}{2}(1+P)q$ contains the even order parts and $\frac{1}{2}(1-P)q$ the odd order parts of q. Similar statements cannot be made about Q and K till certain permissible simplifications are added to our fundamental laws.

The following list of notations habitually used below may prove of service:—

a, b, c, x, y, z	will usually denote		scalars.
$\iota_1, \iota_2, \ldots, \iota, \iota', \ldots$	"	"	fictits.
v, v', v_a, v_b	"	"	multits (v_a, a^{th} order).
$\alpha, \beta, \gamma, \rho, \sigma$	"	"	fictors.
$p, q, r,$	"	"	multenions.
ϖ_a, ϖ_b	"	"	fictor products.
ϖ	"	"	a fictit product.

3. **Permissible simplifications.**—We shall but rarely refer to the continent multiplex, which has been introduced mainly to call attention to the peculiar explicit symmetry of quaternions.

Unless the contrary is stated, we shall always suppose our symbols to be confined in meaning to a given multiplex of order n. A quaternion is still a multenion, namely, one for which $n=2$. The Vq of the quaternions is our present $S_1 q + S_2 q$, and consistently with our fundamental laws we have for quaternions to assume

$$\iota_1^2 = \iota_2^2 = -1.$$

The i, j, k of quaternions may be identified thus with our present symbols

$$i = \iota_1, \quad j = \iota_2, \quad k = \iota_1\iota_2,$$

from which the reader may verify that

$$i^2 = j^2 = k^2 = ijk = -1$$
$$i = jk, \quad j = ki, \quad k = ij.$$

For some purposes it is more convenient to suppose quaternions included in multenions, thus: Do not assume $\iota_1{}^2 = \iota_2{}^2 = -1$, but assume (1) below. Put $i = \iota_3\iota_2{}^{-1}$, $j = \iota_1\iota_3{}^{-1}$, $k = \iota_2\iota_1{}^{-1}$, or

$$w + xi + yj + zk \equiv w + x\iota_3\iota_2{}^{-1} + y\iota_1\iota_3{}^{-1} + z\iota_2\iota_1{}^{-1}.$$

Unless the contrary is stated, we shall always suppose that

(Law A) $$\iota_1{}^2 = \iota_2{}^2 = \ldots\ldots = \pm 1 . \qquad (1)$$

[We leave ourselves the liberty, very rarely availed of, to abandon Law A. For a motor calculus it would be convenient to put

$$\iota_1\iota_2 \ldots\ldots \iota_N = 1 = \iota_1{}^2 = -\iota_2{}^2 = \iota_3{}^2 = -\iota_4{}^2 = \ldots\ldots$$

but we nowhere, except in the Supplement, adopt this below.]

Is there any good reason for choosing the upper or the lower sign in (1)? What is the reason in quaternions why the lower sign is adopted? It is because if we assume $\iota_1{}^2 = \iota_2{}^2 = \pm 1$ we obtain without any ambiguity that $(\iota_1\iota_2)^2 = -1$, so that if we would have

$$\iota_1{}^2 = \iota_2{}^2 = (\iota_1\iota_2)^2$$

we must put each equal to -1.

If we suppose N to be given, we are similarly compelled to accept definitely the upper or the lower sign in (1), as will be shown immediately. Meanwhile we may say that for most purposes N may be taken as large as we please, and in that case there is no reason on present grounds for either sign in preference to the other. I have decided, however (at a little inconvenience in some applications), to leave the sign ambiguous. When the ambiguity affects the form of an expression below, it will be rendered evident by the presence therein of ι^2 which will be put for any one of the equals $\iota_1{}^2, \iota_2{}^2, \ldots\ldots$ Moreover, it is almost invariably true (though generally not obvious) that a formula in which ι^2 does not explicitly occur is true even when Law A is not assumed, but only Laws (1) to (4) of § 2.

To show how the magnitude of N governs the sign of (1) put

$$\varpi = \iota_1\iota_2 \ldots\ldots \iota_n, \quad \varpi_N = \iota_1\iota_2 \ldots\ldots \iota_N \qquad (2)$$

By transpositions it is evident that even when Law A is not assumed

$$\varpi^2 = (-)^{\frac{n(n-1)}{2}} \iota_1{}^2\iota_2{}^2 \ldots\ldots \iota_n{}^2 \qquad (3)$$

or when Law A is assumed

$$\varpi^2 = (-)^{\frac{n(n-1)}{2}} (\pm 1)^n \qquad (4)$$

Hence

$$\varpi_N{}^2 = (-)^{\frac{N(N-1)}{2}} (\pm 1)^N \qquad (5)$$

Since N is odd, it follows that if the scalar ϖ_N is to be real,

$$(-)^{\frac{N-1}{2}}(\pm 1) = 1.$$

Hence we put

$$\iota_1^2 = \iota_2^2 = \ldots. = (-1)^{\frac{N-1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The sign of the scalar ϖ_N is still left arbitrary, and no inconvenience results in leaving it thus, at any rate for the present. [Quaternion assumption is in harmony with either $\varpi_N = -1$ or $\varpi_N = \iota_1^2$. But it must be remembered that the sign of ϖ_N depends on the sequence of the fictits in it, and we have said nothing in our fundamental laws about such an ordained sequence.]

The following results for any two fictors a, a' may be here noted. Let

$$a = x_1\iota_1 + x_2\iota_2 + \ldots., \; a' = x'_1\iota_1 + x'_2\iota_2 + \ldots. \quad . \quad . \quad . \quad (7)$$

then

$$aa' = \iota^2(x_1x'_1 + x_2x'_2 + \ldots.) + \iota_1\iota_2(x_1x'_2 - x_2x'_1) + \ldots. \quad . \quad . \quad (8)$$

Hence aa' contains only zero[th] and second parts (to be easily generalised later), and

$$\mathrm{S}aa' = \mathrm{S}a'a, \quad \mathrm{S}_2aa' = -\mathrm{S}_2a'a \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$a^2 = \mathrm{S}a^2 = \text{scalar} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$aa' + a'a = 2\mathrm{S}aa', \; aa' - a'a = 2\mathrm{S}_2aa' \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\left.\begin{array}{l} aa' = -a'a \quad \text{when} \quad \mathrm{S}aa' = 0 \\ aa' = a'a \quad \text{when} \quad \mathrm{S}_2aa' = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If q be a multenion of order a we have

$$\mathrm{Q}q_a = (-)^{\frac{a(a-1)}{2}}q_a, \quad \mathrm{K}q_a = (\iota^2)^a\mathrm{Q}q_a = (-)^{\frac{a(a-1)}{2}}(\iota^2)^a q_a. \quad . \quad . \quad (13)$$

$$\mathrm{K} = \mathrm{Q} \text{ or } \mathrm{PQ} \quad \text{acc. as} \quad \iota^2 = \pm 1 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Calling $\frac{1}{2}(1+\mathrm{Q})q$ the **Q_0 part** and $\frac{1}{2}(1-\mathrm{Q})q$ the **Q_1 part** of q; and similarly for PQ; we have from (13), on putting $\mathrm{S}_a q = q_a$,

$$\left.\begin{array}{l} \frac{1}{2}(1+\mathrm{Q})q = \mathrm{Q}_0q = q_0 + q_1 + q_4 + q_5 + q_8 + \ldots. \\ \frac{1}{2}(1-\mathrm{Q})q = \mathrm{Q}_1q = q_2 + q_3 + q_6 + q_7 + q_{10} + \ldots. \end{array}\right\} \quad . \quad . \quad . \quad (15)$$

$$\left.\begin{array}{l} \frac{1}{2}(1+\mathrm{PQ})q = (\mathrm{PQ})_0q = q_0 + q_3 + q_4 + q_7 + q_8 + \ldots. \\ \frac{1}{2}(1-\mathrm{PQ})q = (\mathrm{PQ})_1q = q_1 + q_2 + q_5 + q_6 + q_9 + \ldots. \end{array}\right\} \quad . \quad . \quad (16)$$

For convenience of reference we here add [(21) § 2]

$$\left.\begin{array}{l} \frac{1}{2}(1+\mathrm{P})q = \mathrm{P}_0q = q_0 + q_2 + q_4 + \ldots. \\ \frac{1}{2}(1-\mathrm{P})q = \mathrm{P}_1q = q_1 + q_3 + q_5 + \ldots. \end{array}\right\} \quad . \quad . \quad . \quad (17)$$

Also (14), (15), (16) point out the K_0 and K_1 parts. The K_0 and Q_0 parts may appropriately be called the self-conjugate and self-reversate parts respectively. Thus by (19) § 2, $q\mathrm{K}q$ and $q\mathrm{Q}q$ are self-conjugate and self-reversate respectively. [I use the mental phrase ' multenion q such that $Rq=q$ where R is any replacer

4. **The rigid replacement $q(\quad)q^{-1}$. Generalis** must here make a brief digression on the meaning of a given multenion q. We will suppose that q is not multiplex of order n there are 2^n independent mul 2^n+1 multenions

$$1, q, q^2 \ldots q^k,$$

where $k=2^n$ cannot be independent. Let, then, q^m be index which is not independent of those with still low

$$q^m + h'q^{m-1} + \ldots + h^{(m-1)}q + h^{(m)} = 0$$

is an identity satisfied by q, and there is by hypothesi degree so satisfied. When $h^{(m)}$ is not zero we have

$$pq = qp = 1$$

where

$$p = -(h^{(m)})^{-1}(q^{m-1} + h'q^{m-2} + \ldots + h^{(m}$$

Now when $pq=qp=1$ for any multenion p, there is p' such that either $p'q=1$ or $qp'=1$; *e.g.* if $p'q=1$ and therefore (since $qp=1$), $0=(p-p')qp=p-p'$, tha different from p. When, therefore, $h^{(m)}$ is not zero t one multenion q^{-1} such that $qq^{-1}=1$, and for this mu $q^{-1}q=1$. Thus in this case there can be no doubt as q^{-1} of q must mean.

When $h^{(m)}$ is zero let (1), the *minimum* degree id form

$$f(q) \cdot q^a = 0 \quad . \quad .$$

where $f(0)$ is not zero and a is a positive integer not minimum degree identity $f(q) \cdot q^{a-1}$ is not zero. Thus

$$p = [f(q) \cdot (-q)^{a-1}]/f(0) = \underset{x=\infty}{\mathrm{Lt.}}\ x^{1-a}(1 + x$$

we have

$$pq = qp = 0 \quad . \quad .$$

and on account of the limit in the third member of (3 made to depend in a definite way on q.

We see from (3) that p satisfies the quadratic iden

$$p^2 = p \quad \text{or} \quad p^2 = 0 \quad .$$

according as a is equal to or greater than 1.

When q is given, then, q^{-1} may be infinite, but even in this case (4) and (5) hold; in general q^{-1} is finite; and when it is finite it has a unique value such that $qq^{-1} = 1 = q^{-1}q$.

The reader may have a suspicion that $h^{(m)}$ is in no real case zero. That it is sometimes zero can be seen thus: According as $v'v = \pm vv'$ we have

$$(xv + yv')(xv \mp yv') = x^2v^2 \mp y^2v'^2$$

or

$$(xv + yv')^{-1} = \frac{xv \mp yv'}{x^2v^2 \mp y^2v'^2}.$$

Choosing v and v' so that $v^2 = \pm v'^2$, we have

$$(v + v')^{-1} = \infty.$$

Indeed, choosing v, v', so that $v^2 = -v'^2$ and $v'v = -vv'$, we have $(v+v')^2 = 0$. We can clearly choose v, v' in a number of different ways to satisfy these conditions whether $\iota^2 = +1$ or -1 (n being >2 for $(v+v')^{-1} = \infty$ and >3 for $(v+v')^2 = 0$ when $\iota^2 = -1$).

In § 11 below I give an account of my somewhat unsuccessful attempts to simplify the problem of finding q^{-1} when q is given. This much seemed necessary for our immediate purpose.

[One remark seems desirable. When the reciprocal of any self-conjugate or of any self-reversate is known, then also is the reciprocal of any multenion, for

$$q^{-1} = Kq(qKq)^{-1} = Qq(qQq)^{-1} = (Kq.q)^{-1}Kq = (Qq.q)^{-1}Qq].$$

Before considering the rigid replacement, note that $q(\ \ )q^{-1}$ is not in general commutative with S_a, that is, we have not in general

$$q(S_a r)q^{-1} = S_a(qrq^{-1}),$$

as we are tempted to assume from quaternion analogy. We can prove this by the following particular case. Put

$$q = \cos\tfrac{1}{2}\theta + \iota_1 \sin\tfrac{1}{2}\theta,\ q^{-1} = \cos\tfrac{1}{2}\theta - \iota_1 \sin\tfrac{1}{2}\theta \quad \text{when} \quad \iota^2 = -1,$$
$$q = \cosh\tfrac{1}{2}\theta + \iota_1 \sinh\tfrac{1}{2}\theta,\ q^{-1} = \cosh\tfrac{1}{2}\theta - \iota_1 \sinh\tfrac{1}{2}\theta \quad \text{when} \quad \iota^2 = +1.$$

It will be found that we have

$$q\iota_2 q^{-1} = \iota_2 \cos\theta + \iota_1\iota_2 \sin\theta,$$
$$q\iota_2 q^{-1} = \iota_2 \cosh\theta + \iota_1\iota_2 \sinh\theta,$$

respectively.

If, however, q is a *fictor product*, that is, $q = a_1a_2 \ldots . a_a$ where $a_1, a_2 \ldots .$ are fictors, $q(\ \ )q^{-1}$ *is* commutative with S_b. To prove this we have to show that if v_b is a multit of order b, qv_bq^{-1} is of order b, that is,

$$a_1a_2 \ldots . a_a v_b a_a^{-1} a_{a-1}^{-1} \ldots . a_1^{-1}$$

is of order b, that is, $a_a v_b a_a^{-1}$ is of order b. Since by (10) § 3 $a_a^{-1} = a_a \times$ a non-evanescent scalar, we have to show that $a_a v_b a_a$ is of order b. Let

$$a_a = x\iota + x'\iota' + \ldots + y\lambda + y'\lambda' + \ldots,$$

where $\iota, \iota' \ldots, \lambda, \lambda' \ldots$ are fictits; and where $\iota, \iota' \ldots$ occur in the product v_b, whereas $\lambda, \lambda' \ldots$ do not so occur. Passing the second a_a of $a_a v_b a_a$ across the b fictits of v_b we get

$$\begin{aligned} a_a v_b a_a &= (-)^{b-1}(x\iota + \ldots + y\lambda + \ldots)(x\iota + \ldots - y\lambda - \ldots)v_b \\ &= (-)^{b-1}(\iota^2\Sigma x^2 - \iota^2\Sigma y^2 - 2\Sigma xy\iota\lambda)v_b. \end{aligned}$$

Here $\iota\lambda v_b$ is of order b, since ι occurs in v_b and λ does not. Hence $a_a v_b a_a$ is of order b.

When q is a fictor product its reciprocal may be regarded as known by (10) § 3, and

$$(a_1 a_2 \ldots a_a)^{-1} = a_a^{-1} a_{a-1}^{-1} \ldots a_1^{-1}.$$

This reasoning applies to the continent multiplex as well as to any other. This suggests a slight generalisation of the statement that $q(\ \ )q^{-1}$ is commutative with S_a when q is a fictor product. The proof I leave to the reader, as the statement is not subsequently required. If n is even, ϖ behaves like a fictit; and if n is odd, like a scalar. Instead of supposing $a_1, a_2, \ldots$ to be mere fictors, then, we may suppose each a to be a fictor + any multiple of ϖ when n is even; and to be of the form $S_1 q + S_{n-1} q$ when n is odd. Then $q(\ \ )q^{-1}$ is commutative with S_a when $q = a_1 a_2 \ldots a_b$.

It should perhaps be remarked that in our present subject a multenion is not in general a fictor product, whereas a quaternion is always some vector product. Many of the properties of quaternions therefore suggest properties that are not true of multenions in general, but are true of fictor products.

In the above expressions involving a_a, change a_a to a. It is obvious that

$$(x\iota + x'\iota' + \ldots)v_b = S_{b-1} a v_b, \quad (y\lambda + y'\lambda + \ldots)v_b = S_{b+1} a v_b.$$

Hence if q_b be a multenion of order b,

$$\left.\begin{aligned} (-)^b a q_b a &= a^2 q_b - 2aS_{b-1} a q_b = -a^2 q_b + 2aS_{b+1} a q_b \\ &= a^2 q_b - 2S_{b-1} q_b a . a = -a^2 q_b + 2S_{b+1} q_b a . a \end{aligned}\right\} \quad . \quad . \quad . \quad (6$$

Putting $q_b = \beta$ where β is a fictor so that $b = 1$,

$$-a\beta a = a^2\beta - 2aSa\beta = -a^2\beta + 2aS_2 a\beta.$$

Thus (6) is a satisfactory generalisation of two well-known quaternion formulæ. I have not been able to obtain a similarly satisfactory generalisation of the formulæ

$$V\alpha\beta\gamma = \alpha S\beta\gamma - \beta S\gamma\alpha + \gamma S\alpha\beta$$
$$V\alpha V\beta\gamma = \quad - \beta S\gamma\alpha + \gamma S\alpha\beta,$$

but the multenion formulæ immediately suggested are true, namely,

$$\left.\begin{array}{l} S_1\alpha\beta\gamma = \alpha S\beta\gamma - \beta S\gamma\alpha + \gamma S\alpha\beta \\ S_1(\alpha S_2\beta\gamma) = \quad - \beta S\gamma\alpha + \gamma S\alpha\beta \end{array}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

[These are proved below at end of § 5 and the desired "satisfactory generalisation" is found, that is, it is enunciated, but not proved.]

(6) may be put in a variety of other suggestive forms such as (multiply by α^{-1})

$$\left.\begin{array}{l} 2S_{b-1}\alpha q_b = \alpha q_b - (-)^b q_b\alpha,\ 2S_{b+1}\alpha q_b = \alpha q_b + (-)^b q_b\alpha \\ (S_{b-1} + S_{b+1})\alpha q_b = \alpha q_b,\ (S_{b+1} - S_{b-1})\alpha q_b = (-)^b q_b\alpha \end{array}\right\} \quad . \quad . \quad (8)$$

We now pass to the main purpose of this section.

Replacements in general will be considered in a subsequent section. Meanwhile it may be said that the term is meant to connote two different trains of mathematical ideas:—(1) the kinematical ideas of *displacement*, whether (*a*) that of a rigid body, or (*b*) that of a deformable system called homogeneous strain, or (*c*) this combined with perversion such as is produced by a plane mirror, or (*d*) strain in general, including such strain perversions; (2) the algebraic ideas involved in Grassmann's various species of "variation" and my own extensions of them in Octonions. The last are all of the nature of replacing certain given symbols by others according to imposed rules.

The meaning of the (in some respects) simplest of the replacements, the rigid replacement $q(\ \ )q^{-1}$, to be considered, is probably sufficiently explained in the following enunciation.

(1) *The original fictits* $\iota_1, \iota_2, \ldots.$ *of the fictorplex* $\iota_1\iota_2 \ldots. \iota_n$ *may for all the purposes whatsoever of Laws* (1) *to* (4), § 2, *and Law A*, § 3, *be replaced by* n *other multenions* $\lambda_1, \lambda_2, \ldots. \lambda_n$ *where*

$$\lambda_1 = q\iota_1 q^{-1},\ \lambda_2 = q\iota_2 q^{-1}, \ldots. \lambda_n = q\iota_n q^{-1}. \quad . \quad . \quad . \quad . \quad (9)$$

where q *is any given multenion for which* q^{-1} *is finite.*

(2) *By such a replacement any multenion* r *of the fictorplex is replaced by* qrq^{-}.

Law (1), § 2, merely states that N fictits $\iota_1, \iota_2, \ldots.$ are given, and that the laws of ordinary algebra, with an exception, are to apply.

Law (2). A scalar is commutative with the multenion $\Sigma(x_0 + x_1\lambda_1 + \ldots.)(x'_0 + x'_1\lambda_1 + \ldots.)\ldots.$ as with every other multenion.

In order that qp may be equal to $p'q$, q must satisfy the following n linear equations (and that it need satisfy no more conditions will be proved subsequently).

$$q\iota_1 = \lambda_1 q,\ q\iota_2 = \lambda_2 q, \ldots \qquad (11)$$

or

$$(\phi_1 - 1)q = 0,\ (\phi_2 - 1)q = 0, \ldots \qquad (12)$$

where $\phi_1, \phi_2 \ldots$ are the "multilinities" (that is, linear multenion functions of a multenion) given by

$$\phi_1 r = \lambda_1 r \iota_1^{-1},\ \phi_2 r = \lambda_2 r \iota_2^{-1}, \ldots \qquad (13)$$

Here $\iota_1^{-1}, \iota_2^{-1}, \ldots$ have finite meanings for $\iota_1^{-1} = \iota^2.\iota_1$, etc.

$\phi_1, \phi_2, \ldots$ satisfy quadratic equations and are commutative, that is,

$$\phi_1^2 = 1,\ \phi_2^2 = 1, \ldots,\ \phi_1\phi_2 = \phi_2\phi_1,\ \phi_1\phi_3 = \phi_3\phi_1, \ldots \qquad (14)$$

Hence if we put

$$q = \psi r = 2^{-n}(1+\phi_1)(1+\phi_2) \ldots (1+\phi_n)r \qquad (15)$$

q will satisfy all the equations (12). Here r is any multenion whatever.

The doubt remains, of course, whether q^{-1} is finite. We will return to this immediately. Meanwhile note that by (14) and (15) ψ satisfies the quadratic equation

$$\psi^2 = \psi \qquad (16)$$

since from (14) $(1+\phi_1)^2 = 2(1+\phi_1)$. Also notice that by expanding ψ in full in terms of $\phi_1, \phi_2, \ldots$

$$2^n\psi = \Sigma\phi_1\phi_2 \ldots \phi_c$$

or from (13) we have q explicitly in terms of $\iota_1, \iota_2 \ldots, \lambda_1, \lambda_2, \ldots$ thus

$$q = \psi r = 2^{-n}\Sigma v' r v^{-1} \qquad (17)$$

where v is any one of the 2^n multits and v' is what v becomes when in it ι_1 is replaced by λ_1, ι_2 by λ_2, etc.

It is easy to show from (11) that $qp = p'q$. First suppose p is a multit such as $\iota_1\iota_2\iota_3\iota_4$.

$$qp = q\iota_1\iota_2\iota_3\iota_4 = \lambda_1 q\iota_2\iota_3\iota_4 = \lambda_1\lambda_2 q\iota_3\iota_4 = \lambda_1\lambda_2\lambda_3 q\iota_4 = \lambda_1\lambda_2\lambda_3\lambda_4 q = p'q.$$

Next suppose $p = \Sigma xv$,

$$qp = \Sigma xqv = \Sigma xv'q = p'q.$$

It is also easy to show that by changing the sign of λ_c if it should happen that

$$(\phi_c + 1)(\phi_{c-1} + 1) \ldots (\phi_1 + 1)r = 0$$

q will not be zero. Thus put

$$q_1 = (\phi_1 + 1)r,\ q_2 = (\phi_2 + 1)q_1, \ldots,\ 2^n q = (\phi_n + 1)q_{n-1}$$

and suppose that q_{c-1} is not zero but q_c is. Thus

$$q_{c-1} \neq 0,\ q_c = \lambda_c q_{c-1}\iota_c^{-1} + q_{c-1} = 0.$$

Changing the sign of λ_c we get for the new value of q_c

$$-\lambda_c q_{c-1}\iota_c^{-1} + q_{c-1},$$

and this can only be zero if $q_{c-1} = 0$.

Suppose now we have determined q' so that

$$q'\iota_1 = -\lambda_1 q', \ldots, q'\iota_a = -\lambda_a q', q'\iota_{a+1} = +\lambda_{a+1} q', \ldots, q'\iota_n = +\lambda_n q'.$$

If a is even put $q = q'\iota_1\iota_2 \ldots \iota_a$; if a is odd and $n-a$ is odd put $q = q'\iota_{a+1} \ldots \iota_n$; if a is odd and $n-a$ is even put $q = q'\iota_{a+1} \ldots \iota_{n-1}$. We then have in the first two cases

$$q\iota_1 = \lambda_1 q, \; q\iota_2 = \lambda_2 q, \ldots, q\iota_{n-1} = \lambda_{n-1} q, \; q\iota_n = \lambda_n q,$$

and in the third case we have the same, except that the last relation is replaced by $q\iota_n = -\lambda_n q$. This want of harmony in sign, it will be noticed, only occurs when n is odd. And even this can be avoided if we choose to interchange λ_{n-1} and λ_n instead of changing the sign of λ_n; or instead we may change the sign of *every* λ [operate by $\lambda_n(\;)\lambda_n^{-1}$] instead of a single selected one. This change of sign (of one or of every λ) will be called perversion.

These results obviously follow from the relations

$$\iota_1.\iota_1.\iota_1^{-1} = \iota_1, \; \iota_1.\iota_2.\iota_1^{-1} = -\iota_2, \text{ etc.} \qquad (18)$$

And that we can interchange two at the same time as we change the sign of one only, follows from

$$(1+\iota_1\iota_2^{-1})\iota_1(1+\iota_1\iota_2^{-1})^{-1} = -\iota_2, \; (1+\iota_1\iota_2^{-1})\iota_2(1+\iota_1\iota_2^{-1})^{-1} = \iota_1,$$
$$(1+\iota_1\iota_2^{-1})\iota_3(1+\iota_1\iota_2^{-1})^{-1} = \iota_3.$$

The question remains whether q^{-1} is finite; and although I cannot answer this, I think the following remarks may help others desirous to clear up this point. q must satisfy (12), and therefore it must satisfy (15), as may be proved from the general theory of multilinities. We have then only to consider the question whether with q given by (17) [where (a) r is arbitrary, (b) $\psi 1 \neq 0$, (c) $\psi^2 = \psi$ (so that $\psi q = q$)] for some value of r, q^{-1} is finite. It seems to me that some very special condition would have to be satisfied by $\lambda_1, \lambda_2 \ldots \lambda_n$ to render all such values of q^{-1} infinite. At the same time such a condition is

$$\lambda_1\lambda_2 \ldots \lambda_n = \text{scalar},$$

which requires that n is odd. That this condition *does* render $q^{-1} = \infty$ for all values of r appears thus: We have seen in § 2 that the condition may be satisfied by

$$\lambda_n = \lambda_1\lambda_2 \ldots \lambda_{n-}$$

[at present this requires $\lambda_n^2 = \iota^2 = (-1)^{\frac{(n-1)(n-2)}{2}}$], and that when it is satisfied there are but 2^{n-1} independent multiplicative combinations of $\lambda_1, \lambda_2, \ldots \lambda_n$. Now if q^{-1} were finite there would necessarily be 2^n such, corresponding to the 2^n independent multits. But if we impose that $\lambda_1\lambda_2 \ldots \lambda_n$ is not a scalar when n is odd, or if we impose that n is even, I cannot say whether q^{-1} is necessarily finite.*

Note added after conclusion of the paper (*Anno* 1906).—I find that $q = \psi r$ [(15) and (17)] does not render q so arbitrary as I had supposed. If q, q' are two values of q, then $q = xq'$, where when n is even x is a scalar, and when n is odd x is of the form $a + b\varpi$ where a and b are scalars. Thus $q(\ )q^{-1}$ involves $2^n - 1$ or $2^n - 2$ scalars according as n is even or odd. I merely indicate the proof. Unity, and when n is odd ϖ also, is commutative with every multit. This statement is not true of any other multit, but instead we have the rather unexpected simple statement:—υ any such other multit is commutative with half or 2^{n-1} of the multits (including υ and unity), and anti-commutative with the other half. Hence

$$2^{-n}\Sigma\upsilon p\upsilon^{-1} = Sp \quad \text{or} \quad Sp + S_n p \quad . \quad . \quad . \quad . \quad (19)$$

according as n is even or odd. The statement that $q = xq'$ now follows from (17).

The particular case of the rigid replacement when the replacements $\lambda_1, \lambda_2, \ldots$ (of the fictits $\iota_1, \iota_2, \ldots$) are fictors is specially simple and possesses none of the unintelligibilities of the general case; though when n is odd we may still have to "pervert" the system (that is, change the sign at will either of a single selected λ or of every λ).

(10) becomes

$$\lambda_1^2 = \lambda_2^2 = \ldots = \iota^2, \ S\lambda_1\lambda_2 = 0 = S\lambda_1\lambda_3, \text{ etc.} \quad . \quad . \quad . \quad (20)$$

We will show that if we put $r = 1$ in (15) and (17) q becomes an even fictor product. We have already seen that if p is a non-evanescent fictor product, whether even or odd, two great simplicities result: (1) p^{-1} is never infinite; (2) $p.S_a r.p^{-1} = S_a(prp^{-1})$, so that in particular pap^{-1} is a fictor when a is. We proceed by showing in succession that

$$(\phi_1 + 1)1 = p_1, \ (\phi_2 + 1)p_1 = p_2, \ \ldots, \ (\phi_n + 1)p_{n-1} = p_n = 2^n q$$

are even fictor products.

* I am now (1908) able to show that *when* n *is even* q^{-1} *is finite.* If $q^{-1} = \infty$ let $qq_0 = 0 = q_0 q$ where q_0 is not zero. Since $qp = p'q$ we have that $qpq_0 = 0$ for all multenion values of p. In the Supplement below, between equations (44) and (45), it is proved that when n is even this is never true when both q and q_0 differ from zero. Hence q^{-1} is finite. The case of n odd still remains incomplete. qpq_0 may be zero for all multenion values of p when n is odd, as we see by putting $q = 1 + \varpi\sqrt{(\varpi^2)}$, $q_0 = 1 - \varpi\sqrt{(\varpi^2)}$.

$$p_1 = \lambda_1\iota_1^{-1} + 1 = (\lambda_1 + \iota_1)\iota_1^{-1}$$

and ι_1^{-1} is a fictor, viz. $\iota^2.\iota_1$. Let $p_1\iota_2 p_1^{-1} = \mu_2$ so th $p_1\iota_2 = \mu_2 p_1$, and therefore

$$p_2 = (\phi_2 + 1)p_1 = \lambda_2 p_1\iota_2^{-1} + p_1 = (\lambda_2\mu_2^{-1} + 1)p_1 = (\lambda_2 + \mu_2)$$

Let $p_2\iota_3 p_2^{-1} = \mu_3$ so that μ_3 is a fictor. Thus $p_2\iota_3 =$

$$p_3 = (\phi_3 + 1)p_2 = \lambda_3 p_2\iota_3^{-1} + p_2 = (\lambda_3\mu_3^{-1} + 1)p_2 = (\lambda_3 + \mu_3)$$

This reasoning is evidently general, so that p_n is an

[The "circular variation" from $(\iota_1, \iota_2, \iota_3, \ldots . \iota_n)$ t

$$(\iota_1 \cos\theta + \iota_2 \sin\theta, -\iota_1 \sin\theta + \iota_2 \cos\theta, \iota_3, .$$

is effected by the operator $q(\ )q^{-1}$, where $q = (\iota_1 \cos$ fact may be used to establish the properties of when the fictit-replacements are fictors.]

We will henceforth understand the meaning of *fi* such that $\lambda^2 = \iota^2$, and the meaning of *set of fictits* to $\lambda_1, \lambda_2, \ldots .$ which satisfy (20), *i.e.*

$$\lambda_1 = p\iota_1 p^{-1},\ \lambda_2 = p\iota_2 p^{-1}, \ldots .$$

where p is any fictor product whatever.

[We could, of course, mean by "set of fictits" the of (9), but since the replacement of S_a would ther think this undesirable.]

We may now assume from chapter iv. of Octo fictorplex $a_1 a_2 \ldots . a_n$ may be expressed as a comple and therefore that any given multiplex $a_1 a_2 \ldots . a$ a multiplex based on the fictits $\iota_1 \iota_2 \ldots . \iota_n$. [I am the attempted proof of the first italicised statement c quite unsound. The theorem is true, and can indeed algebra. Also the proof *is* sound for the only ca present paper.]

It may be remarked that if Law A, § 3, is not ass not assumed that the signs of $\iota_1^2, \iota_2^2, \ldots . \iota_n^2$ are a fictor may be infinite. Thus if $\iota_1^2 = -\iota_2^2$,

$$(\iota_1 + \iota_2)^2 = 0,\ (\iota_1 + \iota_2)^{-1} = \infty .$$

5. **Fictor Products, Complements** (including G and regressive multiplication).

Let q_a stand for a multenion of order a belonging of order n. We have just seen that this may be taken as the multiplex $\iota_1 \iota_2 \ldots \iota_n$ where $\iota_1, \iota_2, \ldots$ are fictits. Let p_a, p_b be multenions given by

$$p_a = q_a + q_{a-2} + q_{a-4} + \ldots, \quad p_b = q_b + q_{b-2} + \ldots$$

Suppose all these symbols expressed in their simplest extended forms in terms of the multits, and think of the process of picking out the various parts of, say, $q_a q_b$ when $n > a+b$. It is clear that the highest order parts that can occur are of order $a+b$. Lower order parts are in general included, but they only arise by *two* or *four*, etc. fictits, disappearing by fusion with scalars on account of the relations $\iota_1^2 = \text{scalar}$, etc. It follows that the product $p_a p_b p_c$ (of, say, 3, though the reasoning is general) is of the form

$$q'_{a+b+c} + q'_{a+b+c-2} + q'_{a+b+c-4} + \ldots$$

Thus *in* $S_{a+b+c} p_a p_b p_c$ *we may change* p *to the* q *with the same suffix; and* $S_k p_a p_b p_c = 0$ *whenever* k *exceeds* $a+b+c$ *or differs from* $a+b+c$ *by an odd integer.*

Now the fictor products ϖ_a, ϖ_b, ϖ given by

$$\varpi_a = \alpha_1 \alpha_2 \ldots \alpha_a, \quad \varpi_b = \beta_1 \beta_2 \ldots \beta_b, \quad \varpi_c = \gamma_1 \gamma_2 \ldots \gamma_c \quad . \quad . \quad (1)$$

are of this nature (i.e. $\varpi_a = p_a = q_a + q_{a-2} + \ldots$) as we see by expressing each fictor in terms of fictits and each fictor product in terms of multits. Hence we have

$$\varpi_a \varpi_b \ldots = q'_{a+b+\ldots} + q'_{a+b+\ldots-2} + q'_{a+b+\ldots-4} + \ldots \quad . \quad . \quad (2)$$

Hence

$$S_{a+b+\ldots} \varpi_a \varpi_b \ldots = S_{a+b+\ldots} ([S_a]\varpi_a)([S_b]\varpi_b) \ldots \quad . \quad . \quad (3)$$

where the square brackets imply that we may retain or reject the symbol enclosed at will. Also

$$S_k \varpi_a \varpi_b \ldots = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

whenever k exceeds $a+b+\ldots$ or differs from $a+b+\ldots$ by an odd integer.

$S_a \varpi_a$ *is a combinatorial part of the fictor product* ϖ_a. [This means fundamentally that if two consecutive fictors α, α' be interchanged the part alters in sign, but otherwise remains unchanged; derivatively it means that if *any* two α, α' be interchanged the sign thus alters, and that the part $S_a \varpi_a$ is or is not zero according as $\alpha_1, \alpha_2, \ldots \alpha_a$ are not or are independent.] $S_a \varpi_a$ is indeed Grassmann's progressive product of the fictors when a does not exceed n. [When $a > n$, $\alpha_1 \alpha_2 \ldots \alpha_a$ cannot be independent, so that then $S_a \varpi_a = 0$. $S_a \varpi_a$ is therefore not Grassmann's regressive product.]

To prove the theorem let a, a' be two consecutive fictors of ϖ_a. In $S_a a_1 a_2 \ldots . aa' \ldots . a_a$ we may, in accordance with (3), write $S_2(aa')$ instead of aa', and by (9) § 3 the sign is changed when a, a' are interchanged. We may now again suppose the S_2 removed, and the theorem follows.

That these simple and easily proved results contain really complex algebraic truths is evident from the following statements. The particular case of (3),

$$S_n[S_a(a_1a_2 \ldots . a_n)][S_{n-a}(a_{a+1} \ldots . a_n)] = S_n(a_1a_2 \ldots . a_n) \quad . \quad . \quad (5)$$

where $a_1, a_2 \ldots . a_n$ are supposed expressed in terms of n fictits, is an expression of the theorem which develops any determinant of order n in terms of determinant minors of orders a and $n-a$. The equation

$$S_{a+b}\varpi_a\varpi_b = S_{a+b}(S_a\varpi_a S_b\varpi_b) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $a_1, \ldots . a_a, \beta_1, \ldots . \beta_b$ are supposed expressed in terms of n fictits and $n > a+b$ is the equivalent of a series of statements, of the same general nature, having reference to a rectangular array of $(a+b)n$ elements.

A **complement** of a multit of a given multiplex of order n means a product of all the fictits not constituting the given multit. If ϖ is the product of all the fictits and υ is a given multit we shall, consistently with this explanation, call each of the quantities

$$\varpi\upsilon, \ \upsilon\varpi, \ \varpi^{-1}\upsilon, \ \upsilon\varpi^{-1}$$

a complement of υ; or, more generally, if q is any multenion of the multiplex, ϖq, $q\varpi$, $\varpi^{-1}q$, $q\varpi^{-1}$ will each be called a complement of q. It does not seem desirable to restrict more definitely than this the meaning of complement, because these four forms are about equally useful.

Putting, as usual, $q = q_0 + q_1 + q_2 + \ldots .$ where $q_c = S_c q$, we have

$$\varpi q = \varpi q_0 + \varpi q_1 + \varpi q_2 + \ldots .$$

The terms on the right are of orders $n, n-1, n-2, \ldots .$ respectively. Hence the first of the following, the others being proved similarly,

$$S_c\varpi q = \varpi S_{n-c}q, \ S_c q\varpi = S_{n-c}q.\varpi,$$
$$S_c\varpi^{-1}q = \varpi^{-1}S_{n-c}q, \ S_c q\varpi^{-1} = S_{n-c}q.\varpi^{-1},$$

i.e. we may pass ϖ into or out of $S_c(\ )$ provided that at the same time we change S_c to S_{n-c}.

Since in the given multiplex there is but one independent multit of the n^{th} order, these equations are equally true if r_n be substituted for ϖ where $r_n(=x\varpi)$ is any n^{th} order multenion. Thus

$$\left.\begin{array}{l} S_c r_n q = r_n S_{n-c}q, \ S_c q r_n = S_{n-c}q.r_n \\ S_c r_n^{-1}q = r_n^{-1}S_{n-c}q, \ S_c q r_n^{-1} = S_{n-c}q.r_n^{-1} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

A form of (7) which is often useful is

$$S_{n-c}q = r_n^{-1}S_c r_n q = S_c q r_n . r_n^{-1} = r_n S_c r_n^{-1} q = S_c q r_n^{-1} . r_n \quad . \quad . \quad . \quad (8)$$

* We now proceed to prove the theorem numbered (12) below.

[In what immediately follows the reader will probably be assisted by mentally naming ϖ_c "the common product"; to what two expressions it is common is obvious enough.]

Assuming that $\beta_1, \ldots, \beta_b, \gamma_1, \ldots \gamma_c$ are independent we may put $n = b + c$ and $r_n = S_{b+c}\varpi_b\varpi_c$. Thus we have

$$S_{b+c}\varpi_b\varpi_c . S_c\varpi_c = S_b(S_{b+c}\varpi_b\varpi_c . \varpi_c) = S_b(S_{b+c}\varpi_b\varpi_c . S_c\varpi_c) \quad . \quad . \quad . \quad (9)$$

[The middle form is given directly by (7); the third form by noticing that since the first is a b^{th} part, it must be the b^{th} part of itself.] Although we assumed the $b+c$ fictors $\varpi_b\varpi_c$ to be independent, (9) is clearly true also when they are not independent, since $S_{b+c}\varpi_b\varpi_c$ which occurs in each form is then zero.

Changing ϖ_b in (9) to $\varpi_a\varpi_b$ it follows that the first two of the following three equations are true universally,

$$\left.\begin{aligned} S_{a+b+c}\varpi_c\varpi_a\varpi_b . S_c\varpi_c &= S_{a+b}(S_{a+b+c}\varpi_c\varpi_a\varpi_b . \varpi_c) \\ &= S_{a+b}(S_{a+b+c}\varpi_c\varpi_a\varpi_b . S_c\varpi_c) \\ &= S_{a+b}(S_{a+c}\varpi_c\varpi_a . S_{b+c}\varpi_b\varpi_c) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

We proceed to show that the third and fourth of these forms are universally equal to one another.

First suppose the $a+b+c$ fictors $\varpi_a\varpi_b\varpi_c$ are independent. Let a, a' be two of $a_1 \ldots a_a$; β, β' two of $\beta_1 \ldots \beta_b$; γ, γ' two of $\gamma_1 \ldots \gamma_c$. Then linear variations of the following types

$$\begin{aligned} &\gamma \ \text{ to } \ \gamma + x\gamma', \\ &a \ \text{ to } \ a + x\gamma, \quad \text{or to } \ a + xa', \\ &\beta \ \text{ to } \ \beta + x\beta', \ \text{ or to } \ \beta + x\gamma, \quad \text{or to } \ \beta + xa, \end{aligned}$$

do not alter the value of either the third or fourth forms of (10). All these statements are obvious from the properties of combinatorial parts except the last, namely, that changing β to $\beta + xa$ causes no change in the value of $S_{a+b}(S_{a+c}\varpi_c\varpi_a . S_{b+c}\varpi_b\varpi_c) = q$ (say). Variation from β_1 to $\beta_1 + xa_1$ adds to q

$$(-)^{b+c-1}xS_{a+b}(S_{a+c}\varpi_c a_1 \varpi_{a-1} . S_{b+c}\varpi_{b-1}\varpi_c a_1) = r \text{ (say)}$$

* I find (April 1908) that (12) below may be much more simply proved thus: The "transference" of ϖ_x may be effected in x steps by transferring each fictor $\xi_1, \xi_2 \ldots \xi_x$ of ϖ_x in succession; and we have to prove the theorem for one such transference (say of ξ_1) only. The last is quite simply proved by expressing all the fictors in fictits ^{c}, which constitute the fictorplex ϖ_c. [When the fictors ϖ_c are not independent the theorem obviously takes the form $0 = 0$.]

(where $\varpi_{a-1}=\alpha_2\alpha_3 \ldots . \alpha_a, \varpi_b \;=\beta_2\beta_3 \ldots . \beta_b$), and w
is zero. To show this we prove that always

$$S_{a+c}\varpi_c\varpi_a.S_{b+c}\varpi_b\varpi_c=p \text{ (say)}$$

consists of $(a+b)^{th}$ and lower order parts; and ther

$$S_{a+c}\varpi_c\alpha_1\varpi_{a-1}.S_{b+c}\varpi_{b-1}\varpi_c\alpha_1$$

consists of $(a+b-2)^{th}$ and lower order parts; and t

$$p=(S_{a+c}\varpi_c\varpi_a S_{a+b+c}\varpi_a\varpi_b\varpi_c)(S^{-1}_{a+b+c}\varpi_a\varpi_b\varpi_c S$$

Here by (7) the first bracket is of order b and
Hence p consists of $(a+b)^{th}$ and lower order parts.

Now by linear variations of the above six ty
$\alpha_1 \ldots . \alpha_a\beta_1 \ldots . \beta_b\gamma_1 \ldots . \gamma_c$ can be reduced to a s
we can make $\alpha_1=x_1\iota_1, \ldots . , \gamma_c=x_{a+b+c}\iota_{a+b+c}$), and wl
the third and fourth forms of (10) each equals

$$\varpi_c\varpi_a\varpi_b\varpi_c.$$

Exactly similar reasoning, step by step, appl
the fictors $\varpi_a\,\varpi_b\,\varpi_c$ are not independent. [If ϖ_a t
or if $\varpi_b\,\varpi_c$ are not independent, (10) is obviously
independent, and $\varpi_b\;\varpi_c$ are independent but $\varpi_a\;\varpi_b$
then some one β at any rate can be expressed i
$a+b+c-1$ fictors.]

In the equation of the last two forms of (10) f
and again in the same equation change ϖ_b to ϖ
equivalent forms of

$$S_{a+b+x}(S_{a+b+c+x}\varpi_c\varpi_a\varpi_x\varpi_b.S_c\varpi_c),$$

namely

$$S_{a+b+x}(S_{a+c+x}\varpi_c\varpi_a\varpi_x.S_{b+c}\varpi_b\varpi_c)=S_{a+b+x}(S_{a+c}\varpi_c\varpi_a.$$

which shows that in such transferences any product
provided it is not the explicitly common produc
binatorial parts.

(11), again, may be modified by changing $\varpi_c\varpi_a$ to
remembering that the number c is now defined by
c fictors which are common to each of the sets, ϖ_a
b fictors. Thus (11) becomes

$$S_{a+b+x-2c}(S_{a+x}\varpi_a\varpi_x.S_b\varpi_b)=S_{a+b+x-2c}(S_a\varpi_a.S_b$$

(12) appears to me to contain all the essential
mann's progressive ($c=0$) and regressive ($c>0$) mul
We frequently meet with expressions very a

expressions, and desire to know whether without serious alteration they apply in the present subject. The mode of translation is generally by no means obvious. We will therefore give a rule which serves in many cases.

(1) For i, j, k write $\varpi\iota_1, \varpi\iota_2, \varpi\iota_3$ where $\varpi = -\iota_1\iota_2\iota_3 = \iota_3\iota_2\iota_1$. (2) For S, V, K write S, S_2, K. (3) For any vector a write either ϖa where a is a fictor or a' where a' is a second order multenion. (4) Remember that in the third order multiplex when ϖ (which otherwise behaves like a scalar) crosses S_c we must change S_c to S_{3-c}. (5) Any formula so obtained is true in our present subject so long as all the symbols refer to a third order multiplex. Sometimes the theorem so obtained can be generalised so as to apply to a multiplex of any order. [The reason for taking ϖ and i, j, k as above is to ensure that $i^2 = j^2 = k^2 = -1$, $ij = k$, etc., whether $\iota^2 = +1$ or -1.]

As an example take the first of equations (7), § 4. The quaternion analogue is

$$V a\beta\gamma = aS\beta\gamma - \beta S\gamma a + \gamma S a\beta.$$

Hence in our present subject (since a, β, γ *always* belong to a third order fictorplex) we have generally

$$S_2\varpi a\varpi\beta\varpi\gamma = \varpi a S\varpi\beta\varpi\gamma - \varpi\beta S\varpi\gamma\varpi a + \varpi\gamma S\varpi a\varpi\beta.$$

Passing each ϖ to the left and cancelling ϖ^3,

$$S_1 a\beta\gamma = aS\beta\gamma - \beta S\gamma a + \gamma S a\beta.$$

Similarly, the second of (7), § 4, may be proved. As another example, from

$$V\beta\gamma S a\rho + V\gamma a S\beta\rho + V a\beta S\gamma\rho = \rho S a\beta\gamma$$

we at once get for a *third* order multiplex

$$S_2\beta\gamma S a\rho + S_2\gamma a S\beta\rho + S_2 a\beta S\gamma\rho = \rho S_3 a\beta\gamma = S_3 a\beta\gamma.\rho.$$

To generalise this, let the fictorplex $a\beta\gamma$ be that of $\iota_1\iota_2\iota_3$, but let ρ also contain $\iota_4, \iota_5 \ldots.$ The $\iota_4, \iota_5 \ldots.$ parts do not occur on the left, and are cut out from the right by operating by S_2. Hence generally

$$S_2\beta\gamma S a\rho + S_2\gamma a S\beta\rho + S_2 a\beta S\gamma\rho = S_2(\rho S_3 a\beta\gamma) = S_2(S_3 a\beta\gamma.\rho) \quad . \quad . \quad (13)$$

The following generalisation of (13) can be proved from § 6,

$$S_{n-1}a_2a_3 \ldots a_n S a_1\rho - S_{n-1}a_1a_3 \ldots a_n S a_2\rho + \ldots = S_{n-1}(\rho S_n a_1 a_2 \ldots a_n) \quad . \quad (14)$$

whether ρ be wholly, in part, or not at all within the fictorplex $a_1a_2 \ldots a_n$. This theorem is wanted in the generalisation to *any* number of integrations of the well-known line-surface and surface-volume integrals of quaternions [§ 10 below].

A still more general form than (14) is (in the notation of next section)

$$\Sigma a_{n-c}^{(n-c)} S a_c^{(c)} q_c = S_{n-c}(q_c a_n^{(n)}) \quad . \quad . \quad . \quad . \quad . \quad (15)$$

whether q_c be wholly, in part, or not at all within the multiplex $a_1 a_2 \ldots a_n$. This I noticed when treating of the fictorlinity replacement (§ 8).

Putting $n=2$, (14) becomes the second of (7), § 4; so we may say that (15) is the "satisfactory generalisation" then sought, but not found.

6. **A Multenion in terms of given Fictors.**—In this section certain restrictions as to notation will for clearness be strictly adhered to, and the reader is advised to study carefully the preliminary explanation now to be given.

n fictors of a group will be denoted by $a_{-1}, a_{-2} \ldots . a_{-n}$; by $\beta_{-1}, \beta_{-2} \ldots . \beta_{-n}$; by $\bar{a}_{-1}, \bar{\beta}_{-1}$, etc.

Positive *suffixes* will indicate the *order* (e.g. $a_c^{(c)} = \mathrm{S}_c a^{(c)}$).

An *index dash* „ „ „ *number of fictors* in a fictor product (e.g. $a^{(c)}$ is a product of c a's).

The *sequence* of fictors in a fictor product is, for the purposes of this section, frequently of vital importance. We bind ourselves in this matter by no rules of notation.

The following theorem [(1)] is true whether $a_{-1}, a_{-2}, \ldots . a_{-n}$, n given fictors belonging to a given fictorplex of the n^{th} order are or are not independent. It is also true of the continent fictorplex if the summations refer, as usual, only to the odd orders, or only to the even orders. q is any multenion of the given multiplex. Let $a^{(c)}$ be any product of any number c of different fictors $a_{-1}, a_{-2}, \ldots$ and let $a_c^{(c)} = \mathrm{S}_c a^{(c)}$. Let $a^{(n-c)}$ be the product of the rest of the fictors arranged in such a sequence that the (combinatorial part) $\mathrm{S}_n a^{(c)} a^{(n-c)} [= \mathrm{S}_n a_c^{(c)} a_{n-c}^{(n-c)}]$ has the same value for all values of $a^{(c)}$. Then

$$\left. \begin{aligned} q \mathrm{S}_n a^{(c)} a^{(n-c)} &= \Sigma^2 a_c^{(c)} \mathrm{S}_n q a_{n-c}^{(n-c)} \\ &= \Sigma^2 a_{n-c}^{(n-c)} \mathrm{S}_n a_c^{(c)} q \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (1)$$

Here Σ^2 rather than Σ is used to imply two summations, a summation within a given order c, and a summation of the different orders. In the present section Σ will be used for either of these summations singly, and Σ^2 will imply that both have to be made.

The most important case of (1) is when $a_{-1} \ldots . a_{-n}$ are independent. In this case $\mathrm{S}_n a^{(c)} a^{(n-c)}$ is not zero, and (1) is for most purposes more conveniently written

$$\left. \begin{aligned} q &= \Sigma^2 a_c^{(c)} \mathrm{S}_n q a_{n-c}^{(n-c)} . \mathrm{S}_n^{-1} . a^{(c)} a^{(n-c)} \\ &= \Sigma^2 a_c^{(c)} \mathrm{S} . q [a_{n-c}^{(n-c)} \mathrm{S}_n^{-1} a^{(c)} a^{(n-c)}] \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (2)$$

Here, no convention about the sequence of the fictors in the fictor products is necessary. In the last form of (2) $\mathrm{S}_n^{-1}(\ )$ has by (7) § 5 been passed into $\mathrm{S}_n(\ )$, and therefore the last S_n has been changed to S_0 or S. (1) may be similarly modified by passing ϖ (=product of the n fictits) into each $S_n(\ )$, thus

$$q S\varpi\alpha^{(c)}\alpha^{(n-c)} = \Sigma^2 \alpha_c^{(c)} S\varpi q \alpha_{n-c}^{(n-c)} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To prove (1), first suppose $\alpha_{-1} \ldots \alpha_{-n}$ are independent. In the proof it is convenient to say that $\alpha^{(c)}$ and $\alpha^{(n-c)}$ are supplementary products. Suppose $\alpha^{(c)}$ and $\beta^{(e)}$ are two of the products. Then

$$S_n \alpha_c^{(c)} \beta_e^{(e)} = 0 \quad \text{unless} \quad \beta_e^{(e)} = \pm \alpha_{n-c}^{(n-c)} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

for, by the reasoning of § 5, if $\alpha^{(c)}$ and $\beta^{(e)}$ contain no common factor, $\alpha^{(c)}\beta^{(e)}$ consists of $(c+e)^{\text{th}}$ and lower order parts, so that unless $\beta^{(e)}$ is supplementary to $\alpha^{(c)}$, $S_n\alpha_c^{(c)}\beta_e^{(e)} = 0$; and if $\alpha^{(c)}$ and $\beta^{(e)}$ contain any common factor of a fictors, $\alpha^{(c)}\beta^{(e)}$ certainly contains no higher part than the $(n-a)^{\text{th}}$ [and not so high a part unless all n fictors are present in $\alpha^{(c)}$ and $\beta^{(e)}$].

Also (when $\alpha_{-1}, \alpha_{-2}, \ldots$ are independent) the 2^n values of $\alpha_c^{(c)}$ are independent. For suppose, if possible,

$$x\alpha_c^{(c)} + y\beta_e^{(e)} + \ldots = 0.$$

Operating by $S_n\alpha_{n-c}^{(n-c)}(\ )$ we get $x = 0$.

We may then put q in the form $\Sigma^2 x\alpha_c^{(c)}$ and also in the form $\Sigma^2 y\alpha_{n-c}^{(n-c)}$. Operating on these two forms by $S_n(\ )\alpha_{n-c}^{(n-c)}$ and $S_n\alpha_c^{(c)}(\ )$ respectively we get the two forms of (1).

It will be noticed that we may definitely state, consistently with the above,

$$\alpha^{(n)} = \alpha^{(c)}\alpha^{(n-c)} = \alpha_{-1}\alpha_{-2} \ldots \alpha_{-n}, \ \alpha^{(0)} = 1 \quad . \quad . \quad . \quad . \quad (5)$$

When $\alpha_{-1}, \alpha_{-2} \ldots, \alpha_{-n}$ are not independent first suppose $\alpha_{-1}, \alpha_{-2}, \ldots, \alpha_{-(n-1)}$ are independent, and let β be a fictor of the fictorplex which is independent of these $n-1$ fictors. Thus (1) is true when for α_n we write either $\alpha_n + \beta$ or $\alpha_n - \beta$. Adding, (1) follows for this case. We may now suppose $\alpha_{-1}, \alpha_{-2}, \ldots \alpha_{-(n-2)}$ alone to be independent, and so on.

(1) is, of course, a generalisation of the two well-known quaternion theorems,

$$\begin{aligned} \rho S\alpha\beta\gamma &= \alpha S\beta\gamma\rho + \beta S\gamma\alpha\rho + \gamma S\alpha\beta\rho \\ &= V\beta\gamma S\alpha\rho + V\gamma\alpha S\beta\rho + V\alpha\beta S\gamma\rho. \end{aligned}$$

When I started to search for the theorem in our present subject corresponding to these, I confess I scarcely hoped to find so general a result. When it emerged I at once discarded my previous modes of groping after the key to the present methods, and selected the path of "orders" and of "parts." The great powers of the quaternion Kq in the fundamental logic and symbolism of quaternions had previously attracted and, I cannot help thinking, misled me.

The following obvious transformations enable ı number of alternative forms of (1), (2), (3),

$$\left.\begin{array}{l} S_n q r_c = S_n q_{n-c} r = S_n q_{n-c} r_c \\ S q r_c = S q_c r = S q_c r_c \end{array}\right\}$$

These are analogous to the quaternion formulæ

$$S.qVr = S.Vqr = SVqVr.$$

We will now suppose that $a_{-1}, a_{-2} \dots . a_{-n}$ are i use (2). Put

$$a_{n-c}^{(n-c)} S_n^{-1} a^{(c)} a^{(n-c)} = K\bar{a}_c^{(c)} \quad .$$

The expression on the left is clearly of the c^{th} ord of $\bar{a}_c^{(c)}$ is justified. The index dash (c) will be justifie can ignore it for the present. [I have had much hesi $\bar{a}_c^{(c)}$ or $K\bar{a}_c^{(c)}$ on the right of (7). The formulæ of t complicated with the K than they would be witho symmetry between $a^{(c)}$ and $\bar{a}^{(c)}$; and the translatio ordinary algebra is simpler. On the whole, perhaps better than the alternative.]

From (7) and (2) we have

$$S a_c^{(c)} K \bar{a}_c^{(c)} = 1 \quad . \qquad .$$

$$q = \Sigma^2 a_c^{(c)} S \bar{a}_c^{(c)} K q \qquad .$$

$$q_c = \Sigma a_c^{(c)} S \bar{a}_c^{(c)} K q \quad . \qquad .$$

the summation in the last, of course, only referrin which contain c fictors.

Putting c equal to 1 in (7) there are n fictor value which of $a_{-1}, \dots . a_{-n}$ is omitted. These we denote choose them so that (in harmony with (8)) we have

$$S a_{-1} K \bar{a}_{-1} = S a_{-2} K \bar{a}_{-2} = \dots . = 1$$

that is, we put

$$K\bar{a}_{-1} = S_{n-1} a_{-2} a_{-3} \dots . a_{-n} . S_n^{-1} a_{-1} a_{-2} \dots .$$

$$K\bar{a}_{-2} = -S_{n-1} a_{-1} a_{-3} \dots . a_{-n} . S_n^{-1} a_{-1} a_{-2} \dots$$

$$\dots \dots \dots \dots$$

In (12) we may interchange the set $a_{-1}, a_{-2}, \cdot$ $\bar{a}_{-1}, \bar{a}_{-2}, \dots . \bar{a}_{-n}$ that is, the two sets are symmetrically proved directly from § 5, but it is more easily deduced

* There is another and perhaps simpler method of dealing wi of linities. Let ϕ be given by $a_1, a_2, \dots .$ or the latter by the for

$$\phi\rho = a_{-1} S\rho\iota_1^{-1} + a_{-2} S\rho\iota_2^{-1} + \dots .$$

so that $a_{-1} = \phi\iota_1$ etc. ϕ^{-1} is not infinite. Then $\bar{a}_{-1} = \phi'^{-1}\iota_1$, $\bar{a}$. notation of § 8 below $a_c^{(c)} = \phi_c \iota^{(c)}$ etc., $\bar{a}_c^{(c)} = \phi_c'^{-1} \iota^{(c)}$ etc.—[*Note add* case $c=1$ in (10), and the corresponding statement derived from the second form of (1); for this last is

$$q=\Sigma^2\bar{a}_c^{(c)}\mathrm{S}a_c^{(c)}\mathrm{K}q \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and the particular cases of (10) and (13) mentioned are

$$\rho=\Sigma a_{-1}\mathrm{S}\bar{a}_{-1}\mathrm{K}\rho=\Sigma\bar{a}_{-1}\mathrm{S}a_{-1}\mathrm{K}\rho \quad . \quad . \quad . \quad . \quad (14)$$

[That (13) corresponds to the second form of (1) follows easily by noting that this second form clearly gives $q=\Sigma^2 x\bar{a}_c^{(c)}$ and deducing x from (8).] From (10) § 2 it now follows at once that $a_{-1}, a_{-2}, \ldots.$ must be the same functions of $\bar{a}_{-1}, \bar{a}_{-2}, \ldots.$ that the latter are of the former.

We will now show that $\bar{a}_c^{(c)}$ is the same function of $\bar{a}_{-1}, \bar{a}_{-2}, \ldots.$ that $a_c^{(c)}$ is of $a_{-1}, a_{-2}, \ldots.$, and therefore that $\bar{a}^{(c)}$ may be defined as the same product of $\bar{a}_{-1}, \bar{a}_{-2}, \ldots.$ that $a^{(c)}$ is of $a_{-1}, a_{-2}, \ldots.$ [This statement would not be true if the K had been omitted from the right of (7) or the left of (12), though it would if K were replaced by Q. Instead, we should have had to add that the sequence in $\bar{a}_c^{(c)}$ was the opposite of that in $a_c^{(c)}$; that is, $\mathrm{Q}\bar{a}_c^{(c)}$ would have been the same function of $\mathrm{Q}\bar{a}_{-1}, \mathrm{Q}\bar{a}_{-2}, \ldots.$ as $a_c^{(c)}$ is of $a_{-1}, a_{-2}, \ldots.$]

Let $a^{(c-1)}$ stand for any product of $c-1$ of the c fictors constituting $a^{(c)}$ in such sequence that if a is the c^{th} fictor

$$\mathrm{S}_c(a_{c-1}^{(c-1)}a)=a_c^{(c)},$$

and let $\bar{a}$ be the $\bar{a}_{-a}$ which corresponds to a according to (12). If we can show that

$$\mathrm{S}_c(\bar{a}_{c-1}^{(c-1)}\bar{a})=\bar{a}_c^{(c)}$$

it will follow that $\bar{a}_c^{(c)}$ is derived from $\bar{a}_{-1}, \bar{a}_{-2}\ldots.$ just as $a_c^{(c)}$ is from $a_{-1}, a_{-2}, \ldots.$

Now, putting for brevity

$$\mathrm{S}_n^{-1}a_{-1}a_{-2}\ldots.a_{-n}=\xi,$$

we have from (7)

$$\mathrm{K}\bar{a}_c^{(c)}=a_{n-c}^{(n-c)}\xi$$
$$\mathrm{K}\bar{a}_{c-1}^{(c-1)}=\mathrm{S}_{n-c+1}aa^{(n-c)}.\xi$$
$$\mathrm{K}\bar{a}=(-)^{c-1}\mathrm{S}_{n-1}a^{(c-1)}a^{(n-c)}.\xi.$$

Hence

$$\mathrm{S}_c(\mathrm{K}\bar{a}\mathrm{K}\bar{a}_{c-1}^{(c-1)})=(-)^{c-1}\mathrm{S}_c(\mathrm{S}_{n-1}a^{(c-1)}a^{(n-c)}.\xi.\mathrm{S}_{n-c+1}aa^{(n-c)}.\xi)$$
$$=(-)^{n(n-c)}\xi^2\mathrm{S}_c(\mathrm{S}_{n-1}a^{(c-1)}a^{(n-c)}.\mathrm{S}_{n-c+1}aa^{(n-c)})$$

[by transposing ξ with $\mathrm{S}_{n-c+1}(\ )$ and noticing that $\xi^2=$ scalar]

$$=(-)^{n(n-c)}\xi^2\mathrm{S}_n a^{(c-1)}a^{(n-c)}a.\mathrm{S}_{n-c}a^{(n-c)} \ [(10)\ \S\ 5].$$

Putting here

$$S_n a^{(c-1)} a^{(n-c)} a = (-)^{n-c} S_n a^{(c-1)} a a^{(n-c)} = (-)^{n-c} \xi^{-1}$$

and transposing the ξ that remains with $S_{n-c} a^{(n-c)}$ we obtain

$$S_c(\mathrm{K}\bar{a}\mathrm{K}\bar{a}_{c-1}^{(c-1)}) = a_{n-c}^{(n-c)}\xi = \mathrm{K}\bar{a}_c^{(c)}$$

or taking conjugates

$$S_c(\bar{a}_{c-1}^{(c-1)}\bar{a}) = \bar{a}_c^{(c)}.$$

This establishes complete reciprocation between the 2^n independent multenions $a_c^{(c)}$ based on the n fictors $a_{-1}, a_{-2}, \ldots$ and the 2^n independent multenions $\bar{a}_c^{(c)}$ based on the n fictors $\bar{a}_{-1}, \bar{a}_{-2}, \ldots$ Thus corresponding to (7) we have

$$\bar{a}_{n-c}^{(n-c)} S_n^{-1} \bar{a}^{(c)} \bar{a}^{(n-c)} = \mathrm{K} a_c^{(c)} \qquad (15)$$

Putting $c=n$ in (7) or (8) or (15) we have

$$a_n^{(n)}\mathrm{K}\bar{a}_n^{(n)} = 1 \qquad (16)$$

which expresses the reciprocal of a given determinant in terms of it. Moreover, the 2^n relations (8) express similar symmetrical properties of the minors of any order of the two determinants.

Let ρ be an independent variable fictor belonging to the multiplex of order n and ∇ the differential operator defined by

$$\rho = \Sigma\iota x, \quad \nabla = \Sigma\iota \mathrm{D}_x \qquad (17)$$

Thus if q is a function of ρ,

$$dq = \mathrm{S}d\rho\mathrm{K}\nabla . q \qquad (18)$$

and if β be any fictor,

$$\beta = \nabla_1 \mathrm{S}\rho_1 \mathrm{K}\beta = \mathrm{K}\nabla_1 \mathrm{S}\rho_1\beta \qquad (19)$$

Let $\mathrm{L}(\beta, \gamma)$ be any function of β, γ linear in each. Then

$$\mathrm{L}(\nabla_1, \rho_1) = \Sigma\mathrm{L}(\iota, \mathrm{D}_x\rho) = \Sigma\mathrm{L}(\iota, \iota) \qquad (20)$$

so that ∇_1 and ρ_1 may be interchanged. Define $\zeta_1, \zeta_1, \zeta_2, \zeta_2$, etc. by the equations

$$\mathrm{L}(\zeta, \zeta) = \mathrm{L}(\nabla_1, \rho_1), \ \mathrm{L}(\zeta_1, \zeta_1, \zeta_2, \zeta_2) = \mathrm{L}(\nabla_1, \rho_1, \nabla_2, \rho_2) \text{ etc.} \qquad (21)$$

(19) becomes

$$\beta = \zeta\mathrm{S}\zeta\mathrm{K}\beta = \mathrm{K}\zeta\mathrm{S}\zeta\beta \qquad (22)$$

By putting one $\zeta = \Sigma a \mathrm{S}\bar{a}\mathrm{K}\zeta = \Sigma\bar{a}\mathrm{S}a\mathrm{K}\zeta$ we get

$$\mathrm{L}(\zeta, \zeta) = \Sigma\mathrm{L}(a, \bar{a}) = \Sigma\mathrm{L}(\bar{a}, a) \qquad (23)$$

Applying this to each pair of ζ's in

$$\mathrm{S}_c\zeta_1\zeta_2 \ldots \mathrm{S}(\mathrm{S}_c\zeta_1\zeta_2 \ldots \zeta)\mathrm{K}q = \zeta^{(c)}\mathrm{S}\zeta^{(c)}\mathrm{K}q$$

we get from (9) and (10)

$$q_c = (c\,!)^{-1}\zeta_c^{(c)} S \zeta_c^{(c)} K q$$
$$q = \Sigma (c\,!)^{-1}\zeta_c^{(c)} S \zeta_c^{(c} K$$

We shall generally write these equations i

$$q_c = \eta_c^{(c)} S \eta_c^{(c)} K q \;.$$
$$q = \Sigma \eta_c^{(c)} S \eta_c^{(c)} K q$$

It may be noticed that if

$$\alpha_{-1} = \iota_1,\ \alpha_{-2} = \iota_2,\ \ldots\ \alpha_{-}$$

then $$\bar{\alpha}_{-1} = \iota_1,\ \bar{\alpha}_{-2} = \iota_2,\ \ldots\ \bar{\alpha}_{-}$$

Putting $q = \rho\varpi$ where ρ is a fictor in (2
have, after multiplying into ϖ^{-1},

$$\rho = K\eta_{n-1}^{(n-1)}.S_n\eta_{n-1}^{(n-1)}\rho = \Sigma K\alpha_{n-1}^{(n-1)}.S$$

which is wanted below. A similar more
obtained by changing q_c of (26) to $q_c\varpi$ and m

Multiplication of two factors q and Kr an
is so frequent, so simply connected with orc
and so important, that I think it desirable
K, chiefly to be used in this connection.
stroke |, or K = |,

$$S.pqK(rs) = Spq|r$$

In this connection I would re-state a plea
to other users of vector methods. "Scalar
should be restricted to one use, and the qu
in claim. Why not use the term "Energic |
product," or the term "projective product," for
and the corresponding symbol $E\rho_1\rho_2$?

7. Fictorlinities and Multilinities.—
their congeners are vectorlinities. The wc
understood to imply more than merely "line
that fictor or vector, or the like, is mentioı
fictorlinity is a linear *fictor* function of a
conjugate fictorlinity." *Matrix* does not se
properties to be connoted as fictorlinity.]

In this section I put in systematic form
as the best presentation of the properties of
present subject. I shall not in proofs go
chapter iv. of Octonions, and shall be as brie
to present generalised form theorems fr
Quaternions in Physics."

ρ being any fictor of a given fictorplex of order n, $\phi\rho$, a fictorlinity is defined as a function of ρ such that

$$\phi(\beta+\gamma)=\phi\beta+\phi\gamma \text{ always} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and that $\phi\rho$ is a fictor of the given fictorplex. It follows that

$$x\phi\beta=\phi(x\beta) \text{ always} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The general form of ϕ may be made to depend on n given fictors $\beta_1, \beta_2 \ldots.$ according to the first of the following, from which the second is deduced,

$$\left.\begin{array}{l}\phi\iota_1=\beta_1,\ \phi\iota_2=\beta_2,\ \ldots.;\\ \phi\rho=\beta_1 S\iota_1^{-1}\rho+\beta_2 S\iota_2^{-1}\rho+\ldots.\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The following is a fictorlinity in all cases, and if the number of pairs of fictors (β_1, a_1), (β_2, a_2) .... is n or more it is a general form

$$\phi\rho=\beta_1 Sa_1|\rho+\beta_2 Sa_2|\rho+\ldots.=\Sigma\beta Sa|\rho \quad . \quad . \quad . \quad (4)$$

We may apply eq. (22), § 6, to either ρ or $\phi\rho$ thus

$$\phi\rho=\phi\zeta.S\zeta|\rho=\zeta S\zeta|\phi\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The equation

$$S\rho|\phi\sigma=S\sigma|\phi'\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

if true for all fictor values of ρ and σ, *defines* ϕ'; and ϕ' so defined proves to be a fictorlinity; it is called the conjugate of ϕ.

[If Law A is assumed, (6) may be replaced by $S\rho\phi\sigma=S\sigma\phi'\rho$, but the two equations do not mean the same when Law A is not assumed. In the latter case eq. (6) is better than the alternative, since if ϕ is given by

$$\begin{array}{l}\phi\iota_1=x_1'\iota_1+x_2'\iota_2+x_3'\iota_3+\ldots.\\ \phi\iota_2=x_1''\iota_1+\ldots.\\ \phi\iota_3=x_1'''\iota_1+\ldots.\end{array}$$

then, according to (6), to get ϕ' from ϕ we have merely among the scalars x_1', x_2', ... to interchange rows and columns. The alternative $S\rho\phi\sigma= S\sigma\phi'\rho$ produces no such simple statement among n^2 given scalars.]

From (6) and (5)

$$S\sigma|\phi'\rho=S\sigma|\zeta S\zeta|\phi'\rho=S\sigma|(\zeta S\rho|\phi\zeta).$$

Since σ is arbitrary we have [(10) § 2]

$$\phi'\rho=\zeta S\rho|\phi\zeta \quad \text{or} \quad \phi'=\zeta S(\ )|\phi\zeta \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which gives ϕ' explicitly in terms of ϕ. Conversely, (6) follows when ϕ' is defined by (7).

From either (6) or (7)

$$\phi'\rho=\Sigma a S\rho|\beta \quad \text{when} \quad \phi\rho=\Sigma\beta S\rho|a \quad . \quad . \quad . \quad . \quad (8)$$

Hence [since $\Sigma\beta S(\ )|\alpha$ is a general form] if ϕ is self-conjugate and equal to $\Sigma\beta S(\ )|\alpha$, it is also equal to $\Sigma\alpha S(\ )|\beta$, or to half the sum of these two. Hence we have the following: any fictorcolinity (self-conjugate fictorlinity) is given by

$$\psi\rho = \tfrac{1}{2}\Sigma(\alpha S\rho|\beta + \beta S\rho|\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If we put $\bar{\phi}$ for the self-conjugate (or colinity) part $\frac{1}{2}(\phi+\phi')$, as it is called, of ϕ,

$$\bar{\phi}\rho = \tfrac{1}{2}\Sigma(\alpha S\rho|\beta + \beta S\rho|\alpha) \text{ when } \phi\rho = \Sigma\beta S\rho|\alpha \quad . \quad . \quad . \quad (10)$$

If we put ϕ_s for the skew part $\frac{1}{2}(\phi-\phi')$, as it is called, of ϕ,

$$\phi_s\rho = \tfrac{1}{2}\Sigma(-\alpha S\rho|\beta + \beta S\rho|\alpha) = \tfrac{1}{2}\Sigma S_1.(S_2\beta\alpha)|\rho \quad . \quad . \quad (11)$$

[(17) § 4].

Let $L(\gamma, \epsilon)$ be any function (scalar, fictor, multenion, etc.) linear in each of its constituent fictors γ, ϵ. Then with (8)

$$L(\zeta, \phi\zeta) = \Sigma L(\alpha, \beta); \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\text{always} \quad L(\zeta, \phi\zeta) = L(\phi'\zeta, \zeta); \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

with a colinity always

$$L(\zeta, \psi\zeta) = L(\psi\zeta, \zeta); \quad . \quad . \quad . \quad . \quad (14)$$

with a skew linity always

$$L(\zeta, \phi_s\zeta) = -L(\phi_s\zeta, \zeta); \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\text{with (8)} \quad L(\zeta, \bar{\phi}\zeta) = L(\bar{\phi}\zeta, \zeta) = \tfrac{1}{2}\Sigma[L(\alpha, \beta) + L(\beta, \alpha)] \quad . \quad . \quad . \quad . \quad (16)$$

If χ is an arbitrary fictorlinity and $\bar{\chi}$ an arbitrary fictorcolinity, then if ϕ and ψ are fictorlinities,

$$\phi = \psi \text{ when } S\chi\zeta|\phi\zeta = S\chi\zeta|\psi\zeta \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\bar{\phi} = \bar{\psi} \text{ when } S\bar{\chi}\zeta|\phi\zeta = S\bar{\chi}\zeta|\psi\zeta \quad . \quad . \quad . \quad . \quad . \quad (18)$$

[For (17) put $\chi = \beta S(\ )|\alpha$ and for (18) $\bar{\chi} = \frac{1}{2}[\beta S(\ )|\alpha + \alpha S(\ )|\beta]$.]

We will here make a digression to prove that when Law A holds, ζ and ∇ are invariants, that is, that we get the same meanings for them if in their definitions the fictits $q\iota_1q^{-1}, q\iota_2q^{-1}, \ldots$ (where q is a fictor product) be used in place of $\iota_1, \iota_2, \ldots$

Let $\phi = q(\ )q^{-1}$, so that by (6) $\phi' = q^{-1}(\ )q = \phi^{-1}$. [It is here that Law A is assumed. By (6)

$$S\sigma|\phi'\rho = S\rho|\phi\sigma = Sq\sigma q^{-1}|\rho = S(\sigma q^{-1}|\rho.q) = Sq^{-1}\rho q|\sigma$$

because if Law A holds $|\sigma = \iota^2\sigma$, $|\rho = \iota^2\rho$. If Law A were not assumed to hold we should get $\phi' = Kq(\ )Kq^{-1}$ in place of $\phi' = q^{-1}(\ )q$; just as if ϕ be a multilinity given by $\phi r = qrq^{-1}$, its conjugate ϕ' is given by $\phi' r = Kq.r.Kq^{-1}$.] By (13)

$$L(\phi\zeta, \phi\zeta) = L(\phi\phi'\zeta, \zeta) = L(\zeta, \zeta).$$

This shows that ζ is an invariant.

Let $\lambda_1 = q\iota_1 q^{-1}$, $\lambda_2 = q\iota_2 q^{-1}$, ... and let ρ be an independent variable fictor given by

$$\rho = x_1\iota_1 + x_2\iota_2 + \dots = y_1\lambda_1 + y_2\lambda_2 + \dots$$

[Here $x_1, x_2, \dots$ regarded alone are independents; and $y_1, y_2 \dots$ so regarded are independents. Of course each set is expressible in terms of q and the other set.]

Let

$$\nabla = \Sigma\iota D_x, \quad \nabla' = \Sigma\lambda D_y.$$

We have to show that $\nabla = \nabla'$. Now by (18) § 6

$$Sd\rho|\nabla.q = Sd\rho|\nabla'.q$$

where q is any function of ρ. That is

$$Sd\rho|\nabla = Sd\rho|\nabla'$$

for all values of $d\rho$. Hence by (10) § 2 $\nabla = \nabla'$.

The n-tic.—I would here recall the reader's attention to the equations (20) to (27) § 6, where $\zeta^{(c)}$ and $\eta^{(c)}$ first appeared, and would add an obvious deduction from (23) § 6 [on the lines of (24) and (26) § 6], namely,

$$L(\eta_c^{(c)}, \eta_c^{(c)}) = \Sigma L(a_c^{(c)}, \bar{a}_c^{(c)}) = \Sigma L(\bar{a}_c^{(c)}, a_c^{(c)}) \quad . \quad . \quad . \quad (19)$$

ϕ satisfies the n-tic

$$\left.\begin{array}{l} \phi^n - h'\phi^{n-1} + h''\phi^{n-2} - \dots + (-)^n h^{(n)} = 0 \\ \text{where} \quad h^{(c)} = S\eta_c^{(c)}|(\phi\eta)_c^{(c)} \end{array}\right\} \quad . \quad . \quad . \quad (20)$$

We can scarcely hope, I think, to throw this famous theorem into a more compendious form, or one which by (19) can so easily be transformed into many other very general forms (to take but one example: put $a_1 = \bar{a}_1 = \iota_1$, $a_2 = \bar{a}_2 = \iota_2$, etc.), or one which more clearly calls attention to the many invariants associated with it.

To prove (20) we must first enunciate regarding the n-tic what is obvious from chapter iv. of *Octonions*. This part of the chapter is taken almost directly from Grassmann's Ausdehnungslehre.

$h^{(c)}$ is *defined* as the coefficient of $(-x)^{n-c}$ in

$$S_n(\phi - x)a_1(\phi - x)a_2 \dots (\phi - x)a_n . S_n^{-1}a_1 a_2 \dots a_n \equiv \Sigma h^{(c)}(-x)^{n-c} \quad . \quad (21)$$

where $a_1, a_2, \dots a_n$ (our former, § 6, $a_{-1}, a_{-2}, \dots$ which we now discard) are n independent fictors of the given fictorplex of the n^{th} order. [It is not necessary for our present purpose (since it follows by our present method) to show that (21) is independent in meaning of the particular set $a_1, a_2, \dots a_n$ selected, though this is shown in Octonions.]

It is then shown that if the roots of $\Sigma h^{(c)}(-x)^{n-c} = 0$ are a occurring A times only, b occurring B times only, etc., the given fictorplex consists of the following independent fictorplexes,

$$(\lambda_1, \lambda_2, \ldots . \lambda_A), (\mu_1, \mu_2, \ldots . \mu_B), \ldots .$$
$$[\text{or say} \quad (\lambda) \quad , \quad (\mu), \quad \ldots .]$$

which are such that

$$\left.\begin{array}{l} \phi\lambda_1 = a\lambda_1,\ \phi\lambda_2 = a\lambda_2 + \lambda'_1, \ldots . , \phi\lambda_A = a\lambda_A + \lambda'_{A-1} \\ \phi\mu_1 = b\mu_1,\ \phi\mu_2 = b\mu_2 + \mu'_1, \ldots . , \phi\mu_B = b\mu_B + \mu'_{B-1} \\ \ldots \ldots \ldots \ldots \end{array}\right\} \quad . \quad . \quad (22)$$

where λ'_e stands for some fictor of the fictorplex $(\lambda_1, \lambda_2, \ldots . \lambda_e)$, and similarly for μ'_e, etc. Among other things (for which see the chapter) it is further shown (1) that the independent fictorplexes (λ), (μ), ... are definitely determined by ϕ; (2) that $(\phi - a)^A$ kills every fictor of the fictorplex (λ) and no other, $(\phi - b)^B$ kills every fictor of (μ) and no other, etc.; (3) and therefore that

$$(\phi - a)^A(\phi - b)^B \ldots . \equiv \phi^n - h'\phi^{n-1} + \ldots . + (-)^n h^{(n)}$$

kills every fictor of the given fictorplex of order n. [Note that if ϕ be supposed of such form as to be capable of operating on fictors not belonging to the given fictorplex, then even if we impose the condition that ϕ kills every fictor outside the given fictorplex, it is not true in general that $\phi^n - \ldots . + (-)^n h^{(n)}$ kills them. In this case we have to raise the degree of the quantic by 1, and say that $\phi(\phi^n - h'\phi^{n-1} + \ldots .)$ kills every fictor whatsoever.]

To deduce the form of $h^{(c)}$ in (20) from (21) we have

$$h^{(c)} = \Sigma S_n(\phi a)^{(c)} a^{(n-c)} S_n^{-1} a^{(c)} a^{(n-c)} \quad . \quad . \quad . \quad . \quad (23)$$

$$\begin{aligned} h^{(c)} &= \Sigma S.(\phi a)^{(c)}[a^{(n-c)} S_n^{-1}\ a^{(c)} a^{(n-c)}], && [(7)\ \S\ 5] \\ &= \Sigma S.(\phi a)_c^{(c)|}\bar{a}_c^{(c)} && [(7)\ \S\ 6] \\ &= S.(\phi\eta)_c^{(c)|}\eta_c^{(c)} && [(19)\ \S\ 7] \end{aligned}$$

We get a second standard form of $h^{(c)}$ from (23) thus,

$$h^{(c)} = S_n(\phi a)^{(c)} a^{(n-c)} | S_n \bar{a}^{(c)} \bar{a}^{(n-c)} \qquad [(16)\ \S\ 6]$$

whence by (19) § 7

$$\left.\begin{array}{rl} & \phi^n - h'\phi^{n-1} + \ldots . + (-)^n h^{(n)} = 0 \\ \text{where} & h^{(c)} = (c\,!)^{-1}([n-c]\,!)^{-1} S_n(\phi\zeta)^{(c)}\zeta^{(n-c)} | S_n \zeta^{(c)}\zeta^{(n-c)} \\ & \quad = {}_nC_c S(\phi\eta)^{(c)}\eta^{(n-c)} | S_n \eta^{(c)}\eta^{(n-c)} \end{array}\right\} \quad . \quad . \quad (24)$$

These two forms of $h^{(c)}$ are given to prevent ambiguity as to the connection between η and ζ here understood. [For the full quaternion forms of (20) and (24) see *Utility of Quaternions in Physics*, text and footnote, p. 17.]

(23) gives $h^{(c)}$, and therefore the n-tic in terms of any n independent fictors. This is one method familiar to quaternionists. Another is to express $h^{(c)}$ in terms of the a, β of $\phi\rho = \Sigma\beta \mathrm{S}a|\rho$ [(4)]. This is at once given in terms of (12) and (20) thus,

$$h^{(c)} = \Sigma \mathrm{S}\beta_c^{(c)}|a_c^{(c)} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$h^{(n)}$ is the familiar discriminant. This is of such fundamental importance that I propose to reserve the * symbol $[\phi]$ for it, the vertical lines of the square brackets being intended to recall the vertical lines of the determinant form. Thus

$$\left.\begin{aligned}[\phi] = h^{(n)} = \eta_n^{(n)}|(\phi\eta)_n^{(n)} &= \mathrm{S}_n\phi a_1\phi a_2 \ldots . \phi a_n \mathrm{S}_n^{-1} a_1 a_2 \ldots . a_n \\ &= \mathrm{S}_n\phi\iota_1\phi\iota_2 \ldots . \phi\iota_n . (\iota_1\iota_2 \ldots . \iota_n)^{-1}\end{aligned}\right\} \quad . \quad (26)$$

With this notation the scalar n-tic [(21)] itself is

$$0 = [\phi - x] \equiv \Sigma h^{(c)}(-x)^{n-c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and the ϕ n-tic is obtained by replacing the scalar x by the symbol ϕ.

To get the reciprocal ϕ^{-1} of ϕ put in the eq. [(26)],

$$[\phi]\mathrm{S}_n a_1 a_2 \ldots . a_n = \mathrm{S}_n \phi a_1 \phi a_2 \ldots . \phi a_n$$

$a_1 = \rho$, $a_2 = \eta_1$, ... $a_n = \eta_{n-1}$; multiply by $\eta_{n-1}^{(n-1)}$; and apply (29) § 6. Thus

$$\rho[\phi] = \mathrm{K}\eta_{n-1}^{(n-1)}\mathrm{S}_n(\phi\eta)^{(n-1)}\phi\rho \quad . \quad . \quad . \quad . \quad . \quad (28)$$

or when $[\phi]$ is not zero,

$$\left.\begin{aligned}\phi^{-1}\rho &= [\phi]^{-1}\mathrm{K}\eta_{n-1}^{(n-1)}\mathrm{S}_n(\phi\eta)^{(n-1)}\rho \\ &= [\phi]^{-1}\mathrm{K}(\phi'\eta)_{n-1}^{(n-1)}\mathrm{S}_n\eta^{(n-1)}\rho\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

the last form being given by (13). These express ϕ^{-1} explicitly both in terms of ϕ and of ϕ', at any rate if we add

$$[\phi'] = [\phi] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

which is a particular case of the statement that on changing ϕ into ϕ' $h^{(c)}$ is unaltered [(20), (13)], that is, the two n-tics are identical. It also follows from these results that ϕ^{-1} and ϕ'^{-1} are fictorlinities and are conjugate to one another.

We also have

$$[\phi\psi] = [\phi][\psi] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

which contains the usual theorem for the product of two given deter-

* There is here a *most unfortunate* oversight in the notation. In § 8 below, a series of multilinities ϕ_0, ϕ_1, ϕ_2 . . . . ϕ_n of a quite fundamental kind are defined in terms of ϕ; $\phi_1 = \phi$ and ϕ_c is of c dimensions in ϕ. The oversight is that in this series ϕ_n is the *scalar* here denoted by $[\phi]$. ϕ_n is a far more expressive symbol than $[\phi]$, and should, throughout the paper, be read instead of $[\phi]$.—[*Note added April* 1908.]

minants each of the n^{th} order as a determinant of the same order. To prove (31) we have

$$\begin{aligned}[\phi\psi] &= (\phi\psi\eta)_n^{(n)}|\eta_n^{(n)} && [(26)]\\ &= (\phi\eta)_n^{(n)}|(\psi'\eta)_n^{(n)} && [(13)]\\ &= (\phi\eta)_n^{(n)}|\eta_n^{(n)}.[\psi'] && [(26)]\\ &= [\phi][\psi] && [(26), (30)].\end{aligned}$$

By putting $\psi = \phi^{-1}$ in (31) we get

$$[\phi][\phi^{-1}] = [1] = 1$$

$$\text{or} \quad [\phi^{-1}] = [\phi]^{-1} \quad \ldots \ldots \quad (32)$$

We may also note here that the roots of the $f(\phi)$ n-tic are $f(g_1), f(g_2), \ldots.$ where f is a rational integral function of ϕ, and $g_1, g_2, \ldots.$ are the roots of the ϕ n-tic. For, first, if ϕ is given by (22) its n-tic must be $(x-a)^{\text{A}}(x-b)^{\text{B}}\ldots.$ as we see by putting $\lambda_1, \lambda_2, \ldots. \mu_1, \ldots.$ for $a_1, a_2, \ldots.$ in (21). But (*Octonions*, p. 159),

$$\begin{aligned}&f(\phi)\lambda_1 = \lambda_1 f(a), f(\phi)\lambda_2 = \lambda_2 f(a) + \lambda_1'', \ldots.\\ &f(\phi)\mu_1 = \mu_1 f(b), f(\phi)\mu_2 = \mu_2 f(b) + \mu_1'', \ldots.\\ &\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

where λ''_e belongs, like λ'_e, to the fictorplex $\lambda_1, \lambda_2, \ldots. \lambda_e$, etc. Hence the roots of the $f(\phi)$ n-tic are $f(a)$ occurring A times, $f(b)$ occurring B times, etc.

It will be seen that this result may be put

$$\left.\begin{aligned}&[f(\phi) - x] \equiv \{f(g_1) - x\}\{f(g_2) - x\} \ldots.\\ \text{when} \quad &[\phi - x] \equiv (g_1 - x)(g_2 - x) \ldots.\end{aligned}\right\} \quad \ldots \quad (33)$$

From (32) and (33) we easily deduce that

$$[\phi^k] = [\phi]^k \quad \ldots \ldots \quad (33a)$$

where k is any integer positive or negative.

Before proceeding further with fictorlinities it is desirable to ask and answer the question—How far are the results hitherto obtained capable of simple extension from fictorlinities to multilinities? The answer is that all the numbered equations of this section excepting (19), (20), (25) are thus capable of simple extension. The feature of (19), (20), and (25) incapable of the extension is the interpretation of $\text{S}\alpha_c^{(c)}|\beta_c^{(c)}$ (or of $a_c^{(c)}$) except when c is 0, 1, $n-1$, or n. There is no simple close analogy in multenion multiplication to even so simple a fictor product as $\alpha\beta$ of only two fictors.

I will content myself with merely stating (the proof is simple) the principles of extending the meanings of the other equations.

Let $u_1, u_2 \ldots\ldots u$ be k multenions satisfying the conditions

$$\mathrm{S}u_1|u_1 = \mathrm{S}u_2|u_2 = \ldots\ldots = 1,\ \mathrm{S}u_1|u_2 = \mathrm{S}u_1|u_3 = \ldots\ldots = 0 \quad . \quad . \quad (34)$$

It is easy to prove that they must be independent, and are capable of being obtained from the multits by circular variation. The n fictits are one particular case of $u_1, u_2, \ldots\ldots u_k$ and the 2^n multits are another particular case.

Let all the multenions considered belong to the complex $u_1, u_2, \ldots\ldots u_k$.

If $\alpha = \Sigma x\iota$ and $q = \Sigma yu$, x and y may be called a co-ordinate of α and a co-ordinate of q respectively. Thus $\mathrm{S}\alpha|\beta$ is the same function of the co-ordinates of α and β that $\mathrm{S}p|q$ is of the co-ordinates of p, q. Hence we define the conjugate ϕ' of a multilinity ϕ by

$$\mathrm{S}r|\phi' s = \mathrm{S}s|\phi r \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

for all multenion values of r, s. The reader is advised to omit the details from this point to eq. (54) till an occasion when he finds he requires them for other applications. ζ may be paralleled by ξ according to the equations

$$\mathrm{L}(\zeta, \zeta) = \Sigma\mathrm{L}(\iota, \iota),\ \mathrm{L}(\xi, \xi) = \Sigma\mathrm{L}(u, u) \quad . \quad . \quad . \quad . \quad (36)$$

whence parallel with (22) § 6 we have

$$q = \xi\mathrm{S}\xi|q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Hence, omitting the last member of (11), all the equations (1) to (18) may be read as multilinity equations by merely substituting

$$p,\ q,\ t\ ;\ r,\ s\ ;\ \xi,$$

for $\quad \alpha,\ \beta,\ \gamma\ ;\ \rho,\ \sigma\ ;\ \zeta.$

Let

$$\left.\begin{aligned} q &= au_1 + bu_2 + \ldots\ldots + lu_k \\ q_1 &= a_1u_1 + b_1u_2 + \ldots\ldots + l_1u_k \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ q_{k-1} &= a_{k-1}u_1 + b_{k-1}u_2 + \ldots\ldots + l_{k-1}u_k \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (38)$$

be k given multenions. Then define the "combinatorial scalar" $\mathrm{C}_s q^{(k)} = \mathrm{C}_s(qq_1 \ldots\ldots q_{k-1})$ of k multenions $q, q_1 \ldots\ldots q_{k-1}$; and the "combinatorial multenion" $\mathrm{C}_m q^{(k-1)} = \mathrm{C}_m(q_1q_2 \ldots\ldots q_{k-1})$ of $k-1$ multenions by the equations

$$\mathrm{C}_s q^{(k)} = \begin{vmatrix} a & b & \ldots\ldots & l \\ a_1 & b_1 & \ldots\ldots & l_1 \\ \ldots & \ldots & \ldots\ldots & \ldots \\ a_{k-1} & b_{k-1} & \ldots\ldots & l_{k-1} \end{vmatrix} \quad . \quad . \quad . \quad . \quad (39)$$

$$\mathrm{C}_s q^{(k)} = \mathrm{C}_s q q^{(k-1)} = \mathrm{S}q|\mathrm{C}_m q^{(k-1)} \quad . \quad . \quad . \quad . \quad (40)$$

Thus if A, B, ... L are the minors corresponding to the first row of (39)

$$C_m q^{(k-1)} = Au_1 + Bu_2 + \ldots + Lu_k = \begin{vmatrix} u_1 & u_2 & \ldots & u_k \\ a_1 & b_1 & \ldots & l_1 \\ a_2 & b_2 & \ldots & l_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_{k-1} & b_{k-1} & \ldots & l_{k-1} \end{vmatrix} \quad . \quad . \quad . \quad . \quad (41)$$

We may now write down in multenions many analogies to general results above in fictors. Thus if $p_1, p_2 \ldots p_k$ be k given independent multenions and

$$\left.\begin{aligned} \bar{p}_1 &= C_m p_2 p_3 \ldots p_k C_s^{-1} p_1 p_2 \ldots p_k \\ \bar{p}_2 &= -C_m p_1 p_3 \ldots p_k C_s^{-1} p_1 p_2 \ldots p_k \\ & \ldots \ldots \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (42)$$

then
$$\left.\begin{aligned} Sp_1|\bar{p}_1 &= Sp_2|\bar{p}_2 = \ldots = 1 \\ Sp_1|\bar{p}_2 &= Sp_1|\bar{p}_3 = Sp_2|\bar{p}_3 = \ldots = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (43)$$

$$q = \Sigma p S\bar{p}|q = \Sigma \bar{p} Sp|q \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$L(\xi, \xi) = \Sigma L(p, \bar{p}) = \Sigma L(\bar{p}, p) \quad . \quad . \quad . \quad . \quad . \quad (45)$$

In place of (21), (23), (24), (26), (27), (28), (29) we have

$$\left.\begin{aligned} & C_s(\phi - x)p_1(\phi - x)p_2 \ldots (\phi - x)p_k C_s^{-1} p_1 p_2 \ldots p_k \equiv \Sigma h^{(c)}(-x)^{k-c} \\ & \phi^k - h'\phi^{k-1} + \ldots + (-)^k h^{(k)} \equiv (\phi - a)^A (\phi - b)^B \ldots = 0 \end{aligned}\right\} \quad . \quad (46)$$

$$\left.\begin{aligned} h^{(c)} &= \Sigma C_s(\phi p)^{(c)} p^{(k-c)} . C_s^{-1} p^{(c)} p^{(k-c)} \\ &= (c\,!)^{-1}([k-c]\,!)^{-1} C_s(\phi\xi)^{(c)} \xi^{(k-c)} . C_s \xi^{(c)} \xi^{(k-c)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (47)$$

$$\left.\begin{aligned} [\phi] = h^{(k)} &= (k\,!)^{-1} C_s \xi^{(k)} C_s(\phi\xi)^{(k)} = C_s \phi p_1 \phi p_2 \ldots \phi p_k C_s^{-1} p_1 p_2 \ldots p_k \\ &= C_s \phi u_1 \phi u_2 \ldots \phi u_k \end{aligned}\right\} \quad . \quad (48)$$

$$0 = [\phi - x] \equiv \Sigma h^{(c)}(-x)^{k-c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

$$\left.\begin{aligned} \phi^{-1} r &= ([k-1]\,!)^{-1} . [\phi]^{-1} . C_m \xi^{(k-1)} . C_s r(\phi\xi)^{(k-1)} \\ &= [\phi]^{-1} . \Sigma C_m p^{(k-1)} . C_s r(\phi\bar{p})^{(k-1)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (50)$$

$$\left.\begin{aligned} \phi^{-1} r &= ([k-1]\,!)^{-1} . [\phi']^{-1} . C_m(\phi'\xi)^{(k-1)} . C_s r \xi^{(k-1)} \\ &= [\phi']^{-1} . \Sigma C_m(\phi' p)^{(k-1)} . C_s r \bar{p}^{(k-1)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (51)$$

These transformations are effected by first making them for the particular case given by

$$u_1 = \iota_1, \ldots, u_n = \iota_n, \qquad \varpi = \iota_1 \iota_2 \ldots \iota_n \quad . \quad . \quad . \quad . \quad (52)$$

for which we have the transforming formulæ

$$\left.\begin{aligned} C_s a^{(n)} &= a_n^{(n)} \varpi^{-1} = S\varpi|a^{(n)} \\ C_m a^{(n-1)} &= S_1 \varpi|a^{(n-1)} = \varpi K a_{n-1}^{(n-1)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (53)$$

$$a_n^{(n)} = C_s a^{(n)} . \varpi, \quad a_{n-1}^{(n-1)} = K C_m a^{(n-1)} . \varpi \quad . \quad . \quad . \quad . \quad (54)$$

in which it is to be remembered [(41)] that $C_m a^{(n-1)}$ is a fictor.

From the above it appears that the theory of multilinities is virtually included in that of fictorlinities. We shall therefore henceforth only consider fictorlinities, and shall assume the truth of any transformation into multilinity form that we may require.

As this section has extended far beyond my desire, I must content myself with a mere summary of other properties, and proofs thereof, of linities.

To refer to ordinary algebra the following notation will be used,

$$\phi=\begin{bmatrix} a_1\, b_1 \ldots \\ a_2\, b_2 \ldots \\ \ldots \end{bmatrix} \text{ means } \begin{Bmatrix} \phi u_1 = a_1u_1 + b_1u_2 + \ldots \\ \phi u_2 = a_2u_2 + b_2u_2 + \ldots \\ \ldots \end{Bmatrix} \quad . \quad . \quad . \quad (55)$$

and the rows and columns of $[a_1 \ldots]$ will be called the rows and columns of ϕ. Thus the rows and columns of ϕ' are the columns and rows of ϕ respectively.

There are three specially important kinds of real linities. (1) Colinities, $\phi'=\phi$. Columns the same as rows. Colinities the roots of which are all positive or all negative will be called positive or negative colinities respectively. (2) Skewlinities, $\phi'=-\phi$. Columns are the rows with sign changed so that the elements of the principal diagonal are zeros. (3) Rotational linities, $\phi'\phi=1$. A rotational *fictor*linity is a rigid replacement A rotational *multi*linity is in general not a rigid replacement.

Colinities.—A set of fictits can always be found such that

$$\phi\iota_1=a_1\iota_1,\ \phi\iota_2=a_2\iota_2,\ \ldots\ \phi\iota_n=a_n\iota_n \quad . \quad . \quad . \quad . \quad (56)$$

when ϕ is a colinity; where $a_1, a_2, \ldots$ are all real. The n-tic and the discriminant are

$$(\phi-a_1)(\phi-a_2)\ldots(\phi-a_n)=0,\ [\phi]\equiv a_1a_2\ldots a_n \quad . \quad . \quad (57)$$

Proved from (22) thus:—λ'_1 is zero since $S\lambda_1|\phi\lambda_2=S\lambda_2|\phi\lambda_1$; λ'_2 is zero since $S\lambda_3|\phi\lambda_1=S\lambda_1|\phi\lambda_3$, $S\lambda_3|\phi\lambda_2=S\lambda_2|\phi\lambda_3$; etc. The A[th] order fictorplex may be expressed as that of A *fictits* $\lambda_1, \lambda_2, \ldots$ Similarly for $\mu_1, \mu_2, \ldots$ From $S\lambda_1|\phi\mu_1=S\mu_1|\phi\lambda_1$ and $a\neq b$ we get $S\lambda_1|\mu_1=0$. Lastly, a_1 of (56) is real, for putting

$$a_1=x+y\sqrt{(-1)},\quad \iota_1=\rho+\sigma\sqrt{(-1)}$$

where x_1y are real scalars and ρ, σ real fictors, we get from $0=(\phi-a_1)\iota_1$ that

$$(\phi-x)\rho=-y\sigma,\quad (\phi-x)\sigma=y\rho,$$

and now from

$$S\rho|(\phi-x)\sigma=S\sigma|(\phi-x)\rho$$

we get $y=0$, since $S\rho|\rho$ and $S\sigma|\sigma$ are both positive [(18) § 2]. From this it at once follows that the fictorplex (λ) of (22) may be taken as a real one. [Thus from the above if $(\phi-x)\iota_1=0$ *both* $(\phi-x)\rho=0$ *and* $(\phi-x)\sigma=0$.] Hence $\iota_1, \iota_2 \ldots$ of (56) may be taken as real.

We now have the very important facts that if ϕ is any real linity :—(1) $\phi'\phi$ and $\phi\phi'$ are colinities; (2) they are *positive* colinities; (3) $\phi'\phi\rho=0$ if, and only if, $\phi\rho=0$, and $\phi\phi'\rho=0$ if, and only if, $\phi'\rho=0$. We will prove the statements for $\phi'\phi$ only.

$$S\sigma \mid \phi'\phi\rho = S\phi\sigma \mid \phi\rho = S\phi'\phi\sigma \mid \rho.$$

This proves (1). If $\phi'\phi\rho=a\rho$, which is always true for some non-evanescent real ρ, when a is a root of $\phi'\phi$,

$$aS\rho \mid \rho = S\rho \mid \phi'\phi\rho = S\phi\rho \mid \phi\rho,$$

so that a is *positive* or zero according as $\phi\rho$ is not or is zero. This proves (2). If $\phi\rho=0$, then $\phi'\phi\rho$ obviously $=0$. If $\phi\rho$ is not zero, $S\phi\rho \mid \phi\rho$ is not zero; therefore $S\rho \mid \phi'\phi\rho$ is not zero; therefore $\phi'\phi\rho$ is not zero.

The square arrays corresponding to $\phi'\phi$ and $\phi\phi'$ [(55)] are familiar as those occurring in the two symmetric forms of the square of a determinant. If, then, from each diagonal element of a symmetric determinant so formed x be subtracted and the expression so obtained be equated to zero, not only are the roots of the equation real, but they are positive (including zeros).

Thus $\surd(\phi'\phi)$, except as to the *signs* of its roots, is a real colinity without ambiguity. If we impose that the roots are all positive, or else that they are all negative, no ambiguity remains. These two special colinities may be denoted by $(+\surd\phi'\phi)$ [read "the plus radical of $\phi'\phi$"] and $(-\surd\phi'\phi)$ respectively.

Skewlinities.—[I could write down several interesting formulæ involving first and second order multenions which are analogous to quaternion formulæ involving vectors. The most important statement of the kind for our immediate purpose is that though $S\alpha\beta\gamma$ is necessarily zero when α, β, γ are fictors, yet $S\rho\omega\sigma$ is not in general zero, but is a combinatorial part of ρ, ω, σ when ρ and σ are fictors and ω is a second order multenion. Thus

$$S\rho\omega\sigma = S\omega\sigma\rho = S\sigma\rho\omega = -S\sigma\omega\rho = -S\rho\sigma\omega = -S\omega\rho\sigma.$$

In treating of skew fictorlinities and rotational fictorlinities it is the general second order multenion, ω, with its $\frac{1}{2}n(n-1)$ scalars which appears as the analogue of a vector, rather than the general fictor, a, with its n scalars.]

Putting in (11) $\frac{1}{2}\Sigma S_2\beta|a=\omega$ so that ω is any second order multenion we obtain that

$$\chi\rho = S_1\omega\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

is the most general type of skewlinity. It can easily be proved that given (58), then $\chi' = -\chi$, and that given $\chi' = -\chi$, then χ involves $\frac{1}{2}n(n-1)$ arbitrary scalars.

From the definition $S\sigma|\chi\rho = S\rho|\chi'\sigma$, $\chi' = -\chi$ we get by putting $\sigma = \rho$ that

$$S\rho|\chi\rho = 0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

for every ρ. This also follows from (58).

Conversely if (59) holds for every ρ, χ is skew. For $S\iota_1|\chi\iota_1 = S\iota_2|\chi\iota_2 = \ldots . = 0$; and from

$$S(x\iota_1 + y\iota_2)|\chi(x\iota_1 + y\iota_2) = 0$$

it follows that $S\iota_1|\chi\iota_2 = -S\iota_2|\chi\iota_1$; and from this it follows that $S\rho|\chi\sigma = -S\sigma|\chi\rho$ for every ρ and σ; or $\chi' = -\chi$.

The fictits $\iota_1, \iota_2 \ldots .$ of (56) may be called the undeviated fictits of the colinity ϕ. $\chi'\chi = -\chi^2$ is a positive colinity, so that χ^2 is a negative colinity. Let $\chi^2\iota_1 = -a^2\iota$, and first let a be zero. Since $\chi\rho = 0$ if, and only if, $\chi'\chi\rho = 0$, $\chi\iota_1$ is in the present case zero.

Next let a be not zero, and put $a^{-1}\chi = \chi_1$, and $\chi_1\iota_1 = \alpha$. Then

$$\chi_1{}^2\iota_1 = -\iota_1, \; \chi_1\iota_1 = \alpha, \; \chi_1\alpha = -\iota, \; \chi_1{}^2\alpha = -\alpha.$$

From $S\iota_1|\chi_1\iota_1 = 0$ and $S\iota_1|\chi_1\alpha = -S\alpha|\chi_1\iota_1$ we get

$$S\iota_1|\alpha = 0, \; S\alpha|\alpha = S\iota_1|\iota_1.$$

Hence we may take $\alpha = \iota_2$; and we see that the root $-a^2$ (when not zero) must occur at least twice in the χ^2 n-tic. If it occurs more than twice we have now only to take ι_3 in the fictorplex of undeviated fictors of χ^2 corresponding to this repeated root $-a^2$; and the same reasoning will apply to ι_3 as to ι_1.

Therefore when χ is skew, there always exist n fictits $\iota_1, \iota_2, \ldots . \iota_n$ such that

$$\chi\iota_1 = a\iota_2, \; \chi\iota_2 = -a\iota_1, \; \chi\iota_3 = b\iota_4, \; \chi\iota_4 = -b\iota_3 \ldots . \quad . \quad . \quad . \quad (60)$$

and when n is odd, $\chi\iota_n = 0$. Thus with the notation of (55) we have

$$\chi = \begin{bmatrix} 0 & a & 0 & 0 & \ldots . \\ -a & 0 & 0 & 0 & \ldots . \\ 0 & 0 & 0 & b & \ldots . \\ 0 & 0 & -b & 0 & \ldots . \\ . & . & . & . & . \; . \; . \end{bmatrix} \quad . \quad . \quad . \quad . \quad (61)$$

In the notation of our present subject the following is the appropriate form,

$$\chi\rho = S_1\omega\rho \quad \text{where} \quad \omega = a\iota_2\iota_1{}^{-1} + b\iota_4\iota_3{}^{-1} + \ldots . \quad . \quad . \quad . \quad (62)$$

According as n is even or odd the n-tic is

$$(\chi^2 + a^2)(\chi^2 + b^2) \ldots . = 0, \quad \text{or} \quad \chi(\chi^2 + a^2)(\chi^2 + b^2) \ldots . = 0 . \quad . \quad (63)$$

This proves that a skew determinant of even order is a perfect square, $a^2b^2 \ldots$, and one of odd order is $0a^2b^2=0$. (The determinant in question is $[\chi]$.)

The vertical line $| = K$ is not wanted with fictorlinities, though it is with multilinities. We will cease to use it.

Rotational linities.—If ϕ is a rotational fictorlinity, $\phi'\phi=1$, and ϕ is really the rigid replacement (where *fictors* replace fictors) under a new name, and we might help ourselves somewhat by previous results, but refrain.

Since $\phi'\phi=1$, $\phi\rho$ is never zero. Operating then by $\phi(\ )\phi^{-1}$ we get also $\phi\phi'=1$. Put ψ for the colinity part $\frac{1}{2}(\phi+\phi')$ of ϕ and χ for the skew part $\frac{1}{2}(\phi-\phi')$. Since $\phi'\phi=1=\phi\phi'$,

$$(\phi'-\phi)(\phi'+\phi)=\phi'^2-\phi^2=(\phi'+\phi)(\phi'-\phi)$$

and $$(\phi'+\phi)^2-2=\phi'^2+\phi^2=(\phi'-\phi)^2+2,$$

or $$\psi\chi=\chi\psi,\ \psi^2-\chi^2=1 \quad . \quad . \quad . \quad . \quad . \quad (64)$$

Let χ be given by (60). If $a=0$, $\chi\iota_1=\chi\iota_2=0$. Hence from (64)

$$\psi^2\iota_1=\iota_1,\ \psi^2\iota_2=\iota_2.$$

Hence $$\phi\iota_1=\psi\iota_1=\pm\iota_1,\ \phi\iota_2=\psi\iota_2=\pm\iota_2.$$

[There is actually an ambiguity of sign, but it need only affect a single ι, as I leave the reader to verify at the end of the treatment of the rotational linity. Strictly, we ought to consider the necessity in certain cases of perversion with respect to a single ι, but for simplicity I will ignore this.]

When a is not zero, it is easy to verify from (60) and (64) that without exception

$$\psi\iota_1=\sqrt{(1-a^2)}.\iota_1,\ \psi\iota_2=\sqrt{(1-a^2)}.\iota_2,$$

or putting $a=\sin\theta$, $b=\sin\theta'$, ...,

$$\left.\begin{array}{ll}\phi\iota_1=\iota_1\cos\theta+\iota_2\sin\theta & \phi'\iota_1=\iota_1\cos\theta-\iota_2\sin\theta\\ \phi\iota_2=-\iota_1\sin\theta+\iota_2\cos\theta & \phi'\iota_2=\iota_1\sin\theta+\iota_2\cos\theta\end{array}\right\} \quad . \quad . \quad (65)$$

(opposite circular variations). Hence we have

$$\phi\rho=p\rho p^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

where p is the even fictor product

$$p=(\cos\tfrac{1}{2}\theta+\iota_2\iota_1^{-1}\sin\tfrac{1}{2}\theta)(\cos\tfrac{1}{2}\theta'+\iota_4\iota_3^{-1}\sin\tfrac{1}{2}\theta') \ldots . \quad . \quad . \quad (67)$$

Since $(\iota_2\iota_1^{-1})^2=-1$, we have, putting $\frac{1}{2}\theta\iota_2\iota_1^{-1}=\omega_1$, $\frac{1}{2}\theta'\iota_4\iota_3^{-1}=\omega_2$, ... that

$$p=e^{\omega_1}e^{\omega_2}\ldots .$$

Also, though in general $e^{\omega_1}e^{\omega_2}$ would not be equal to $e^{\omega_1+\omega_2}$, yet in the present case the second order multenions $\omega_1, \omega_2, \ldots$ are all commutative with each other and with scalars; and since, therefore, among themselves and scalars they obey *all* the laws of ordinary algebra, we have here $p = e^{\omega}$ or

$$\left.\begin{array}{l} \phi\rho = e^{\omega}\rho e^{-\omega} \\ \text{where} \quad \omega = \frac{1}{2}(\theta\iota_2\iota_1^{-1} + \theta'\iota_4\iota_3^{-1} + \ldots) \end{array}\right\} \quad . \quad . \quad . \quad . \quad (68)$$

This, without doubt, is to be regarded as the standard form of the fictor-into-fictor rigid replacement. It seems to me a rather remarkable generalisation of a well-known quaternion result.

The following summarise the connections between rotational linities (ϕ) and skewlinities (χ).

$$\left.\begin{array}{l} \phi = \chi + \sqrt{(1+\chi^2)} = e^{\omega}(\quad)e^{-\omega},\ \omega = \frac{1}{2}(\theta\iota_2\iota_1^{-1} + \theta'\iota_4\iota_3^{-1} + \ldots) \\ \chi = \frac{1}{2}(\phi - \phi^{-1}) = S_1\omega'(\quad),\ \omega' = (\sin\theta.\iota_2\iota_1^{-1} + \sin\theta'.\iota_4\iota_3^{-1} + \ldots) \end{array}\right\} \quad . \quad (69)$$

$$(\phi^2 - 2\phi\cos\theta + 1)(\phi^2 - 2\phi\cos\theta' + 1)\ldots = 0 \quad . \quad . \quad . \quad (70)$$

with an additional factor $\phi \pm 1$ when n is odd. [Proved by finding $\phi(\iota_1 + \iota_2\sqrt{[-1]})$ from (65).]

$$(\chi^2 + \sin^2\theta)(\chi^2 + \sin^2\theta')\ldots = 0 \quad . \quad . \quad . \quad . \quad (71)$$

with an additional factor χ when n is odd.

When we are given ϕ alone, we may permit ourselves to interchange ι_1 and ι_2. Thus the range of θ need only be from 0 to π; that is, the range of each angle $\frac{1}{2}\theta, \frac{1}{2}\theta', \ldots$ of (68) need only be from 0 to $\frac{1}{2}\pi$. It is best to suppose this permissible convention in connection with (68), for thereby ω and ϕ are each uniquely given by the other. [It is best to ignore the limiting cases, corresponding to the semi-revolution of a rigid body, which may be supposed effected either in the one direction or the opposite.]

It may be noticed that since for any second order multenion $K\omega = -\omega$, $e^{-\omega} = Ke^{\omega}$. More generally, when q is such that $Kq = -q$ and $p = e^q$, then

$$p^{-1} = Kp,$$

though in general pKp is not a scalar.

$e^{\omega}ve^{-\omega}$ can be shown, directly, to be of the same order as v by putting $e^{\omega} = \underset{n=\infty}{\text{Lt}}\,(1 + n^{-1}\omega)^n$; and thus $e^{\omega}(\quad)e^{-\omega}$ may be shown to be commutative with S_a.

(68) suggests that any second order multenion may be transformed by $p(\quad)p^{-1}$ to the form there given, though, of course, θ, θ' must now be unrestricted. This is the case. [To convert $a\iota_2\iota_3 + b\iota_3\iota_1 + c\iota_1\iota_2 = \iota_1^{-1}\iota_2^{-1}\iota_3^{-1}(a\iota_1 + b\iota_2 + c\iota_3)$ to $\iota_1\iota_2\sqrt{(a^2+b^2+c^2)}$, remember that $\iota_1^{-1}\iota_2^{-1}\iota_3^{-1}$ behaves like a scalar with reference to $\iota_1, \iota_2, \iota_3$, and therefore put

$$p^{-1} = (a\iota_1 + b\iota_2 + c\iota_3)\iota_3^{-1} + \sqrt{(a^2 + b^2 + c^2)}.$$

Thus $\iota_1\iota_3$, $\iota_2\iota_3$, and thereafter $\iota_1\iota_4$, $\iota_2\iota_4$, etc., may be got rid of from $\Sigma x\iota'$, etc.]

We have seen that, as is well known, every fictorlinity can be expressed uniquely as the *sum* of a colinity and a skewlinity. Similarly, every fictorlinity ϕ can be expressed as the *product* $\psi\chi$ and also as the product $\chi\psi$ where ψ is a colinity and χ is a rotational linity. This is not unique, as ψ is any value of $\surd.\phi\phi'$ in the first case, and is any value of $\surd.\phi'\phi$ in the second (except possibly as to the sign of *one* root). It is unique if we impose that $\surd.\phi'\phi$ or $\surd.\phi\phi'$ shall be the positive value when possible, and shall be the negative value when, by reason of perversion, this is necessary. This can be proved from the facts (1), (2), (3) above stated for $\phi'\phi$ and $\phi\phi'$.

A similar analysis of a real ϕ into a real product $\psi\chi$ or $\chi\psi$ where ψ is still a colinity but now χ is a *skew* linity cannot be made in general; that is, ϕ must for this satisfy special conditions. For if ρ is any one of the undeviated fictors of ψ (and there are always n and sometimes an infinite number), $S\rho\phi\rho$ would have to be zero. Hence for some complete set of fictits we should require $S\iota\phi\iota$ to be zero. Thus if $\phi = 1$, although there are imaginary fictors for which $\rho^2 = 0$, there are no real non-evanescent ones.

8. **Replacements.—The fictorlinity replacement.**—Replacements are subdivided into proplacements and retroplacements, that is, *every* replacement is either a proplacement ($Rqr = RqRr$) or a retroplacement ($Rqr = RrRq$).

A very important class of replacements is that of unireplacements ($R^2q = q$). Some of these are proplacements and others retroplacements.

We may use the following notation. Any replacement may be denoted by R; proplacement by R_p; retroplacement by R_r; uni-replacement by R_u; uni-proplacement by R_{up}; uni-retroplacement by R_{ur}. These symbols will be understood and not explained on each occasion of using them. Other replacement symbols will be explained as required. K, P, Q will be exclusively used as explained in § 3.

Before proceeding to details, I will make an important but decidedly fine distinction, partly to render my meanings clearer, partly to warn the reader that it is necessary to walk warily.

Let R stand for the rigid replacement. That is, among other things, if $\lambda_1 = q\iota_1q^{-1}$, $\lambda_2 = q\iota_2q^{-1}$ we shall say that ι_1 is replaced by λ_1 and ι_2 by λ_2, and we shall denote this by

$$\lambda_1 = R\iota_1,\ \lambda_2 = R\iota_2.$$

It appears as if no misleading would occur by saying that $R=q(\ )q^{-1}$. But consider the question, are the replacements R and K commutative?

$$KR(\iota_1\iota_2)=K(\lambda_1\lambda_2)=(\lambda_1\lambda_2)^{-1},$$
$$RK(\iota_1\iota_2)=R.(\iota_1\iota_2)^{-1}=(\lambda_1\lambda_2)^{-1},$$

so they apparently are. But putting $R=q(\ )q^{-1}$,

$$K.q(\iota_1\iota_2)q^{-1}=Kq^{-1}.(\iota_1\iota_2)^{-1}.Kq,$$
$$q(K.\iota_1\iota_2)q^{-1}=q.(\iota_1\iota_2)^{-1}.q^{-1}.$$

[In Quaternions these two right-hand members would be equal.] Now these two are not equal in general, as can be shown by particular cases. The apparent inconsistency arises from K being able to affect q. We lay down the following, then, as a rule governing our meanings.

A replacement symbol R must be supposed not explicitly to involve multenions of any kind without justification by examination. We must suppose R to denote an actual replacing of certain symbols by others, the new ones taking the place of the old for all purposes whatsoever. We must examine the new meaning of every single symbol, such as S_c. At the same time the new meanings of formulæ are capable of representation in the old dress; and indeed this is the main object of the process of replacement. Thus in the fictor-to-fictor rigid replacement, the mainly useful feature is that certain fictors not originally called fictits were found to obey among themselves and scalars all the laws that the original fictits obeyed among themselves and scalars.

With this warning I shall continue to call the rigid replacement $q(\ )q^{-1}$, but shall refrain (when it is to be regarded *as* a replacement, and not as a multilinity) from using $q(\ )q^{-1}$ in the equations.

A replacement is a process of (1) replacing n specified original fictits by other multenion symbols, and (2) in the case of retroplacements writing every product of the new fictits in the reversed sequence of the original fictits. Laws (1) to (4), § 2 (in so far as they apply to the n fictits), are *invariably* to hold with the new meanings; and law A, § 3, will only occasionally be violated. We may further impose that

$$R(x_1q_1+x_2q_2+\ldots)=x_1Rq_1+x_2Rq_2+\ldots \qquad (1)$$

I consider the following [(2) (3)] to be really involved in the above prescriptions, but place them as definitions in order to avoid a doubtful argument, and also to render matters clear.

$$R_pqr=R_pq.R_pr,\ R_rqr=R_rr.R_rq \qquad (2)$$
$$R(S_cq)=(RS_c)(Rq) \qquad (3)$$

With regard to unireplacements a preliminary restriction is, for simplicity, required; that is, to ensure (5) below universally. When a unireplacement does not treat all the fictits similarly; when, for instance, it changes the sign of some and not that of others; then no unireplacement is to be made except on that one system (and its replacements). The unireplacements are Q or K; certain others to be defined; and the last in combination with Q or K. Any such other is defined as a replacement in which any assigned fictits of a given set are negatived, that is, have their signs changed. Thus P is such a unireplacement, namely, the one in which all the fictits are negatived. QK is another, since acc. as $\iota^2 = \pm 1$, QK = 1, or QK = P.

For some purposes it is desirable to extend the meaning of unireplacement to allowing all or some of the fictits to be multiplied by $\sqrt{(-1)}$. Thus for some purposes it would be convenient to consider the replacement in which every fictit is multiplied by the scalar $\sqrt{(\iota^2)}(-)^{\frac{n(n-1)}{4}}$; since, if we were dealing with a multiplex of order n, this would make ϖ precisely homologous with a scalar or a fictit according as n is odd or even. But I have thought it best to eschew imaginaries, especially since, for practical purposes, we can choose ι^2 to be either $+1$ or -1.

With this meaning accepted above,

$$R_u^2 q = q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$RR_u q = R_u R q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$S.pR_u q = S.qR_u p \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(6) is proved by putting $p = x_1\upsilon_1 + x_2\upsilon_2 + \ldots, q = y_1\upsilon_1 + \ldots$, $S.pR_u q = \Sigma x_1 y_1 S.\upsilon_1 R_u \upsilon_1$.

A restriction that is *not* imposed refers to the independence of the 2^n multits (other than the fictits which are covered by the fundamental laws). When n is odd the replaced meanings of S_c and S_{n-c} are sometimes identical, namely, when the replacement is a complementary replacement.

The special simplicities of unireplacements are then (1) $R_u^2 = 1$; (2) $RR_u = R_u R$; (3) $R_u S_c = S_c R_u$; (4) a product of any number of unireplacements is a uniproplacement or a uniretroplacement according as the number of constituent retroplacements is even or odd; (5) $SpR_u q = SqR_u p$.

Any replacement R is converted from retro- to pro- or else from pro- to retro- by a uniretroplacement such as K; and its species is left unchanged by a uniproplacement such as P. Or generally

$$\left.\begin{array}{l} R_p R_{up} = R'_p, \; R_p R_{ur} = R'_r \\ R_r R_{up} = R'_r, \; R_r R_{ur} = R'_p \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In a certain obvious sense indeed uniretroplacements behave like *minus* signs and uniproplacements like *plus* signs.

As we may remember by aid of the rigid replacement we have for replacements in general:—(1) $RS_c(\ )$ is not equal to $S_cR(\ )$; (2) R^2 is not equal to unity; (3) RR' is not equal to $R'R$.

When it is proposed to find whether some operator X is a replacement the principal questions to answer are:—(1) Is $(X\iota_1)^2$ a scalar differing from zero? (2) Is $X\iota_1 X\iota_2 = -X\iota_2 X\iota_1$? (3) Is Xqr equal to $XqXr$ or else to $XrXq$?

When X has been shown to be a replacement R the more important questions are: (1) What is the new meaning Rq in the old dress, q being any multenion? (2) Are the new multits equal in number to the old, or are they only half as numerous? (3) What is the interpretation of RS_c?

The unireplacements which merely negative fictits (that is, the uniproplacements) depend on n independent such replacements, viz. on $I_1, I_2, \ldots$, which negative ι_1 only, ι_2 only, ... For to negative $\iota_1, \iota_2, \iota_3$ (say) we have to make the replacement $I_1 I_2 I_3$. Thus corresponding to each multit there is one definite such replacement, namely, the one that negatives the fictit constituents of the multit, and no others. Thus P corresponds to the multit ϖ, and this is the main reason for calling it P. Also there are $2^n - 1$ such replacements. These replacements I made some use of in the early stages of the work, and their properties are very simple, so I have described them. I think they would be occasionally useful, for instance, in a calculus of motors, or if attention were directed to a set of commutative second order multenions $x\iota_2\iota_1^{-1} + y\iota_4\iota_3^{-1} + \ldots$ (and their products) where $x, y \ldots$ are arbitrary scalars. Nevertheless, as they do not (except P) treat different fictit sets impartially, they have no very extended applications. Moreover, in the same sense that the rigid replacement may be said to be $q(\ )q^{-1}$, the replacement PI_1 may be said to be $\iota_1(\ )\iota_1^{-1}$. I_1 is what we have called a perversion with respect to ι_1. By three-dimensional geometry (plane mirror reflections) we see that two such perversions with reference to fictits not belonging to one set (successive reflections in mirrors not perpendicular to one another) produce a rotation. Thus we see the necessity of restricting R_{up} to one set of fictits if (5) is to hold universally.

We are now about to consider the most general replacement which has the property that fictits are replaced by fictors, and we shall call it the fictorlinity replacement. The fictors which thus replace the fictits do not necessarily belong to the fictorplex $\iota_1\iota_2 \ldots \iota_n$ which is replaced. They may form any fictorplex of order n whatever. Clearly all the replacements hitherto considered except rigid replacements which are incapable of being expressed as $e^w(\ )e^{-w}$ (that is, except such rigid replacements as are not commutative with S_c) must be included in the fictorlinity replacement.

Law A is not assumed to hold with the general fictorlinity replacement. *If Law A is imposed, then the fictorlinity replacement* (*when real*) *reduces to a combination of a unireplacement as defined above and the rigid replacement* $e^{\omega}(\ )e^{-\omega}$. I shall not formally prove this below, but it will be quite obvious from the general argument. The fact shows that the above definition of unireplacement covers the most general case for which $R_u^2=1$; so that this last may be taken as the definition of a unireplacement, though I have thought it clearer to use the full description of properties of R_u as its definition.

The fictorlinity replacement may be regarded as the generalisation of homogeneous strain to Euclidean space of n dimensions. In three dimensions if a body is homogeneously strained:—

(1) Any vector *line* λ of it becomes (is replaced by) $\phi\lambda$ where ϕ is a general vectorlinity.

(2) Any vector *area* $\sigma=V\lambda\mu$ of it becomes (is replaced by) $[\phi]\phi'^{-1}\sigma$, though this has to be modified or further interpreted when $[\phi]=0$.

(3) Any *volume*, v, of it becomes (is replaced by) $[\phi]v$.

The fictorlinity replacement similarly shows for n dimensions how every region of c dimensions, included, is strained, and furnishes simple expressions for the strained region. ["Region" is the generalisation of the three terms, (1) vector line, (2) vector area, (3) volume.]

Let Φ, Ψ be (what may be called *ordinal*) multilinities having the property that, acting on a multenion of order c, they convert it into another multenion of order c; c being any positive integer. Thus

$$\left.\begin{aligned}\Phi S_c q = S_c\Phi q = \Phi_c q \quad (\text{say})\\ \Psi S_c q = S_c\Psi q = \Psi_c q \quad (\text{say})\end{aligned}\right\} \qquad (8)$$

Thus an ordinal multinity is commutative with S_c. We clearly have

$$\Phi=\Phi_0+\Phi_1+\Phi_2+\ldots,\ \Psi=\Psi_0+\Psi_1+\Psi_2+\ldots \qquad (9)$$

$$\Phi\Psi=\Sigma\Phi_c\Psi_c \qquad (10)$$

Let ϕ, ψ be any two given fictorlinities and let $\{\phi\}$, $\{\psi\}$ be multilinities defined by

$$\{\phi\}q=\Sigma(\phi\eta)_c^{(c)}Sq|\eta_c^{(c)}=\Sigma\eta_c^{(c)}Sq|(\phi'\eta)_c^{(c)} \qquad (11)$$

the second form being given by (13) § 7. We shall eventually show that $\{\phi\}$ may always be regarded as a replacement (with qualifications as to meaning similar to those explained for the rigid replacement), and that this replacement is the most general form of fictorlinity proplacement as already described. Meanwhile, we regard it merely as a mutilinity whose properties are to be investigated.

$\{\phi\}$ is obviously an ordinal multilinity for which therefore denote $\{\phi\}_c$, $\{\phi\}'_c$ by ϕ_c, ϕ'_c respectively, t ourselves to return to the original notation if we ple

We clearly have

$$\{\phi\} = \phi_0 + \phi_1 + \phi_2 + \ldots + \phi_n$$

where $$\phi_0 q = Sq,\ \phi_1 q = \phi\eta Sq|\eta, \ldots, \phi_c q = (\phi$$

$$\phi_1 \rho = \phi\rho \quad . \quad . \quad .$$

$$\{\phi\}' = \Sigma\phi'_c = \{\phi'\} \quad . \quad .$$

By § 6 and (17) § 7 transform (11) by substit Thus

$$\left.\begin{array}{l}\{\phi\}q = \Sigma\phi_c q \\ \phi_c q = \Sigma(\phi a)_c^{(c)} Sq|\bar{a}_c^{(c)}\end{array}\right\}$$

In (16) put $q = a_c^{(c)}$, and to fix the ideas, consi $c = 4$. Thus

$$\left.\begin{array}{l}\phi_c a_c^{(c)} = (\phi a)_c^{(c)} \\ \phi_4 S_4 a_1 a_2 a_3 a_4 = S_4 \phi a_1 \phi a_2 \phi a_3 \phi a_4\end{array}\right\}$$

by (8) and (4) § 6. This equation, which is obvi etc., are not independent, justifies the statement t operator for space of n dimensions.

It follows from (17) that

$$\{\psi\phi\} = \{\psi\}\{\phi\} \quad . \quad .$$

for $$\psi_4\phi_4 S_4 a_1 a_2 a_3 a_4 = \psi_4 S_4 \phi a_1 \phi a_2 \phi a_3 \phi a_4 = S_4 \psi\phi a_1 \psi$$

Summing for all such sets of four fictors out of n gi

$$\psi_4\phi_4 = (\psi\phi)_4,$$

and (18) now follows from (10).

Putting $\psi = \phi^{-1}$ in (18) we have

$$\{\phi\}\{\phi^{-1}\} = \{1\} = 1$$

or $$\{\phi\}^{-1} = \{\phi^{-1}\} \quad . \quad .$$

From these we clearly have

$$\{\phi^a\} = \{\phi\}^a \quad . \quad .$$

for all integral values of a, positive or negative; an

$$\{f(\phi)\} = f\{\phi\} \quad . \quad .$$

when $f(x)$ is a given rational integral function of x.

An interesting generalisation of $Sa\phi\beta = S\beta\phi' a$ we might have c in place of 4.

$$\begin{aligned} &S.S_4 a_1 a_2 a_3 a_4 S_4 \phi a_1 \phi a_2 \phi a_3 \phi a_4 \\ &= S.S_4 a_1 a_2 a_3 a_4 \phi_4 S_4 a_1 a_2 a_3 a_4 \\ &= S.(\phi_4{}' S_4 a_1 a_2 a_3 a_4 . S_4 a_1 a_2 a_3 a_4 \\ &= S.S_4 \phi' a_1 \phi' a_2 \phi' a_3 \phi' a_4 S_4 a_1 a_2 a_3 a_4, \end{aligned}$$

The middle of these three transformations is merely a particular case of the general statement in multilinities

$$Sq|\chi r = Sr|\chi' q.$$

$\{\phi\}$ furnishes us with yet another standard form of $h^{(c)}$, the general coefficient in the ϕ n-tic, namely

$$h^{(c)} = S\eta_c^{(c)}|\{\phi\}\eta_c^{(c)} = S\eta_c^{(c)}|\phi_c\eta_c^{(c)} \quad . \quad . \quad . \quad . \quad (23)$$

for by (16)

$$S\eta_c^{(c)}|\phi_c\eta_c^{(c)} = \Sigma S\eta_c^{(c)}|(\phi a)_c^{(c)}.S\eta_c^{(c)}|\bar{a}_c^{(c)}$$
$$= \Sigma S(\phi a)_c^{(c)}\bar{a}_c^{(c)} \quad [(26)\ \S\ 6]$$

and this last is the form of $h^{(c)}$ given two lines below (23) § 7.

(17) shows at once that $\{\phi\}$ may always be regarded as a replacement, namely one which replaces the definite set of fictits which are not killed by $\phi'\phi$ (or ϕ [§ 7]) and are undeviated by $\phi'\phi$. For let these fictits be $\iota_1, \iota_2 \ldots . \iota_n$ and let $\phi = e^{\omega}\sqrt{(\phi'\phi)}(\ ).e^{-\omega}$. Then since $\{\phi\}\iota_1 = \phi\iota_1$, etc., the conditions are satisfied: (1) that $(\{\phi\}\iota_1)^2$, etc. are all scalars differing from zero; (2) that $(\{\phi\}\iota_1.\{\phi\}\iota_2 = -\{\phi\}\iota_2.\{\phi\}\iota_1$, etc.; $[\{\phi\}q.\{\phi\}r$ is not in general equal to $\{\phi\}qr$, as we see by putting $\phi = x$. In this respect, then, the multilinity $\{\phi\}$ is not strictly a replacement.

Moreover, from (1) we see that the most general replacement which replaces fictits by fictors is that which replaces any fictor by the most general linear fictor function of itself. Hence $\{\phi\}$ is the most general proplacement of the kind; and $\{\phi\}$ and $Q\{\phi\}$ together form the most general replacement of the kind.

The replacement here considered consists of first replacing the fictits of a given set by (non-zero) scalar multiples thereof; and then superposing an arbitrary fictor-to-fictor rigid replacement $e^{\omega}(\ )e^{-\omega}$. The second operation in general changes the multiplex operated on to a second multiplex of the same order; but this is not contrary to the conditions we have laid down for the constitution of a replacement.

Law A will only be retained if the multiplication of each fictit is by ± 1; hence the most general type of unireplacement that converts fictors into fictors is that defined above.

It may be shown that the replacement may always be effected by first making an arbitrary rigid replacement. This is done by operating on the undeviated fictits of $\phi\phi'$.

I may put here what perhaps should have been placed in the previous section. $\phi'\phi$ and $\phi\phi'$ have the same n-tic. Putting $\phi = e^{\omega}\psi(\ ).e^{-\omega} = \chi\psi$ where ψ is a colinity, $\chi' = \chi^{-1}$ and

$$\phi'\phi = \psi^2,\ \phi\phi' = \chi.\psi^2.\chi^{-1} = \chi.\phi'\phi.\chi^{-1}.$$

Now, without exception, that is whatever be ϕ ı we can show from (22) § (7) that

$$\chi\phi\chi^{-1}.\chi\lambda_1 = a\chi\lambda_1,\ \chi\phi\chi^{-1}.\chi\lambda_2 = a\chi\lambda_2 + (\chi$$

and therefore that the n-tics of ϕ and $\chi\phi\chi^{-1}$ have

9. **Complementary Replacements, or the ı** By the principle of duality I mainly mean tl translating properties of plane areas, their iı orientations, etc., into properties of straight l magnitudes, orientations, etc. In quaternion ; striking applications of this principle, (1) tha mathematically identified with vector lines, and (2 be identified with a quadrantal versor.

In our present subject the principle, no doubt tions, and one is, so to speak, on the surface. The has a definite complementary multit associated wit of the properties of multenions are identical with] mentary multenions. We are led naturally to in technical use of "replacement," a multenion ı complement (it may *not*), or whether there is an connected with complements (there *is*).

Before entering on this matter of complementa an allied problem.

Corresponding to the equation [(1) § 6]

$$q.S_n a^{(c)} a^{(n-c)} = \Sigma^2 a_c^{(c)} S_n q a_{n-c}^{(n-c)}$$

$$\text{or} \qquad q.S_n a^{(c)} a^{(n-c)} = \Sigma^2 S_c a^{(c)}.S_n(q.S_{n-c} a^{(n}$$

there is clearly another equation in which occur β_1 are arbitrary $(n-1)^{\text{th}}$ order multenions, in place o say $a_1, a_2, \ldots.$) which are first order multenions (this other equation.

To avoid undue complexity I find it necessa ${}_l S_k$ *means* S_k *when* l *is even, and means* S_{n-k} *whe* of an equation this is

$${}_l S_k = \tfrac{1}{2}[1 + (-)^l]S_k + \tfrac{1}{2}[1 - (-)^l]S$$

Change the a of (1) into $\varpi\beta$ so that, as is re $(n-1)^{\text{th}}$ order multenion. Here ϖ as usual stan sequence, of $\iota_1, \iota_2, \ldots. \iota_n$, ($n$ fictits constituting a $a_2, \ldots. a_n$ belong, whether or not $a_1, a_2, \ldots. a_n$ are

The following are easy to prove,

$$\varpi a = (-)^{n-1} a\varpi,\ \varpi\beta = (-)^{n-1}\beta\varpi,\ \varpi\cdot{}_t S_n r = {}_t S_n r.\varpi \quad . \quad . \quad . \quad (3)$$

Pass all the ϖ's of (1) to the end and then cancel ϖ^n. In this process the left S_n becomes ${}_n S_n$, the right S_n becomes ${}_{n-c}S_n$, S_c and S_{n-c} become ${}_c S_c$ and ${}_{n-c}S_{n-c}$ respectively. Also the left is multiplied $\frac{1}{2}n(n+1)$ times and the right $\frac{1}{2}c(c+1)+\frac{1}{2}(n-c)(n-c+1)$ times by $(-)^{n-1}$. Hence

$$q\cdot{}_n S_n \beta^{(c)}\beta^{(n-c)} = \Sigma^2(-)^{c(n-1)}.{}_c S_c \beta^{(c)}.{}_{n-c}S_n(q\cdot{}_{n-c}S_{n-c}\beta^{(n-c)}) \quad . \quad . \quad (4)$$

This may be modified in several simple ways. The following statements may be proved by the reader:

$$P^{n-1}q = \varpi q \varpi^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$(-)^{c(n-1)}.{}_c S_c \beta^{(c)} = P^{n-1}{}_c S_c \beta^{(c)} = \varpi_c S_c \beta^{(c)}.\varpi^{-1} \quad . \quad . \quad . \quad (6)$$

The following may be taken as the standard form of (4),

$$q\cdot{}_n S_n \beta^{(c)}\beta^{(n-c)} = \Sigma^2{}_{n-c}S_{n-c}\beta^{(n-c)}.{}_c S_n(q\cdot{}_c S_c \beta^{(c)}) \quad . \quad . \quad . \quad . \quad (7)$$

It may be obtained thus: Multiply each side of (4) into ${}_n S_n^{-1}\beta^{(c)}\beta^{(n-c)}$ so as to be free as to the sequence of factors in $\beta^{(c)}$ and $\beta^{(n-c)}$; transpose $\beta^{(c)}$ and $\beta^{(n-c)}$; change c to $n-c$.

We now turn to complementary proplacements. Let every fictit ι be replaced by $\varpi\iota$ where ϖ is a product of all the fictits in any sequence. On account of (5) it will be seen that with ϖ thus to a certain extent arbitrary $\varpi\iota$ can be made any one of the complements of ι and we will at any stage suppose ϖ to be replaced by some other such product, for instance by ϖ^{-1} or by $\iota^2\varpi$. This substitution of $\varpi\iota$ for ι obviously constitutes a proplacement for which Law A as well as Laws (1) to (4) are retained, for

$$\varpi\iota_1.\varpi\iota_2 = -\varpi\iota_2.\varpi\iota_1,\ (\varpi\iota_1)^2 = (-)^{n-1}\varpi^2\iota^2 = (\varpi\iota_2)^2 = \ldots . \quad . \quad . \quad (8)$$

Thus we may put

$$R\iota_1 = \varpi\iota_1,\ R\iota_2 = \varpi\iota_2, \ldots . \quad . \quad . \quad . \quad . \quad (9)$$

where R is a proplacement in our technical sense.

If $\upsilon_c = \iota'\iota'' \ldots \iota^{(c)}$ it is easy to find an expression for $R\upsilon_c = \varpi\iota'\varpi\iota'' \ldots \varpi\iota^{(c)}$, for every transposition of ϖ with an ι requires us to multiply by $(-)^{n-1}$. Hence

$$R\upsilon_c = (-)^{\frac{1}{2}(n-1)c(c-1)}\varpi^c\upsilon_c,$$

or if q_c is $S_c q$

$$Rq_c = (-)^{\frac{1}{2}(n-1)c(c-1)}\varpi^c q_c \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Remember that for different forms of ϖ, ϖ^2 has but one definite meaning, namely [(3) § 3] $(-)^{\frac{1}{2}n(n-1)}(\iota^2)^n$. From (10) we clearly have the following fully extended forms:

$$\begin{pmatrix} n \text{ odd} \\ \varpi^2 = \pm 1 \end{pmatrix} \qquad \left.\begin{aligned} \mathrm{R}q &= (q_0 \pm q_2 + q_4 \pm q_6 + \dots) \\ &\quad + x\varpi(q_1 \pm q_3 + q_5 \pm q_7 + \dots) \end{aligned}\right\} \qquad (11)$$

$$\begin{pmatrix} n \text{ even} \\ \varpi^2 = \pm 1 \end{pmatrix} \qquad \left.\begin{aligned} \mathrm{R}q &= (q_0 \mp q_2 + q_4 \mp q_6 + \dots) \\ &\quad + x\varpi(q_1 \mp q_3 + q_5 \mp q_7 + \dots) \end{aligned}\right\} \qquad (12)$$

Here x is understood to be $+1$ or -1. $x\varpi$ is put instead of ϖ to enable us to pass from one ϖ to another.

(11) and (12) may be supposed to give the general form of any complementary proplacement and RK any complementary retroplacement. There is a very important difference of property between (11) and (12). Even order multenions alone occur on the right of (11), whereas Rq in (12) is a perfectly general multenion.

When n is odd a complementary replacement reduces the multiplex of order n to one with the properties of the continent multiplex.

Let now [(17) § 3] $\mathrm{P}_0 = \frac{1}{2}(1+\mathrm{P})$, $\mathrm{P}_1 = \frac{1}{2}(1-\mathrm{P})$ and define Bq by

$$\mathrm{B}q = (\mathrm{P}_0 + \varpi\mathrm{P}_1)\mathrm{K}^{n-\frac{1}{2}(1+\varpi^2)}q \qquad (13)$$

It will be found that B is the value of R when we give x a special value, thus

$$\mathrm{R} = \mathrm{B} \text{ when } x = (\iota^2)^{\frac{1}{2}(n+1)(n+2)} \qquad (13a)$$

This statement may be verified by finding from (13) the values of $\mathrm{B}q_0$, $\mathrm{B}q_1$, $\mathrm{B}q_2$, $\mathrm{B}q_3$. At first B will be looked upon as a multilinity whose properties are to be investigated.

By examination of the four cases $n = 0, 1, 2, 3$ it is easy to show that invariably

$$\mathrm{K}^{n-\frac{1}{2}(1+\varpi^2)} = \mathrm{K}^{\frac{1}{2}(n-\iota^2)(n-2} \qquad (14)$$

It is convenient to put

$$n - \tfrac{1}{2}(1+\varpi^2) = m \qquad (15)$$

By means of [(5)] $\varpi(\ )\varpi^{-1} = \mathrm{P}^{n-1}$ and $\mathrm{PP}_1 = -\mathrm{P}_1$ it is easy to see the effect of transposing K with $\varpi\mathrm{P}_1$ thus

$$\begin{aligned} \mathrm{K}(\varpi\mathrm{P}_1 q) &= \mathrm{KP}_1 q.\mathrm{K}\varpi = \mathrm{P}_1\mathrm{K}q.\varpi^{-1} \\ &= \varpi^{-1}\mathrm{P}^{n-1}\mathrm{P}_1\mathrm{K}q \end{aligned}$$

$$\text{or} \qquad \mathrm{K}(\varpi\mathrm{P}_1 q) = (-)^{n-1}\varpi^2.\varpi\mathrm{P}_1(\mathrm{K}q) \qquad (16)$$

To transpose K^m with $\varpi\mathrm{P}_1$ then we have to multiply by $[(-)^{n-1}\varpi^2]^m$. The value of this is easily found by use of the relations established by putting $\varpi^2 = \pm 1$

$$(\varpi^2)^{\frac{1}{2}(1-\varpi^2)} = \varpi^2 = (-1)^{\frac{1}{2}(1-\varpi^2)},\ (\varpi^2)^{\frac{1}{2}(1+\varpi^2)} = 1,\ (-1)^{\frac{1}{2}(1+\varpi^2)} = -\varpi^2 \qquad (17)$$

We could clearly here write any symbol X for ϖ^2, provided $X^2 = 1$. Thus we might write any one of ι^2, P, Q, K. We thus find that

$$[(-)^{n-1}\varpi^2]^m = (-)^{n-1}\varpi^2 \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Thus K^{m-1} is commutative with ϖP_1, and if we pass either K or K^m across ϖP_1 we must multiply by $(-)^{n-1}\varpi^2$. Hence

$$Bq = (P_0 + \varpi P_1)K^m q = K^{m-1}(P_0 + \varpi P_1)Kq = K^m[P_0 + (-)^{n-1}\varpi^{-1}P_1]q \quad . \quad (19)$$

By aid of the definition of B′ the conjugate of B, namely

$$SqKB'r = SrKBq,$$

we now have

$$B'q = P_0 K^m q + P_1(\varpi P^{n-1} K^m q) \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Hence

$$[n \text{ even}] \quad B'q = (P_0 - \varpi P_1)K^m q = PBq \quad . \quad . \quad . \quad . \quad . \quad (21)$$
$$[n \text{ odd}] \quad B'q = (1 + \varpi)P_0 K^m q = K^m(1 + \varpi^{-1})P_0 q \quad . \quad . \quad . \quad . \quad (22)$$
$$[n \text{ even}] \quad B'Bq = (P_0 - \varpi P_1)(P_0 - \varpi^{-1}P_1)q = q \quad . \quad . \quad . \quad . \quad (23)$$
$$[n \text{ odd}] \quad B'Bq = (1 + \varpi)(P_0 + \varpi^{-1}P_1)q = (1 + \varpi P^{\frac{1}{2}(1-\varpi^2)})q \quad . \quad . \quad (24)$$
$$[n \text{ even}] \quad B^2 q = Pq \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$
$$[n \text{ odd}] \left\{ \begin{array}{ll} B^2 q = (P_0 + \varpi^{-1}P_1)q = K^{\frac{1}{2}(1-\varpi^2)}Bq & . \quad . \quad . \quad . \quad (26) \\ BB' = 2P_0,\ BB'B = 2B,\ B'^2 = (1 + \varpi)P_0,\ B'BB' = 2B' & . \quad (27) \\ B^3 = B & . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28) \end{array} \right.$$

Of these the most important are the last three for n odd and the, here re-collected, still simpler corresponding ones for n even;

$$[n \text{ even}] \quad B'B = 1 = BB',\ B' = PB,\ B^2 = P,\ B^4 = 1 \quad . \quad . \quad . \quad . \quad (29)$$

Thus for n even B is a rotational multilinity, and may (a rather unexpected result) be said to be a square root of P.

In the equation

$$B\iota_1 = (\iota^2)^{\frac{1}{2}(n+1)(n+2)}\varpi\iota_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

we may suppose

$$\varpi = \iota_1\iota_2\iota_3 \ldots . \iota_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Supposing now an exactly similar replacement is made again, what is the new value of ι_1? that is, what is the value of

$$[(B\iota_1)^2]^{\frac{1}{2}(n+1)(n+2)}B\varpi . B\iota_1 \,?$$

It will be found that it is $-\iota_1$ when n is even, and it is $B\iota_1$ when n is odd. Thus when the replacement is made twice or several times exactly according to (30), (31) we get successively for q

[n even] q, Bq, Pq, PBq, q, Bq, ...
[n odd] q, Bq, Bq, Bq, ...

Let us call the case when n is odd and $\varpi^2 = -1$ "anomalous"; when

n is odd the multiplex may when $\varpi^2 = +1$ be ident multiplex, but not when $\varpi^2 = -1$.

We now have what seems to me a very remark *the anomalous case,* the effect of making the rep change q to $B^k q$. In the anomalous case the result ment once, twice, etc. is to change q to

$$Bq, Bq, Bq, \ldots .$$

whereas the effect of operating by B, B^2, B^3 .... is t

$$Bq, KBq, Bq, KBq, \ldots .$$

Addition to § 9, *April* 1908.—Since in the last c order terms, we have

$$KBq = QBq = PQBq.$$

The anomalous case can be avoided by use of t inclined to regard the following as the standard ment. Let

$$\left.\begin{array}{ll} [n \text{ even}] & C\iota_1 = (\iota^2)^{\frac{1}{2}(n+2)}\varpi\iota_1 \\ [n \text{ odd}] & C\iota_1 = \sqrt{(\varpi^2)}.\varpi\iota_1 \\ \text{where} & \varpi = \iota_1\iota_2 \ldots\ldots \iota_n \end{array}\right\}.$$

These give $Cq = Bq$, except in the anomalous case pure imaginary. The effect of making the replac times is the same as what is denoted by C^k wl multilinity. Also without exception

$$\left.\begin{array}{ll} [n \text{ even}] & C^2 = P,\ C^4 = 1 \\ [n \text{ odd}] & C^2 = C \end{array}\right\}$$

The *minimum* degree identity satisfied by C except for $n = 0$ and $n = 2$, when we have

$$C - 1 = 0,\ (C - 1)(C^2 + 1) = 0$$

respectively for this identity.

(11) becomes

$$(n \text{ odd}) \qquad Cq = (q_0 + q_2 + q_4 + \ldots\ldots) + \sqrt{(\varpi^2)}.\varpi(q_1 + q_3 + q_5 +$$

and (12) remains unaltered, with C in place of R an

At the end of the supplement it is recommen multiplex be ignored, and that instead it be repl even order multenions of an odd order multiplex.

(7) above furnishes us with the fundamental treatment, and shows us how to treat such a mult the n independent $(n-1)^{\text{th}}$ order units $\varpi\iota_1$, $\varpi\iota_2$, ...

In (7) let n be odd; divide the summation into two parts for which $c=2a$ and $c=n-2b$ where a and b are positive integers; then without restricting q to being of even order we have

$$\left.\begin{aligned} q = &\Sigma^2[\mathrm{S}_{2a}\beta^{(n-2a)}.\mathrm{S}_n(q.\mathrm{S}_{2a}\beta^{(2a)}).\mathrm{S}^{-1}\beta^{(2a)}\beta^{(n-2a)}] \\ &+ \Sigma^2[\mathrm{S}_{2b}\beta^{(2b)}.\mathrm{S}(q.\mathrm{S}_{2b}\beta^{(n-2b)}).\mathrm{S}^{-1}\beta^{(n-2b)}\beta^{(2b)}] \end{aligned}\right\} \quad . \quad . \quad . \quad (35)$$

Here no convention as to the sequence of the β factors is required.

When q is of even order $\mathrm{S}_n(q.\mathrm{S}_{2a}\beta^{(2a)})$ is zero, and we get

$$q\mathrm{S}\beta^{(n-2b)}\beta^{(2b)} = \Sigma^2\mathrm{S}_{2b}\beta^{(2b)}.\mathrm{S}(q.\mathrm{S}_{2b}\beta^{(n-2b)}) \quad . \quad . \quad . \quad (36)$$

Here, of course, the convention is required that the sequence is such that $\mathrm{S}\beta^{(n-2b)}\beta^{(2b)}$ has the same value for all values of b.

(36), which is true whether β_1, β_2, ... are or are not independent, develops q in terms of the products $\beta^{(c)}$ of an even number of factors. It may be regarded as the natural simplest generalisation of

$$\rho\mathrm{S}\alpha\beta\gamma = \mathrm{V}\beta\gamma\mathrm{S}\alpha\rho + \mathrm{V}\gamma\alpha\mathrm{S}\beta\rho + \mathrm{V}\alpha\beta\mathrm{S}\gamma\rho.$$

The corresponding generalisation of

$$\rho\mathrm{S}\alpha\beta\gamma = \alpha\mathrm{S}\beta\gamma\rho + \beta\mathrm{S}\gamma\alpha\rho + \gamma\mathrm{S}\alpha\beta\rho$$

may be obtained by changing q of (35) to $q\varpi$, and supposing the new q to be of even order. Thus by (7) § 5

$$q\mathrm{S}\beta^{(2a)}\beta^{(n-2a)} = \Sigma^2\mathrm{S}_{2a}\beta^{(n-2a)}.\mathrm{S}(q.\mathrm{S}_{2a}\beta^{(2a)}) \quad . \quad . \quad . \quad (37)$$

It seems scarcely necessary to remark that ${}_c\mathrm{S}_c\beta^{(c)}$ is a combinatorial part of $\beta^{(c)}$ with reference to the factors β, because $\mathrm{S}_c\alpha^{(c)}$ is one with reference to the factors α.

All that it is necessary to *remember* about (36) and (37) is that q *can* be expanded in the form $\Sigma x\mathrm{S}_{2b}\beta^{(2b)}$ and also in the form $\Sigma y\mathrm{S}_{2a}\beta^{(n-2a)}$. The coefficients are easy enough to obtain even mentally.

If we put

$$\varpi_0 = \varpi\sqrt{(\varpi^2)}, \quad \varpi_1 = \varpi\sqrt{(-\varpi^2)} \quad . \quad . \quad . \quad . \quad (38)$$

we obtain several convenient ways of treating p the general multenion of an odd order multiplex in terms of two even order multenions, q, r of the multiplex. Thus we may put

$$p = q + \varpi r, \quad \text{or instead} \quad q + \varpi_0 r, \quad \text{or} \quad q + \varpi_1 r, \quad \text{or} \quad \xi q + \eta r \quad . \quad (39)$$

where

$$\left.\begin{aligned} &\xi = \tfrac{1}{2}(1+\varpi_0),\ \eta = \tfrac{1}{2}(1-\varpi_0) \\ \text{and therefore} \quad &\xi^2 = \xi,\ \eta^2 = \eta,\ \xi\eta = 0 = \eta\xi,\ \xi + \eta = 1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (40)$$

Here ϖ, and therefore ϖ_0, ϖ_1, ξ, η, are all commutative with p. Thus $q + \varpi_1 r$ is the analogue of Hamilton's bi-quaternion.

The form $\xi q + \eta r$ is especially simple for some purposes. Thus

$$p^{-1} = \xi q^{-1} + \eta r^{-1},\ p_1 p_2 = \xi q_1 q_2 + \eta r_1 r_2,$$

or more generally if $f(p_1, p_2 \ldots)$ is a given function of $p_1, p_2, \ldots$

$$f(p_1 \ldots) = \xi f(q_1 \ldots) + \eta f(r_1 \ldots) \qquad (41)$$

The above will, I think, bear out the contention that the concept of the continent multiplex is a mischievous excrescence* on the present method; the reason why I was initially misled into contemplating it will be obvious enough to quaternionists.

From (34) we have, when q and r are of even order,

$$C(q + \varpi_0 r) = q + r$$

and therefore $$(1 - C)(q + \varpi_0 r) = (\varpi_0 - 1)r = -2\eta r.$$

Now $Cp = p_0$ and $(1 - C)p = p_1$ are two parts of p, such that

$$p = p_0 + p_1,\ Cp_0 = p_0,\ Cp_1 = 0 \qquad (42)$$

Hence another convenient bi-form of p is

$$p = q + \eta r \qquad (43)$$

The complementary replacement (C) of q is q.

10. **Differentiation, Integration, Jacobians.**—If the reader thinks that

$$\frac{q}{r} = q/r$$

ought to mean qr^{-1}, even when q and r are differentials, and therefore demurs to the meaning about to be given of

$$\frac{d\sigma}{d\rho} = d\sigma/d\rho,\ \frac{\partial\epsilon}{\partial\rho} = \partial\epsilon/\partial\rho,$$

he may bring into use the old significance of the colon, and may mentally read $d\sigma : d\rho$ or

$$\frac{d\sigma}{\overline{\overline{d\rho}}}$$

wherever $d\sigma/d\rho$ occurs below.

If $y_1, y_2, \ldots, y_n$ are n scalar functions of n scalar variables $x_1, x_2, \ldots x_n$, in our present subject we should take account of this by saying that the fictor σ is a function of the fictor ρ, where

$$\left.\begin{aligned}\rho &= x_1\iota_1 + \ldots + x_n\iota_n\\ \sigma &= y_1\iota_1 + \ldots + y_n\iota_n\end{aligned}\right\} \qquad (1)$$

* Mischievous because it hides some of the inherent simplicities of the method.

Differentiations are effected by aid of

$$\left.\begin{array}{l} {}_\rho\nabla = \nabla = \Sigma\iota D_x \\ {}_\sigma\nabla = \nabla' = \Sigma\iota D_y \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When it is desirable to emphasise the distinction between partial and total differentiation in some defined sense, we may in manuscript use $\bar{\nabla}$ for partial differentiation, and in print we may use Θ if there is need of *frequent* indications. Below I have decided to use $\bar{\nabla}$.

By (18) § 6

$$d\sigma = Sd\rho|\nabla.\sigma, \; d\rho = Sd\sigma|\nabla'.\rho \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus the fictorlinity S( ) | $\nabla.\sigma$ converts $d\rho$ into $d\sigma$. We therefore define $d\sigma/d\rho$ *to mean a fictorlinity,** namely S( )|$\nabla.\sigma$. Or

$$d\sigma/d\rho = S(\ )|\nabla.\sigma, \; d\rho/d\sigma = S(\ )|\nabla'.\rho = (d\sigma/d\rho)^{-1} \quad . \quad . \quad . \quad (4)$$

Since [(26) § 7] we always denote the discriminant of ϕ by $[\phi]$, $[d\sigma/d\rho]$ means the discriminant of $d\sigma/d\rho$. Or from $[\phi] = \eta_n^{(n)}|(\phi\eta)_n^{(n)}$ and by (12), § 7

$$\left.\begin{array}{rl} [d\sigma/d\rho] & = (n\,!)^{-1}S_n\nabla_1\nabla_2 \ldots \nabla_n|S_n\sigma_1\sigma_2 \ldots \sigma_n \\ & = (n\,!)^{-1}\nabla_n^{(n)}|\sigma_n^{(n)} \\ & = \Lambda_n^{(n)}|\sigma_n^{(n)} \end{array}\right\} \quad . \quad . \quad . \quad (5)$$

(5) sufficiently defines $\nabla_n^{(n)}$, $\Lambda_n^{(n)}$. More generally we may say that

$$\left.\begin{array}{l} S_c\psi\nabla_1\psi\nabla_2 \ldots \psi\nabla_c = (\psi\nabla)_c^{(c)} \\ (\Lambda\psi)_c^{(c)} = (c\,!)^{-1}(\psi\nabla)_c^{(c)} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(5) at once shows that $[d\sigma/d\rho]$ is a Jacobian, namely

$$[d\sigma/d\rho] = \partial(y_1, y_2, \ldots, y_n)/\partial(x_1, x_2, \ldots, x_n) \quad . \quad . \quad . \quad (7)$$

From the definition that $d\sigma/d\rho$ is the fictorlinity that converts $d\rho$ into $d\sigma$ we at once have

$$\frac{d\sigma}{d\rho} \cdot \frac{d\rho}{d\tau} = \frac{d\sigma}{d\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

but we have not in general $d\rho/d\tau . d\sigma/d\rho = d\sigma/d\tau$; though the corresponding statement in discriminants, or Jacobians, is true because $[\phi\psi] = [\phi][\psi] = [\psi\phi]$. Taking the discriminant of (8) we at once have the well-known theorem

$$\frac{\partial(y_1, y_2 \ldots y_n)}{\partial(x_1, x_2 \ldots x_n)} \cdot \frac{\partial(x_1, x_2 \ldots x_n)}{\partial(z_1, z_2 \ldots z_n)} = \frac{\partial(y_1, y_2 \ldots y_n)}{\partial(z_1, z_2 \ldots z_n)},$$

but this is clearly only a very particular property of the much more general meaning of (8).

* I find (April 1908) on p. 116 of *Sci. Papers* that Professor Gibbs recommended emphatically this notation in 1886.

The conjugate $(d\sigma/d\rho)'$ of $d\sigma/d\rho$ may conveniently be written $d'\sigma/d'\rho$ or

$$\left(\frac{d\sigma}{d\rho}\right)' = \frac{d'\sigma}{d'\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Change of independent variable in differentiations is straightforward in the present notation. Thus

$$S d\rho \nabla = S d\sigma \nabla' = S \cdot \frac{d\sigma}{d\rho} d\rho . \nabla'$$

$$= S d\rho \frac{d'\sigma}{d'\rho} \nabla'$$

$$\text{or} \qquad \nabla = \frac{d'\sigma}{d'\rho} \nabla' = \nabla_1 S \nabla' | \sigma_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This is analogous to

$$D_x = D_x y . D_y,$$

and of course is all that is needed in some cases of change of variable. But the more complicated explicit analogue of

$$D_y = (D_x y)^{-1} D_x$$

is more frequently wanted.

By (29) § 7 this analogue is

$$\left.\begin{aligned} \nabla' &= [d\sigma/d\rho]^{-1} K \sigma_{n-1}^{(n-1)} . S_n \Lambda_{n-1}^{(n-1)} \nabla \\ &= K \Lambda_{n-1}^{(n-1)} . S_n \Lambda_{n-1}^{(n-1)} \nabla \div \sigma_n^{(n)} | \sigma_n^{(n)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In accord with our definition of $d\rho/d\sigma$ and with (10) we may appropriately assert that

$$\frac{d'\sigma}{d'\rho} = \frac{{}_\rho\nabla}{{}_\sigma\nabla} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The necessary and sufficient condition that the n fictors $\phi\iota_1, \phi\iota_2, \ldots . \phi\iota_n$ are not independent, is that $[\phi]=0$. Thus the necessary and sufficient condition that n scalars $dx_1, dx_2 \ldots . dx_n$ can be found such that

$$\frac{d\sigma}{d\rho}(\iota_1 dx_1 + \iota_2 dx_2 + \ldots . + \iota_n dx_n) = 0,$$

that is, such that

$$\iota_1 dy_1 + \ldots . + \iota_n dy_n = 0,$$

is that the Jacobian $[d\sigma/d\rho]$ vanishes. This condition then (that $[d\sigma/d\rho]=0$) is the necessary and sufficient condition that the x's may be varied while the y's are not. If an identity holds between the y's it is obvious that such variation of the x's is possible; and if no such identity holds, the x's can be expressed in terms of the y's, so that when $dy_1 = \ldots . = dy_n = 0$, so also are $dx_1 = \ldots . = dx_n = 0$, and such variation of the x's is impossible. This establishes the well-known theorem that if $[d\sigma/d\rho]=0$ then $f(y_1, y_2, \ldots .) = 0$, and conversely.

If $y_1 \ldots . y_n$ are given implicitly as functions of $x_1 \ldots . x_n$ by the n equations

$$f_1(x_1 \ldots . x_n y_1 \ldots . y_n) = 0, \quad f_2 = 0 \ldots . f_n = 0,$$

we put

$$\iota_1 f_1 + \ldots . + \iota_n . f_n = \epsilon$$

and say that $\epsilon(\rho, \sigma)$ is a given fictor function of ρ, σ and the implicit relation is $\epsilon = 0$. Thus

$$0 = d\epsilon = (\mathrm{S}d\rho|\nabla . + \mathrm{S}d\sigma|\nabla' .)\epsilon$$

$$= \frac{\partial\epsilon}{\partial\rho}d\rho + \frac{\partial\epsilon}{\partial\sigma}d\sigma$$

$$\text{or} \quad d\sigma = -\left(\frac{\partial\epsilon}{\partial\sigma}\right)^{-1}\left(\frac{\partial\epsilon}{\partial\rho}\right)d\rho$$

$$\text{or} \quad \frac{d\sigma}{d\rho} = -\left(\frac{\partial\epsilon}{\partial\sigma}\right)^{-1}\left(\frac{\partial\epsilon}{\partial\rho}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Taking discriminants and remembering that $[x\phi] = x^n[\phi]$, and therefore that $[-\phi] = (-)^n[\phi]$, we have the well-known Jacobian theorem

$$\left[\frac{d\sigma}{d\rho}\right] = (-)^n\left[\frac{\partial\epsilon}{\partial\sigma}\right]^{-1}\left[\frac{\partial\epsilon}{\partial\rho}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This seems to me a great improvement on the ordinary algebraic proof, and it must be remembered that (13) is a much more general statement than (14).

I think the above fully justifies the importance of the notation that $d\sigma/d\rho$, or something very analogous, such as $d\sigma : d\rho$, should mean a fictor-linity. It strikes me with much surprise that quaternionists have not adopted this notation long ago. It seems to me immensely to decrease the complexities of change of variables in general.

We now pass to integrations. The well-known theorem of change of variables from $x_1, x_2, \ldots . x_n$ to $y_1 \ldots . y_n$ in

$$\int \ldots . \int \mathrm{V}(x_1, x_2 \ldots .)dx_1 dx_2 \ldots . dx_n$$

at once suggests itself, but the integration implied here is of a very specialised type, and it is best to regard that theorem as a special case. Integration "along a path" in Thermodynamics, along curves and over surfaces in Physics, integrations connected with "actual" paths of dynamical systems and along "varied" paths thereof, suggest what we ought to regard as the general type of integration in our present system. Let ρ (and therefore σ) be a function of (say four) scalar parameters u, v, w, z. We may think if we like of $n-4$ other parameters $e_1 \ldots . e_{n-4}$, of which ρ and σ are in general functions, but for the present $e_1 \ldots . e_{n-4}$ are to have given constant values. The general typ considered is sufficiently illustrated by supposing obtained by varying u, v, w, z independently betwee

Any such integral may in the first instance be su

$$\int du \int dv \int dw \int dz \mathrm{V}(u, v, w, z)$$

(where V is not necessarily a scalar) and thereafter increments $d\rho_1, d\rho_2, \ldots$ of ρ. Let $\partial\rho/\partial u = \rho_u$. Thu

$$d\rho = \rho_u du + \rho_v dv + \rho_w dw + \rho_z dz$$
$$= d_u\rho + d_v\rho + d_w\rho + d_z\rho$$

which is meant to define the independent i $[d_u\rho = \rho_u du]$. Here

$$\mathrm{S}_4\rho_u\rho_v\rho_w\rho_z . dudvdwdz = \mathrm{S}_4(d_u\rho . d_v\rho . d_w\rho .$$

or $$dudvdwdz = \mathrm{S}_4^{-1}\rho_u\rho_v\rho_w\rho_z . \mathrm{S}_4(d_u\rho . d_v\rho . d_w\rho$$

This shows that in such integrations as I have the "element" of integration may always be taken as more generally as

$$\phi \mathrm{S}_c d\rho_1 d\rho_2 \ldots d\rho_c$$

where ϕ is any (scalar, fictor, or the like) linear multenion and $d\rho_1, d\rho_2, \ldots d\rho_c$ are c independent i

The integral (of c integrations) itself may be den change of variable from ρ to σ is at once given by (

$$\int^{(c)}\int \phi d\rho_c^{(c)} = \int^{(c)}\int \phi\left(\frac{d\rho}{d\sigma}d\sigma\right)_c^{(c)} = \int^{(c)}\int \phi . \left\{ \frac{d}{d} \right.$$

or [(19) § 8]

$$\int^{(c)}\int \phi d\rho_c^{(c)} = \int^{(c)}\int \phi . \left\{ \frac{d\sigma}{d\rho} \right\}^{-1} . d\sigma$$

Put $c = n$ in (15). Thus

$$\phi . \left\{ \frac{d\rho}{d\sigma} \right\} . d\sigma_n^{(n)} = \phi\left(\frac{d\rho}{d\sigma}\eta\right)_n^{(n)} \mathrm{S} d\sigma_n^{(n)} | \eta_n^{(n)}$$
$$= \left[\frac{d\rho}{d\sigma}\right] \phi \eta_n^{(n)} \mathrm{S} d\sigma_n^{(n)} | \eta_n^{(n)}$$
$$= \left[\frac{d\rho}{d\sigma}\right] \phi d\sigma_n^{(n)}$$

Finally putting $\phi = \mathrm{XS}\varpi|(\ \ )$ where X is a scal $\varpi = \iota_1\iota_2 \ldots \iota_n$ we get the usual theorem in the form

$$\int^{(n)}\int \mathrm{X}\varpi | d\rho_n^{(n)} = \int^{(n)}\int \mathrm{X}\left[\frac{d\rho}{d\sigma}\right]\varpi | d\sigma_n^{(n)}$$

where we are at liberty to suppose

$$d\rho_1 = \iota_1 dx_1, \ldots, d\rho_n = \iota_n dx_n, \; d\sigma_1 = \iota_1 dy_1, \ldots, d\sigma_n = \iota_n dy_n.$$

Our typical quadruple integral has a boundary which in a definite sense may be said to be of three dimensions or trebly infinite in third order infinitesimals. In our mode of picturing the integration this three-dimensional boundary consists of eight parts corresponding to the upper and lower limits of each of the four parameters u, v, w, z. Now a triple integral over this complete boundary can always be expressed as a quadruple integral extending between the same limits as our original quadruple integral. For if we sum the triple integral over the boundary of each of the quadruply infinite four-dimensional elements we shall get by cancellings the integral for the original boundary only.

If the element of the triple integral is $\phi S_3 d\rho_1 d\rho_2 d\rho_3$ the element $(d\rho_1, d\rho_2, d\rho_3, d\rho_4)$ contributes four pairs of elements whose sum can be shown to be

$$\begin{aligned}\phi_b(S_3 d\rho_2 d\rho_3 d\rho_4 S d\rho_1|\nabla_b - S_3 d\rho_1 d\rho_3 d\rho_4 S d\rho_2|\nabla_b \\ + S_3 d\rho_1 d\rho_2 d\rho_4 S d\rho_3|\nabla_b - S_3 d\rho_1 d\rho_2 d\rho_3 S d\rho_4|\nabla_b) \\ = \phi_b S_3(|\nabla_b S_4 d\rho_1 d\rho_2 d\rho_3 d\rho_4)\end{aligned}$$

by (14) § 5. Thus

$$\int^{(c-1)}\!\!\int \phi d\rho_{c-1}^{(c-1)} = \int^{(c)}\!\!\int \phi_1 S_{c-1}(|\nabla_1 . d\rho_c^{(c)}) \quad . \quad . \quad . \quad . \quad (17)$$

the $(c-1)$-ple integral on the left extending over the complete boundary of the c-ple integral on the right. This is, of course, the generalisation of the quaternion line-surface and surface-volume integrals, and includes both.

Putting $\phi = 1$ in (17) we get $\int^{(c-1)}\!\!\int d\rho_{c-1}^{(c-1)} = 0$ for such a complete boundary. This is a fact, but cannot be regarded as a proof, as the fact is required for the quadruple element in the steps indicated by the words "can be shown to be."

Since $\{d\sigma/d\rho\}_c$ converts $d\rho_c^{(c)}$ to $d\sigma_c^{(c)}$ we may extend the $d\sigma/d\rho$ notation according to the following formulæ,

$$\frac{d\sigma_c^{(c)}}{d\rho_c^{(c)}} = \left\{\frac{d\sigma}{d\rho}\right\}_c = S(\;)|\Lambda_c^{(c)}.\sigma_c^{(c)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\frac{d'\sigma_c^{(c)}}{d'\rho_c^{(c)}} = \left\{\frac{d'\sigma}{d'\rho}\right\}_c = S(\;)|\sigma_c^{(c)}.\Lambda_c^{(c)} = \frac{\rho\Lambda_c^{(c)}}{\sigma\Lambda_c^{(c)}} = \frac{\rho\nabla_c^{(c)}}{\sigma\nabla_c^{(c)}} \quad . \quad . \quad . \quad . \quad (19)$$

$$\left\{\frac{d\sigma}{d\rho}\right\} = S(\;) + \frac{d\sigma}{d\rho} + \frac{d\sigma''_2}{d\rho''_2} + \ldots + \frac{d\sigma_n^{(n)}}{d\rho_n^{(n)}} \quad . \quad . \quad . \quad . \quad (20)$$

These serve to imply how intimately the fictorlinity replacement is connected with general strain (σ an arbitrary function of ρ) in Euclidean space of n dimensions.

The generalisation of the pure mathematical properties of $\mathfrak{q}$, "fluxes" and "intensities" as presented in "The Mathematical Theory of Electromagnetism" [*Phil. Trans.* A, 1893, pp. 685 *et seq.*] to Multenions is now so much a mere matter of easy detail that to save space I will leave it alone. It will be time to record the formulæ and ideas when useful applications happen to present themselves. [At this date I would re-name the "fluxes" and "intensities" *perductors*, and *tractors*. I would call Maxwell's **B**, **D**, **E**, **H**, the magnetoductor, electroductor (or electrostatic perductor), electrotractor, magnetotractor; the corresponding integrals would be magnetoduction, electroduction, electrotraction, magnetotraction; the path of a line integral might be called its track, the surface (or sometimes the ring boundary thereof) of a surface integral might be called its perduit (*cf.* circuit, conduit); I would banish for ever the word force or the indefinite "intensity" from such connections as electromotive force, magnetic force; I would distinguish always in language between a vector and its line or surface integral, reserving the termination -tor for the former, and the termination -tion for the latter.]

If we put σ above $=\nabla v$ where v is a given scalar function of ρ, $d\sigma/d\rho$ becomes what we may call the Hessian colinity, namely

$$\frac{d\sigma}{d\rho}=\frac{d\nabla v}{d\rho}=\mathrm{S}(\ )|\nabla.\nabla v \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and taking discriminants we have the Hessian determinant

$$\left[\frac{d\nabla v}{d\rho}\right]=(n\,!)^{-1}\nabla_n^{(n)}|\nabla_n^{(n)}v \quad . \quad . \quad . \quad . \quad . \quad (22)$$

11. **Miscellaneous concluding remarks.**—The ambiguity of sign of ι^2 is an inconvenience, though not so serious as might have been anticipated. I have experimented with various possible systems of fictits, such as

$$1=\iota_1^2=-\iota_2^2=\iota_3^2=-\iota_4^2=\ldots\ldots=\iota_1\iota_2\ldots\ldots\iota_N=(\iota_1\iota_2\ldots\ldots\iota_n)^2$$
$$\iota_1^2=\iota_2^2=\ldots\ldots=1$$
$$\iota_1^2=\iota_2^2=\ldots\ldots=-1.$$

In correcting proofs (April 1908) I have left the next four paragraphs as they were despatched in 1906, because some important questions are here raised, but they do not correctly express my present views. The latter are given at the end of the Supplement below.

Quaternions is undoubtedly the simplest three-dimensional geometrical method akin to our present algebra. Equally certain is it that for general algebraic purposes it is more convenient to suppose that $\iota^2=1$ than that either $\iota^2 = -1$ or that the sign of ι^2 is ambiguous. Must we or must we not assert that economy of thought demands that $\iota^2 = 1$? Must we or must we not assert that the same economy demands the simplest possible three-dimensional geometrical method? I am really quite doubtful, but rather incline to choose $\iota^2 = 1$. This makes $(\iota_1\iota_2\iota_3)^2 = -1$ and so prevents us from regarding $\iota_1\iota_2\iota_3$ as a real scalar. Nevertheless, even with $\iota^2 = 1$ and $n = 3$, Quaternions is still a real particular symmetrical case of Multenions; quaternions is the calculus of multenions $q, r, \ldots$ such that $S_1q = S_3q = 0$, $S_1r = S_3r = 0, \ldots$ And further, we may still, if we please, assert that $\iota_1\iota_2\iota_3$ is a scalar, namely the scalar $\surd(-1)$. Thus with $n = 3$, $\iota^2 = 1$, $\iota_1\iota_2\iota_3 =$ scalar, a multenion becomes identical in properties with Hamilton's bi-quaternion $q + r\surd(-1)$ where q and r are real quaternions.

At the same time I do not recommend this course when $\iota^2 = 1, n = 3$. Instead, I recommend that $\iota_1\iota_2\iota_3$ be not regarded as a scalar but as an independent real multit. Further, I recommend the following as the *standard* geometrical interpretation. Identify fictors with vector lines, and therefore second order multenions with vector areas, and third order multenions $(x\iota_1\iota_2\iota_3)$ with volumes. [At the same time remember that fictors *may* be identified with vector areas, and second order multenions with vector lines.]

I think there can be no question that the geometrical method we thus get is distinctly inferior to the quaternion method; but for all that, if we *must* choose between $\iota^2 = +1$ and $\iota^2 = -1$, I am inclined to choose $\iota^2 = +1$ in spite of its inferior geometry.

Perhaps the best plan is that adopted in this paper of leaving the sign of ι^2 doubtful. If we make our primary $\iota^2 = 1$, the complementary replacement produces cases where $\iota^2 = -1$.

The treatment in § 4 above of q^{-1} is not complete. I have failed to make it reasonably simple. The reader will probably find little difficulty in supplying proofs of the assertions now to be made.

What is to be desired is some simple function Xq of q (simple in the sense that it can be easily written down when q is given) such that qXq is a scalar which is only zero when q^{-1} is infinite.

If q is a given multenion, r an arbitrary one, and x *any* scalar, the multilinity ϕ given by

$$\phi r = \tfrac{1}{2}(qr + rq) + \tfrac{1}{2}x(qr - rq) \quad . \quad . \quad . \quad . \quad (1)$$

has the property that for all positive integral values of a

$$\phi^a 1 = q^a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence the 2^n-tic satisfied by ϕ is also satisfied by q. Also in general this 2^n-tic in q involves even positive powers of the arbitrary scalar x. The coefficient (a rational integral function of $q-Sq$) of every such power is separately zero. From any such equation satisfied by q, q^{-1} may be derived, and this is a solution of the problem, given q, required q^{-1}. But that it is practically a very clumsy one is obvious from taking $n=10$, in which case the 2^n-tic is of degree higher than a thousand, and it involves a determinant the number of whose constituents is over a million.

We saw in § 4 that to find q^{-1} we need only find $(Kq.q)^{-1}$. The problem of finding p^{-1} when p is self-conjugate is slightly but not much simpler than the general problem. Thus let p be a given self-conjugate multenion, r an arbitrary self-conjugate multenion, and let

$$\psi r = \tfrac{1}{2}(pr + rp). \qquad (3)$$

Then ψ is a self-conjugate multilinity and ψr a self-conjugate multenion. Such a multilinity clearly satisfies an equation of degree equal to the number of self-conjugate multits, that is

$$2^{n-1} + 2^{\frac{1}{2}(n-1)} \cos \tfrac{1}{4}(n - \iota^2)\pi.$$

p satisfies the same identity. The roots are all real. When they recur, if $a_1, a_2, \ldots . a_k$ are the different roots, then

$$(p-a_1)(p-a_2) \ldots . (p-a_k) = 0 \qquad (4)$$

with no repetitions of roots. If we put

$$p_1 = (p-a_2)(p-a_3) \ldots . (p-a_n)/(a_1-a_2)(a_1-a_3) \ldots . (a_1-a_n) \qquad (5)$$

and similarly for $p_2, p_3, \ldots .$, then the following are true,

$$pp_1 = a_1 p_1, pp_2 = a_2 p_2, \ldots . \qquad (6)$$

$$p_1^2 = p_1, p_2^2 = p_2, \ldots ., p_1 p_2 = p_2 p_1 = 0 \ldots . \qquad (7)$$

$$1 = p_1 + p_2 + \ldots . + p_k \qquad (8)$$

$$p = a_1 p_1 + a_2 p_2 + \ldots . + a_k p_k \qquad (9)$$

more generally; always for positive integral values of m; also for negative integral values of m, when no a is zero;

$$p^m = a_1^m p_1 + \ldots . + a_k^m p_k . \qquad (10)$$

If $a_1 = 0$

$$\underset{x=\infty}{\mathrm{Lt}}(1+xp)^{-1} = p_1$$

or generally

$$\underset{x=\infty}{\mathrm{Lt}}[1 + x(p-a_1)]^{-1} = p_1 \qquad (11)$$

Up to $n=4$, $(p-Sp)^2$ is a scalar, so that the inversion of p is simple. This suggests that given n, the maximum value of k is decidedly smaller generally than the number $2^{n-1} + 2^{\frac{1}{2}(n-1)} \cos \frac{1}{4}(n-\iota^2)\pi$ mentioned above. Thus with $n=4$ this last is 6, whereas k is 2. What the maximum value of k is I do not know. [This value ought apparently to be deducible from the great peculiarities of (7) and (8).]

Up to $n=4$ we have for any multenion q,

$$(n<5) \qquad q^{-1} = \frac{\mathrm{K}q(q\mathrm{K}q - 2\mathrm{S}q\mathrm{K}q)}{q\mathrm{K}q(q\mathrm{K}q - 2\mathrm{S}q\mathrm{K}q)} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the denominator on the right being a scalar, because $(q\mathrm{K}q - \mathrm{S}q\mathrm{K}q)^2$ is a scalar. Thus in the present case $(n<5)$ the $\mathrm{X}q$ spoken of above is the numerator on the right of (12).

Even when $n=5$, q^{-1} can be explicitly exhibited although there are then 32 independent multits as opposed to the quaternion 4. Thus putting

$$q\mathrm{Q}q = p$$

since $\mathrm{Q}p = p$, p contains no S_2 or S_3 parts. Putting its S_1 and S_4 parts in the form $\rho + \varpi\sigma$ where ρ and σ are fictors, it is easy to see that the square of $\mathrm{S}_1 p + \mathrm{S}_4 p$ is a scalar $+ 2\mathrm{S}_5(\mathrm{S}_1 p \mathrm{S}_4 p)$. Hence

$$(p - \mathrm{S}p - \mathrm{S}_5 p)^2 = \text{scalar} + 2\mathrm{S}_5(\mathrm{S}_1 p \mathrm{S}_4 p)$$

$$\text{or} \qquad p^2 - 2p(\mathrm{S}p + \mathrm{S}_5 p) = x + 2\mathrm{S}_5(\mathrm{S}_1 p \mathrm{S}_4 p - \mathrm{S}p\mathrm{S}_5 p)$$

where x is a scalar. Thus

$$\begin{aligned} y = \text{scalar} &= [x + 2\mathrm{S}_5(\mathrm{S}_1 p \mathrm{S}_4 p - \mathrm{S}p\mathrm{S}_5 p)]\,[x - 2\mathrm{S}_5(\mathrm{S}_1 p \mathrm{S}_4 p - \mathrm{S}p\mathrm{S}_5 p)] \\ &= p(p - 2\mathrm{S}p - 2\mathrm{S}_5 p)[(p - 2\mathrm{S}p)(p - 2\mathrm{S}_5 p) - 4\mathrm{S}_5.\mathrm{S}_1 p \mathrm{S}_4 p], \end{aligned}$$

$$\left.\begin{aligned} \text{or} \qquad y &= q\mathrm{Q}q(q\mathrm{Q}q - 2\mathrm{S}q\mathrm{Q}q - 2\mathrm{S}_5 q\mathrm{Q}q)[(q\mathrm{Q}q - 2\mathrm{S}q\mathrm{Q}q)(q\mathrm{Q}q - 2\mathrm{S}_5 q\mathrm{Q}q) \\ &\qquad - 4\mathrm{S}_5.(\mathrm{S}_1 q\mathrm{Q}q)(\mathrm{S}_4 q\mathrm{Q}q)] \\ [n<6] \quad q^{-1} &= y^{-1}\mathrm{Q}q(\quad \text{''} \quad \text{''} \quad)[\quad \text{''} \quad \text{''} \quad] \end{aligned}\right\} \quad . \quad (13)$$

In (12) we may write PK, Q, or PQ in place of K.

Similar to (12) and (13) we have

$$[n<3] \qquad q\mathrm{P}\mathrm{Q}q = \text{scalar},\ q^{-1} = \mathrm{P}\mathrm{Q}q/q\mathrm{P}\mathrm{Q}q \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This is, of course, virtually the quaternion case modified so as to be true when $\iota^2 = +1$ as well as when $\iota^2 = -1$.

SUPPLEMENT.

The central proposition of *this supplement*, t
regarded as a linity, was sent to the Royal Society,
1906. The present form was given in the correcti

As originally despatched, the final sentence
speculations as to the bearings of equations (7),
If the paper were re-cast, the substance of the latt
ought to occupy a prominent early position, say in

I have now succeeded in simplifying the proble
The hint came from Cayley's § (*c*), p. 148 of Tai
where he shows that a quaternion is a linity of
now able to show in the same sense that a n
multiplex of order $2m-1$, or of order $2m$, is a lini
a particular consequence, it satisfies an identity of
2^{2m-1} or 2^{2m} contemplated in § 11.

I may explain that the reason for leaving unp
whose treatment I have not succeeded in reducing
I have hesitated to further amplify this already to

The linity theorems of § 7 above down to *eq.* (
are true of linities in general, whether real or imag
of § 7 from *eq.* (56) to the end depends on assum
Corresponding theorems, when the restriction is r
so simple. It is necessary, for application to a
consider now these generalisations, and some allied

We begin by developing the fundamental
equations (21), (22), (23), § 7 above, and assume the
We have

$$(\phi-a)^{A}(\phi-b)^{B}\ldots\ldots=0 \quad .$$

where a, b, ... are all different. Let the m
corresponding to this be

$$(\phi-a)^{g}(\phi-b)^{h}(\phi-c)^{i}\ldots\ldots=$$

so that g, h, etc. are positive integers, of which g t
than A, the second h equal to or less than B, and
stand for any positive integers equal to or greater t
so that in particular we may put either $g+=A$ et

x being any symbol, expand the fraction

$$1/(x-a)^{g+}(x-b)^{h+}\ldots\ldots$$

in the usual way in partial fractions, so that we get

$$\left.\begin{array}{l} 1\equiv(x-b)^{h+}(x-c)^{l+}\ldots\ldots[A_0+A_1(x-a)+\ \ldots\ldots\ +A_{g-1}(x-a)^{-1+g+}]+\ \ldots\ldots \\ \ \equiv f_a(x)+f_b(x)+\ \ldots\ldots \end{array}\right\} \quad (17)$$

where $f_a(x)$ is put for brevity instead of $(x-b)^{h+}\ \ldots\ldots\ [A_0+\ \ldots\ldots]$. Define the linities ξ, ξ_0, η, η_0, etc. by the equations

$$\left.\begin{array}{l} \xi=f_a(\phi),\ \eta=f_b(\phi),\ \ldots\ldots \\ \xi_0=f_a(\phi).(\phi-a),\ \eta_0=f_b(\phi).(\phi-b),\ \ldots\ldots \end{array}\right\} \quad . \quad . \quad . \quad (18)$$

Since $\xi, \xi_0, \ldots\ldots$ are integral functions of ϕ they are all commutative with ϕ and each other. Also from (16), (17), (18) the following are obvious,

$$\left.\begin{array}{l} \xi\eta=\eta\xi=\ \ldots\ldots\ =0,\ \xi_0\eta=\eta\xi_0=\eta_0\xi=\ \ldots\ldots\ =0 \\ \xi+\eta+\ \ldots\ldots\ =1 \\ 0=\xi_0^{\,g}=\eta_0^{\,h}=\ \ldots\ldots \end{array}\right\} \quad . \quad . \quad (19)$$

Also from the equation $1-\xi=\eta+\zeta+\ \ldots\ldots$ multiplying by ξ and by ξ_0 we get

$$\left.\begin{array}{l} \xi^2=\xi,\ \xi\xi_0=\xi_0=\xi_0\xi, \\ \eta^2=\eta,\ \eta\eta_0=\eta_0=\eta_0\eta,\ \text{etc.} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (20)$$

Also $\phi=(\xi+\eta+\ \ldots\ldots\ )\phi$, and $\xi\phi=a\xi+\xi_0$ by the definition of ξ_0. Hence

$$\left.\begin{array}{l} \phi=(a\xi+\xi_0)+(b\eta+\eta_0)+\ \ldots\ldots \\ \ \ =\xi(a+\xi_0)+\eta(b+\eta_0)+\ \ldots\ldots \end{array}\right\} \quad . \quad . \quad . \quad (21)$$

Conversely, *if the linities* $\xi, \xi_0, \eta\ \ldots\ldots$ *are related to* ϕ *by the equations* (19), (20), (21), *they must satisfy equations* (17) *and* (18); *and* (15) *and* (16) *must also be true.* From this it follows that they are *definite* linities given by ϕ, in spite of the arbitrariness of the integers $g+, h+, \ldots\ldots$ and the consequent arbitrariness of the functional forms $f_a(x), f_b(x), \ldots\ldots$ To prove the italicised statement, we first have from (19), (20), (21)

$$\phi^M=\xi(a+\xi_0)^M+\eta(b+\eta_0)^M+\ \ldots\ldots$$

where M is any positive integer; or more generally by the combination of powers such as ϕ^M into an integral function

$$f(\phi)=\xi f(a+\xi_0)+\eta f(b+\eta_0)+ \quad . \quad . \quad . \quad . \quad (22)$$

where $f(\phi)$ is any integral function of ϕ. Giving to f the particular form f_a defined by (17) we get [since $f_a(b+\eta_0)=0$, etc. on account of $\eta_0^{\,h}=0$]

$$f_a(\phi)=\xi f_a(a+\xi_0).$$

Putting $x=a+\xi_0$ in the identity $1\equiv f_a(x)+f_b(x)+\ \ldots\ldots$, we get

$$1=f_a(a+\xi_0).$$

Hence $f_a(\phi)=\xi$. Also from (20) and (21)

$$\xi(\phi-a)=\xi\xi_0=\xi_0.$$

Hence ξ and ξ_0 satisfy (18).

The following is easily proved, and is required below,

$$\left.\begin{array}{c}\xi \text{ of } f(\phi)=\xi,\ \xi_0 \text{ of } f(\phi)=f(a+\xi_0)-f(a)\\ [f(a+\xi_0)-f(a)]^p=0\end{array}\right\} \quad . \quad . \quad . \quad (23)$$

provided $f(a), f(b), \ldots.$ are all different from one another. Here "ξ of $f(\phi)$" means the particular ξ which corresponds to the root $f(a)$ of $f(\phi)$, and similarly, of course, for ξ_0.

I have thought the above the best introduction to the standard analysis of ϕ implied by (21), if only because it enables us by the simple rules implied by (17), (18) to determine ξ, ξ_0, etc., when either the minimum degree identity or the Grassmann identity is known for ϕ. But the definitions of ξ, etc. might, instead, have been made to depend on the fundamental Grassmann theorem (22) of § 7. We proceed to show with the notation of that equation: if ρ is any fictor

$$\left.\begin{array}{l}\xi\rho=\lambda_1 S\rho|\bar{\lambda}_1+\lambda_2 S\rho|\bar{\lambda}_2+\ldots.+\lambda_A S\rho|\bar{\lambda}_A\\ \xi_0\rho=\lambda_1' S\rho|\bar{\lambda}_2+\lambda_2' S\rho|\bar{\lambda}_3+\ldots.+\lambda_{A-1}' S\rho|\bar{\lambda}_A\end{array}\right\} \quad . \quad . \quad (24)$$

where the bar refers to the set of independent fictors $\lambda_1, \lambda_2, \ldots, \mu_1, \mu_2, \ldots.$ (24) is equivalent to

$$\left.\begin{array}{l}\xi\lambda_1=\lambda_1,\ \xi\lambda_2=\lambda_2, \ldots., \ \xi\mu_1=\xi\mu_2=\ldots.=0\\ \xi_0\lambda_1=0,\ \xi_0\lambda_2=\lambda_1' \ldots., \ \xi_0\mu_1=\xi_0\mu_2=\ldots.=0\end{array}\right\} \quad . \quad . \quad (25).$$

To prove these equivalent sets (24) and (25), first note that ξ kills every $\mu, \nu, \ldots, \eta$ kills every $\lambda, \nu, \ldots,$ because of the factor $(\phi-b)^{A+}$ in ξ and the like. Thus putting

$$\begin{aligned}\rho&=\xi\rho+\eta\rho+\ldots.\\ &=\lambda+\mu+\ldots.\end{aligned}$$

then $\lambda=\xi\rho$ is a fictor in the fictorplex (λ) [and may be called the component of ρ in (λ) with reference to $(\lambda), (\mu), \ldots$], and similarly for $\mu=\eta\rho$ etc. It also follows that

$$(\phi-a)\lambda=(\phi-a)\xi\lambda=\xi_0\lambda=\xi_0\rho.$$

The second of (24) now follows from the fact that $(\phi-a)\lambda_1=0$, $(\phi-a)\lambda_2=\lambda_1'$, etc.

Equation (22) has useful applications to many other cases than when f is an integral function with a finite number of terms. Thus we may clearly put for $f(x)$ any one of the following direct algebraic functions (supposed defined by convergent integral expansions),

$$e^x, \cosh x, \sinh x, \cos x, \sin x.$$

All the expressions for $f(a+\xi_0)$, such as $e^{a+\xi_0}$, form finite series because $\xi_0{}^q=0$. When we deal with many valued functional forms of $f(x)$, such as $\log x$, $x^{\frac{1}{2}}$, we have to face formidable difficulties when the general values of $f(\phi)$ are required; but the difficulties vanish in most useful cases when we restrict ourselves in some defined way to one of the many values. Thus we will here consider what may be called the *principal* logarithm $\log_0 \phi$ of ϕ, and the *principal* square root $\surd_0 \phi$ of ϕ.

Guided by (22) and by familiar theorems of ordinary algebra we put

$$\left.\begin{aligned} &\log_0 \phi = \xi \log_0 (a+\xi_0) + \eta \log_0 (b+\eta_0) + \ldots . \\ \text{where} \quad &\log_0 (a+\xi_0) = \log_0 a + [a^{-1}\xi_0 - \tfrac{1}{2}(a^{-1}\xi_0)^2 + \tfrac{1}{3}(a^{-1}\xi_0)^3 - \ldots .] \end{aligned}\right\} \quad . \quad (26).$$

[The principal logarithm $\log_0 a$ of the scalar a may be defined in various simple ways. Thus putting $a = e^{x+y\surd(-1)}$ where x and y are real and y is between $\pm\pi$, we may put $\log_0 a = x + y\surd(-1)$. Similarly, below, the principal square root $\surd_0 a$ may be understood to mean $e^{\frac{1}{2}(x+y\surd(-1))}$.]

By (26) $\log_0 \phi$ becomes unintelligible when, and only when, one of the quantities a is zero; that is, ϕ^{-1} is infinite; that is, the discriminant of ϕ vanishes. A similar remark applies to the forms below given for ϕ^{-1} and $\surd_0 \phi$.

From (26) it is a simple matter to show that

$$\left.\begin{aligned} \phi = e^{\log_0 \phi},\ \log_0 f(\phi)\mathrm{F}(\phi) = \log_0 f(\phi) + \log_0 \mathrm{F}(\phi) \\ \log_0 \phi^{-1} = -\log_0 \phi \end{aligned}\right\} \quad . \quad . \quad . \quad (27).$$

Similar to (26) we have

$$\left.\begin{aligned} &\phi^{-1} = \xi(a+\xi_0)^{-1} + \eta(b+\eta_0)^{-1} + \ldots . \\ \text{where} \quad &(a+\xi_0)^{-1} = a^{-1}[1 - (a^{-1}\xi_0) + (a^{-1}\xi_0)^2 - \ldots .] \end{aligned}\right\} \quad . \quad . \quad (28)$$

and

$$\left.\begin{aligned} &\surd_0 \phi = \xi \surd_0 (a+\xi_0) + \eta \surd_0 (b+\eta_0) + \ldots . \\ \text{where} \quad &\surd_0 (a+\xi_0) = \surd_0 a \cdot [1 + \tfrac{1}{2}a^{-1}\xi_0 - \tfrac{1}{8}(a^{-1}\xi_0)^2 + \ldots .] \end{aligned}\right\} \quad . \quad . \quad (29).$$

The last square bracket contents mean the familiar algebraic development of $(1+a^{-1}\xi_0)^{\frac{1}{2}}$.

N.B.—Although when $\phi^{-1}=\infty$ the value here given of $\surd_0 \phi$ is unintelligible, it does not follow that ϕ has no square root, as is obvious from a consideration of the square root of ϕ^2; but I have verified that cases occur where there is no square root.

Clearly (26), (28), (29) are but special examples of a whole class of cases. It is perfectly easy, for instance, to write down the form of $\phi^{\frac{x}{y}}$ where x and y are integers prime to one another, corresponding to the form of $\surd_0 \phi$ in (29).

There is never any question of convergency or divergency to raise respecting the series in ξ_0, because they all terminate.

We now introduce a generalisation of the conjugate ϕ' of a linity ϕ. [Read ϕ' as "ϕ dash" and $\phi^{\backprime}$ as "ϕ shad"]. Let ν be a given linity such that

$$\left.\begin{array}{ll} & \nu' = \nu^{-1} = \pm\nu \\ \text{and let} & \phi^{\backprime} = \nu\phi'\nu^{-1} \end{array}\right\} \qquad (30).$$

[When ϕ is identified with a multenion q, then Rq where R is any uniretroplacement becomes a case of $\phi^{\backprime}$; and R_q where R is any uniproplacement becomes a case of $\nu\phi\nu^{-1}$.]

$\phi^{\backprime}$ will be called the ν-conjugate of ϕ. ϕ is said to be ν-self-conjugate when $\phi^{\backprime}=\phi$; ν-skew when $\phi^{\backprime}=-\phi$; ν-rotational when $\phi^{\backprime}\phi=1$, and therefore $\phi\phi^{\backprime}=1$.

ν-conjugacy is almost identical in properties with conjugacy, which, of course, is a particular case given by $\nu=1$. Thus

$$(\phi^{\backprime})^{\backprime}=\phi,\ (\phi\psi)^{\backprime}=\psi^{\backprime}\phi^{\backprime},\ (\phi^{\backprime-1})^{\backprime}=\phi^{\backprime-1} \qquad (31)$$

Also $\phi^{\backprime}$ has precisely the same n-tic as ϕ; because $\nu\phi'\nu^{-1}$ has the same n-tic as ϕ', and ϕ' the same n-tic as ϕ. Also since $(\phi^{M})^{\backprime}=(\phi^{\backprime})^{M}$ when M is an integer, positive or negative; we have with all the meanings of f considered above

$$f(\phi^{\backprime})=[f(\phi)]^{\backprime} \qquad (32)$$

In particular:—*the* ξ, ξ_0 *and* $\xi(\mathrm{a}+\xi_0)$ *of* $\phi^{\backprime}$ *corresponding to the root* a [which occurs A times in the ϕ n-tic and g times in the minimum degree identity of ϕ; and therefore occurs A times in the $\phi^{\backprime}$ n-tic and g times in the minimum degree identity of $\phi^{\backprime}$] *are* $\xi^{\backprime}$, $\xi_0^{\backprime}$ *and* $\xi^{\backprime}(\mathrm{a}+\xi_0^{\backprime})$ *respectively.* The statement in square brackets is true, because whenever $f(\phi)\equiv 0$, then $f(\phi^{\backprime})\equiv 0$; and whenever $f(\phi^{\backprime})\equiv 0$, then $f(\phi)\equiv 0$.

(32) shows that when ϕ is ν-self-conjugate, then also is $f(\phi)$. When ϕ is ν-skew; $f(\phi)$ is ν-self-conjugate when f is an even function and is ν-skew when f is an odd function.

More generally, the ν-self-conjugate part of $f(\phi)$ is $\frac{1}{2}[f(\phi)+f(\phi^{\backprime})]$ and the ν-skew part is $\frac{1}{2}[f(\phi)-f(\phi^{\backprime})]$. When ϕ is ν-skew these parts become the even and odd parts of $f(\phi)$, that is $\frac{1}{2}[f(\phi)\pm f(-\phi)]$; and when ϕ is ν-self-conjugate they become the whole and zero respectively.

The general form of a ν-rotational function is e^{χ} where χ is any ν-skew linity; for, in the first place, e^{χ} is ν-rotational since

$$(e^{\chi})^{\backprime}=e^{\chi^{\backprime}}=e^{-\chi}=(e^{\chi})^{-1};$$

and in the second place, when $\phi^{\backprime}\phi=1$

$$\log_0\phi^{\backprime}+\log_0\phi=0$$

[never unintelligible when $\phi^{\backprime}\phi=1$ since this gives that the (discriminant)² of ϕ is 1], so that $\log_0 \phi$ is ν-skew, and therefore $\phi = e^{\log_0 \phi}$ is in the desired form e^{χ}.

If we try to put $\phi = \chi\psi$ where ψ is ν-self-conjugate and χ is ν-rotational we must have

$$\phi`\phi = \psi`\chi`\chi\psi = \psi^2 .$$

Hence if $\phi`\phi$ has no square root (and such cases do occur), the resolution is impossible. If, however, $\phi^{-1}$ is not infinite, we may put $\psi = \sqrt{}_0(\phi`\phi)$; and this form of ψ is ν-self-conjugate by (32); and now we have

$$\chi = \phi\psi^{-1}$$

is necessarily ν-rotational, since we have

$$\chi\chi` = \phi\psi^{-2}\phi` = \phi\phi^{-1}\phi`^{-1}\phi` = 1.$$

Thus when ϕ^{-1} is not infinite we have a solution of the problem, but there are always many solutions if there is a single one. Under the same circumstances the resolution $\phi = \psi\chi$ is possible. We get $\psi = \sqrt{}_0(\phi\phi`)$, $\chi = \psi^{-1}\phi$.

I have found, by an argument too long to reproduce here, standard forms for a ν-skew and for a ν-self-conjugate linity which reduce to those considered at the end of § 7 when $\nu = 1$ and ϕ is real.

It is not difficult to show that invariably the $\lambda_1, \lambda_2 \ldots$ of (22) § 7 may be so taken that

$$\lambda_1' = x_1\lambda_1,\ \lambda_2' = x_2\lambda_2,\ \ldots .\ \mu_1' = y_1\mu_1,\ \ldots .$$

where every $x, y \ldots .$ is either unity or zero. [Think of the fictorplex of independent fictors $\lambda_g, \lambda_g' \ldots .$ which survive the operator ξ_0^{g-1}; take

$$\lambda_1 = \xi_0^{g-1}\lambda_g,\ \lambda_2 = \xi_0^{g-2}\lambda_g,\ \ldots .$$
$$\lambda_1' = \xi_0^{g-1}\lambda_g',\ \ldots . ;$$

then think of the fictorplex not included in the above, which survives ξ_0^{g-2}, and so on.] $\xi, \xi_0, \eta, \eta_0, \ldots .$ are thus subdivided into parts which themselves satisfy all the conditions of (20), (21) above. But $\lambda_1, \lambda_2 \ldots .$ still remain to a certain extent arbitrary. This arbitrariness is further diminished in the standard forms referred to. In the following enunciation, g is the number of fictors in one of these subdivisions, and not the original value, in general, of the minimum degree. Similarly, ξ and ξ_0 refer to such a subdivision instead of to the sum of the subdivisions.

When $\phi` = \pm\phi$ we may put

$$\left.\begin{aligned} \xi\rho &= (a_1 S\rho|\nu\beta_g \pm \beta_g S\rho|\nu^{-1}a_1) + (a_2 S\rho|\nu\beta_{g-1} \pm S\rho|\nu^{-1}a_2) + \ldots . \\ \xi_0\rho &= (a_1 S\rho|\nu\beta_{g-1} \pm \beta_{g-1} S\rho|\nu^{-1}a_1) + (a_2 S\rho|\nu\beta_{g-2} \pm \beta_{g-2} S\rho|\nu^{-1}a_2) + \ldots . \\ \textit{where}\quad \bar{a}_1 &= \nu\beta_g,\ \bar{\beta}_g = \nu^{-1}a_1,\ \bar{a}_2 = \nu\beta_{g-1},\ \bar{\beta}_{g-1} = \nu^{-1}a_2,\ \ldots . \end{aligned}\right\} \quad (33)$$

where $\beta_1, \beta_2, \ldots .$ are the same as $a_1, a_2, \ldots .$ in two cases only; namely

(1) *when* $\phi' = \phi$ *and* $\nu' = \nu$; *and* (2) *when* $\phi' = -\phi$, $a = 0$, $\nu' = (-)^{p-1}\nu$; *and where in all other cases the sets* $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ *are independent.*

The most important of these cases, in multenion applications, we will partially consider without proving (33). This is when $\phi' = \phi$ and $\nu' = -\nu$; the sets mentioned are then independent; it easily follows from (33) that not only do the roots then occur in pairs of equals, but ϕ satisfies an equation of degree only half as great as usual. It will be noticed that this necessitates that n the order of the fictorplex must be even. This is explained by the fact, which can be proved easily, that it is impossible for the conditions $\nu' = \nu^{-1} = -\nu$ to be satisfied when n is odd. Indeed, it was the peculiarities of behaviour $q\mathrm{K}q$ and $q\mathrm{PQ}q$ in (12) and (14) respectively that led me to (33).

To prove what has just been mentioned independently of (33), we first show that when $\phi' = \phi$, $\nu' = -\nu$ there will in general be $\frac{1}{2}n$ different roots occurring in equal pairs. If

$$\phi\alpha = a\alpha,\ \phi\beta = b\beta,\ \text{etc.}$$

then $$\phi\rho = a\alpha\mathrm{S}\rho|\bar{\alpha} + b\beta\mathrm{S}\rho|\bar{\beta} + \ldots$$

and therefore $$\phi'\rho = a\bar{\alpha}\mathrm{S}\rho|\alpha + b\bar{\beta}\mathrm{S}\rho|\beta + \ldots$$

Since $\phi = \phi'$,

$$\phi\nu\bar{\alpha} = \nu\phi'\bar{\alpha} = \nu\phi'\bar{\alpha} = a\nu\bar{\alpha}.$$

Thus not only does $\phi - a$ kill α but it also kills $\nu\bar{\alpha}$. Now α cannot be a multiple of $\nu\bar{\alpha}$, for $\mathrm{S}\alpha|\bar{\alpha} = 1$ and $\mathrm{S}\bar{\alpha}|\nu^{-1}\alpha = 0$ because ν is skew. Hence the root a occurs twice, and $\phi - a$ kills the fictors of a *second* order fictorplex (namely $\alpha, \nu\bar{\alpha}$).

When the root a occurs more than twice, take a second ν-self-conjugate linity ψ such that $\phi + x\psi$ has $\frac{1}{2}n$ different roots. Thus $\phi + x\psi$ satisfies an identity of degree $\frac{1}{2}n$, and this holds up to the limit $x = 0$; the statement is therefore true of ϕ.

We will now make some statements about multilinities. The following is not actually required below, but very directly bears on what is to come. *The most general form of a multilinity* ϕ *is given by*

$$\left.\begin{array}{ll}(n \text{ even}), & \phi q = p_1 q p_1' + p_2 q p_2' + \ldots = \Sigma p q p' \\ (n \text{ odd}), & \phi q = \Sigma p q p' + \Sigma p_0 \mathrm{P} q p_0' \end{array}\right\} \quad . \quad . \quad (34)$$

The fact that when n is odd $\Sigma pqp'$ is not the most general form is connected with the other fact that then ϖ is commutative with q.

To prove (34), first put [(2) § 2] $q = \Sigma\upsilon\mathrm{S}\upsilon^{-1}q$ and therefore $\phi q = \Sigma\phi\upsilon\mathrm{S}\upsilon^{-1}q$. Then transform $\mathrm{S}\upsilon^{-1}q$ by aid of the following,

$$\mathrm{S}r = 2^{-n}(1 + \mathrm{I}_1)(1 + \mathrm{I}_2) \ldots r \quad . \quad . \quad . \quad . \quad (35)$$

That this is true for every multenion r appears when it is noted that $\frac{1}{2}(1+I_1)$ kills every multit containing ι_1 and leaves every other multit unchanged. Hence $2^{-n}(1+I_1)(1+I_2)\ldots\ldots$ kills every part of r except the scalar, which it leaves unaltered. In (35) put

$$I_1 = P.\iota_1(\ )\iota_1^{-1} = \iota_1 P(\ ).\iota_1^{-1}.$$

Thus Sr is expressed as $\Sigma uru' + \Sigma vPrv'$ where u, v, u', v' are multenions. Applying this to $S\upsilon^{-1}q$ we get the second of (34) as the general form of ϕq whatever be n. When n is even we may put $P = \varpi(\ )\varpi^{-1}$ and get the first of (34).

The following is required below. *The root sum of the multilinity* $\upsilon(\ )\upsilon'$ *is zero except when each of the two multits is unity, and if* n *is odd when each is* ϖ; *in these two cases the root sum is* 2^n *and* $2^n\varpi^2$ *respectively; the root sum of the multilinity* $\upsilon P(\ ).\upsilon'$ *is in every case zero.* This may be proved by putting

$$\begin{aligned}\phi v_1 &= av_1 + bv_2 + \ldots\ldots\\ \phi v_2 &= a'v_1 + b'v_2 + \ldots\ldots\\ &\ldots\ldots\ldots\ldots\end{aligned}$$

and examining the scalars $a, b' \ldots\ldots$ in the principal diagonal when $\phi = \upsilon(\ )\upsilon'$ and when $\phi = \upsilon P(\ )\,\upsilon'$; each such scalar is zero or ± 1; the root sum is $a + b' + \ldots\ldots$

Law A will be violated in certain cases below. In all cases, however, the definition of ϕ' as follows,

$$SrK\phi q = SqK\phi' r,$$

leads to the fundamental property that

$$\text{when}\qquad \phi = \begin{bmatrix} a, & b, & \ldots\ldots \\ a', & b', & \ldots\ldots \\ \cdot & \cdot & \cdot \end{bmatrix} \text{ then } \phi' = \begin{bmatrix} a, & a', & \ldots\ldots \\ b, & b', & \ldots\ldots \\ \cdot & \cdot & \cdot \end{bmatrix}$$

and also gives

$$\left.\begin{aligned}\phi' q &= \Sigma(Kp.q.Kp') + \Sigma P(Kp_0.q.Kp_0'\\ \text{when}\qquad \phi q &= \Sigma pqp' + \Sigma p_0 Pqp_0'\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (36)$$

or more simply:—the conjugate of $p(\ )p'$ is $Kp.(\ ).Kp'$ and $\phi q = Pq$ is self-conjugate. [The last is true of any unireplacement when regarded as a multilinity.]

We will now prove that a multilinity belonging to a given multiplex of order n may always be itself regarded as a multenion belonging to a multiplex of order $2n$, and we will at the same time find a set of fictits belonging to this higher order multiplex. Conversely, it will appear that a multenion may always be regarded as a linity.

Thus the linear quaternion function of a quaternion, of which Professor Joly has made such admirable use, is a multenion expressible by four fictits. These fictits may be taken as the linear quaternion functions (quaternion linities)

$$i(\ ),\ j(\ ),\ k(\ )i,\ k(\ )j,$$

or if law A is to hold, as

$$i(\ ),\ j(\ ),\ k(\ )i\surd(-1),\ k(\ )j\surd(-1).$$

These are obtained by putting below:—

$$c=1,\ n=2\ ;\ \epsilon_1=i\surd(-1),\ \epsilon_2=j\ ;\ \text{and therefore } \varpi_0=k\surd(-1),$$

and they may be taken to illustrate the meanings of the proof of the general proposition, now to be given.

Law A could be assumed and the imaginary made use of, but it is much simpler to violate law A and to connect therewith when required by aid of $\surd(-1)$. It should be remarked that taking $\iota_1\surd(-1)$ instead of ι_1 as a fictit changes the meaning of K to KI_1, but does not change the meaning of any other symbol of replacement such as the following and combinations of them,

$$P,\ Q,\ I_1,\ I_2\ \ldots\ .$$

According as n is even or odd, let its value be $2c$ or $2c+1$, and let the fictits of the given multiplex be denoted by $\epsilon_1,\ \epsilon_2,\ \ldots\ \epsilon_{2c},\ \epsilon$, the last being omitted when $n=2c$. Let

$$1=\epsilon_1^2=-\epsilon_2^2=\epsilon_3^2=\ \ldots\ =-\epsilon_{2c}^2=\epsilon^2 \qquad (37)$$

Let

$$\varpi_0=\epsilon_1\epsilon_2\ \ldots\ \epsilon_{2c}\,. \qquad (38)$$

so that when $n=2c$, ϖ_0 is the product of the fictits, and when $n=2c+1$, $\varpi_0\epsilon$ is the product.

It is convenient to note that (37) and (38) give

$$1=\epsilon_1^2=(\epsilon_1\epsilon_2)^2=(\epsilon_1\epsilon_2\epsilon_3)^2=\ \ldots\ =\varpi_0^2=(\varpi_0\epsilon)^2 \qquad (39)$$

Define the *multilinities* $\lambda_1, \lambda_2, \ldots\ \lambda_{4c+2}$ (omitting the two $\lambda_{4c+1}, \lambda_{4c+2}$ when $n=2c$) by the equations

$$\left.\begin{array}{l}\lambda_1=\epsilon_1(\ ),\ \lambda_2=\epsilon_2(\ ),\ \ldots,\ \lambda_{2c}=\epsilon_{2c}(\ )\\ \lambda_{2c+1}=\varpi_0(\ )\epsilon_1,\ \lambda_{2c+2}=\varpi_0(\ )\epsilon_2,\ \ldots,\ \lambda_{4c}=\varpi_0(\ )\epsilon_{2c}\\ \lambda_{4c+1}=P(\ ),\ \lambda_{4c+2}=\varpi_0\epsilon P(\ )\end{array}\right\} \qquad (40)$$

It is a quite simple matter to verify that these linities are all anti-commutative with one another; to take account of λ_{4c+2} it is perhaps easiest first to verify that λ_{4c+2} is anti-commutative with λ_{4c+1}, and then to verify that $\lambda_{4c+2}\lambda_{4c+1}=\varpi_0\epsilon(\ )$ is commutative with all the rest. It is equally easy to verify that

$$1=\lambda_1^2=-\lambda_2^2=\lambda_3^2=\ldots\ldots=-\lambda_{4c}^2=\lambda_{4c+1}^2=-\lambda_{4c+2}^2 \quad . \quad . \quad (41)$$

In other words $\lambda_1, \lambda_2 \ldots\ldots$ obey all the laws of fictits and form a set of precisely the same type as (37).

Since the number of λ's is even (whether $n=2c$ or $2c+1$), their multiplicative combinations are independent by the first theorem of the paper in § 2 above, and the number of these combinations is 2^{2n}. Thus between them they must furnish the most general multilinity of (34).

We have now established that a multilinity may always be looked upon as a multenion, and that $\lambda_1, \lambda_2, \ldots\ldots \lambda_{2n}$ are a set of fictits from which it may be constructed.

Conversely, it is obvious that a multenion belonging to an even order multiplex may be regarded as a linity, and in particular as a multilinity in a multiplex of half its own order. Since a multenion of a $(2m-1)^{\text{th}}$ order multiplex may always be regarded as one of a $2m^{\text{th}}$ order multiplex, any multenion may be regarded as a linity. But the multenions of an odd order multiplex do not form a self-contained system of linities; they form half the members of such a self-contained system.

I have thought (37) the best to start from, but it is desirable *temporarily* to suppose (37) replaced by

$$\epsilon_1^2=x_1,\ \epsilon_2^2=x_2,\ \ldots\ldots,\ \epsilon_{2c}^2=x_{2c},\ \epsilon^2=x$$

where every x is an arbitrary constant scalar differing from zero. Retain (38) and (40) to define $\varpi_0, \lambda_1, \lambda_2, \ldots\ldots$ The modifications of (39) and (41) are not wanted here, but are quite easy to write down if required. In this general case the conjugates, *in a linity sense,* of $\lambda_1, \lambda_2, \ldots\ldots$ would by (36) be $\lambda_1^1, \lambda_2^1, \ldots\ldots$ where

$$\lambda_a^1=\lambda_a^{-1}=K\lambda_a,$$

where K has its original defined meaning, *in a multenion sense,* when applied to the multiplex $\lambda_1, \lambda_2 \ldots\ldots$ These two meanings, then, of the conjugate of a multenion are identical. Either may be taken as the definition, and then the fact that the other applies is a theorem, true, but by no means obviously true.

We will now again suppose (37) and (41) to hold, but we will cease to speak of $\lambda_1, \lambda_2 \ldots\ldots$ as the fictits of the multiplex $\lambda_1\lambda_2 \ldots\ldots$ Instead we will suppose $\iota_1, \iota_2, \ldots\ldots$ to be the fictits where

$$\iota_1=y_1\lambda_1,\ \iota_2=y_2\lambda_2,\ \ldots\ldots,\ \iota_{2m}=y_{2m}\lambda_{2m} \quad . \quad . \quad . \quad (42)$$

and we note that the replacement A which is to be identified with conjugacy in the linity sense is

$$A = QI_2I_4I_6 \dots \qquad (43)$$

It may be called the alternate replacement or retroplacement because alternate fictits are negatived. Similarly, when $\epsilon_1, \epsilon_2 \dots$ in (37) or $\lambda_1, \lambda_2 \dots$ in (41) are taken as fictits we may call the arrangement an alternate fictit arrangement.

If with our new fictits $\iota_1, \iota_2, \dots$ we adopt law A, we have

$$\text{Law A} \qquad \left.\begin{aligned} \pm 1 &= \iota_1^2 = \iota_2^2 = \iota_3^2 = \iota_4^2 = \dots \\ &= y_1^2 = -y_2^2 = y_3^2 = -y_4^2 = \dots \end{aligned}\right\} \qquad (44)$$

We now have when q is a multenion of the multiplex $\iota_1\iota_2 \dots \iota_{2m}$:—

(A) Aq the alternate replacement of q is the conjugate of q in the linity sense. [It is obvious that q may be identified with a linity in a multitude of ways. With *one* of those ways Aq is the linity conjugate.]

(B) Therefore any uniretroplacement of q which [since $I_1 = \varpi\iota_1(\ )\iota_1^{-1}\varpi^{-1}$] may always be expressed as $v\mathrm{A}qv^{-1}$ is a ν-conjugate of q in the sense applied to linities above. And the two species of ν-conjugate above, depending on whether ν' is $+\nu$ or $-\nu$, depend now on whether Av is $+v$ or $-v$.

(C) From what was said above about the root sum of the multilinity $v(\ )v'$ it follows that the root sum of q is $2^m\mathrm{S}q$. Changing q to q^c we get that the sum of the c^{th} powers of the roots of the identity of degree 2^m which q satisfies is 2 Sq^c.

We may now transfer to multenions the theorems proved above for linities. Thus ignoring (24), (25), every equation from (15) to (32) may be taken as a multenion equation by reading

$$\begin{aligned} &v,\ q,\ r,\ \mathrm{A}q,\ \mathrm{R}q = v\mathrm{A}qv^{-1},\ \mathrm{R}(\ ) \text{ or "R-conjugate"} \\ \text{for}\quad &\nu,\ \phi,\ \psi,\ \phi',\ \phi^{\backprime},\ \text{"}\nu\text{-conjugate."} \end{aligned}$$

We can now prove the statement made in the footnote of § 4 above. We have to show that if $qpq_0 = 0$ for all multenion values of p, belonging to a given even order multiplex, then one of the multenions q, q_0 is zero; or, what is the same since such a multenion is a linity, restricted to a given order of linities; if $\phi\chi\psi = 0$ for all linity values of χ, then one of the linities ϕ, ψ is zero. Take the linities to be fictorlinities. If ψ is not zero, there is at least one non-evanescent fictor $\beta = \psi a$ obtained by operating by ψ finite fictor a. By proper choice of χ, β may be converted into any fictor $\chi\beta = \gamma$ required. Hence, that $\phi\chi\psi$ may be zero, ϕ must kill every fictor γ whatever; *i.e.* when ψ is not zero, ϕ is zero.

We will now find in terms of c, the number of symbols I_a, I_b, ... in the equation

$$[c = \text{number of operators } I_a, I_b, \ldots.], \quad Rq = vAqv^{-1} = QI_aI_b \ldots. q \quad . \quad (45)$$

the condition that $Av = -v$. The interest of this lies in the fact that when $Av = -v$ and $p = Rp$ (for instance when $p = qRq$) then p satisfies a simpler equation than usual, that is an equation of degree 2^{m-1} instead of degree 2^m as usual.

Let I_aI_b .... contain x of the symbols I_1, I_3, I_5, ... and y of the symbols I_2, I_4, I_6, ... so that $c = x + y$. Putting each I as $\varpi\iota(\ )\iota^{-1}\varpi^{-1}$ we have

$$v = \text{product of } (x \text{ of } \varpi\iota_1, \varpi\iota_3, \ldots) \text{ and } (y \text{ of } \varpi\iota_2, \varpi\iota_4, \ldots.).$$

Hence noting that

$$A(\varpi\iota_1) = -\varpi\iota_1, \quad A(\varpi\iota_2) = +\varpi\iota_2$$

we obtain

$$Av = (-)^{x + \frac{1}{2}(m-y+x)(m-y+x-1)}v = (-)^{\frac{1}{2}(m-c)(m-c-1)}v.$$

Hence $Av = -v$ *when* c *is* m+1, *or* m+2 *or any integer differing from either of these by a multiple of* 4. In other cases, of course, $Av = +v$.

When $R = Q$, $c = 0$, and when $R = PQ$, $c = 2m$. Hence

$$\left.\begin{array}{llll} \text{For } R = Q, & Av = -v, & \text{we must have} & n = 2m = 4, 6, 12, 14, 20, 22, \ldots. \\ \quad ,, \ R = PQ, & Av = -v, & \quad ,, \quad ,, & n = 2m = 2, 4, 10, 12, 18, 20, \ldots. \end{array}\right\} \quad . \quad (46).$$

Multiples of eight occur in neither list, so that to obtain the lowering of the degree, in those cases, some other retroplacement than Q or PQ must be used. Other multiples of four occur in both lists; (12) above is a case of this.

If q is unrestricted let $k = 2^m$; if $q = Rq$ and $Av = -v$ let $k = 2^{m-1}$. Thus in both cases k is the degree of the identity satisfied by q. Also in both cases

$$kSq^c = \text{sum of } c^{\text{th}} \text{ powers of the roots.}$$

This has already been stated in (C) above for the first case, and the second case is easily deduced from the first, thus. The Grassmann identity, when $q = Rq$ and $Av = -v$, has roots which consist of pairs of equals, say a, a, b, b, ... and the first case gives

$$2^m Sq^c = 2(a^c + b^c + \ldots.)$$

The roots of the lower degree identity satisfied by q in this case are a, b, ... so that

$$\begin{aligned} kSq^c = 2^{m-1}Sq^c &= a^c + b^c + \ldots. \\ &= \text{sum of } c^{\text{th}} \text{ powers of roots.} \end{aligned}$$

The connection between these sums and the coefficients is well known and is as follows. If the identity is

$$q^k - h'q^{k-1} + h''q^{k-2} - \ldots\ldots = 0$$

then

$$kSq - h' = 0$$
$$kSq^2 - h'.kSq + 2h'' = 0$$
$$kSq^3 - h'.kSq^2 + h''.kSq - 3h''' = 0.$$

Eliminating the coefficients we get

$$\begin{vmatrix} 1 & q & q^2 & q^3 \ldots\ldots & q^{k-2} & q^{k-1} & q^k \\ 1 & Sq & Sq^2 & Sq^3 \ldots\ldots & Sq^{k-2} & Sq^{k-1} & Sq^k \\ 0 & 1-k^{-1} & Sq & Sq^2 \ldots\ldots & Sq^{k-3} & Sq^{k-2} & Sq^{k-1} \\ 0 & 0 & 1-2k^{-1} & Sq \ldots\ldots & Sq^{k-4} & Sq^{k-3} & Sq^{k-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \ldots\ldots & 0 & k^{-1} & Sq \end{vmatrix} = 0 \quad . \quad (47)$$

or

$$\left.\begin{array}{l} 0 = \left(\frac{q}{k}\right)^k - \left(\frac{q}{k}\right)^{k-1}Sq + \frac{1}{2!}\left(\frac{q}{k}\right)^{k-2}\begin{vmatrix} Sq & Sq^2 \\ k^{-1} & Sq \end{vmatrix} \\ -\frac{1}{3!}\left(\frac{q}{k}\right)^{k-3}\begin{vmatrix} Sq & Sq^2 & Sq^3 \\ 2k^{-1} & Sq & Sq^2 \\ 0 & k^{-1} & Sq \end{vmatrix} + \frac{1}{4!}\left(\frac{q}{k}\right)^{k-4}\begin{vmatrix} Sq & Sq^2 & Sq^3 & Sq^4 \\ 3k^{-1} & Sq & Sq^2 & Sq^3 \\ 0 & 2k^{-1} & Sq & Sq^2 \\ 0 & 0 & k^{-1} & Sq \end{vmatrix} - \ldots\ldots \end{array}\right\} \quad . \quad (48)$$

With a multiplex of the 4th order we find from the above that $A\upsilon = -\upsilon$ in the cases when

$$R = Q,\ QI_1I_2I_3,\ PQ.$$

Hence in (12) above we may put R instead of K with any one of these meanings. The condition that $(q-x)^{-1}$ may be infinite is that the denominator on the right of (12) is zero, when in that denominator we put R for K and $q-x$ for q. This namely,

$$(q-x)R(q-x)[(q-x)R(q-x) - 2S.(q-x)R(q-x)] = 0 \quad . \quad . \quad (49)$$

which is a quartic in x, must by linity theory be the quartic of q. The coefficient of each power of x must, of course, be a scalar. If we substitute q for x we get another form in the present case for (47) or (48). I have not been able to generalise this to a multiplex of any order.

In conclusion, I will return to the matters discussed in the first complete page of § 11 above. I no longer (April 1908) have doubts as to the sign of ι^2 to be taken as a standard, but my present views are not correctly described on that page.

I think we ought definitely to accept the convention that *for a real fictit* $\iota^2 = -1$. [We may still call ι an *imaginary* fictit when $\iota^2 = +1$.]

It is a decided convenience, of course, to be able to put the i, j, k of quaternions equal to ι_1, ι_2 and $\iota_1\iota_2$ respectively, but my main reasons are purely algebraic.

In an even order multiplex the product ϖ is, except for the value of ϖ^2, virtually a fictit. We cannot make $\varpi^2=\iota^2$ without using $\surd(-1)$ or making ι^2 depend on the order of the multiplex. But we can make $\varpi^2=(\varpi\iota_1)^2$, namely by taking $\iota^2=-1$. Thus the n fictit complements and the product $\iota_1\iota_2 \ldots . \iota_n$ (or what is the same when, as at present, n is even, the product $\varpi\iota_1.\varpi\iota_2 \ldots . \varpi\iota_n$) are not only all anti-commutative, but are also in harmony as to their squares. As a consequence, as we should expect, we can very simply identify without introducing $\surd(-1)$ the 2^{2m} even order multenions of a multiplex of order $2m+1$ with the complete set of 2^{2m} multenions of a multiplex of order $2m$. Thus if $\epsilon_1, \epsilon_2, \ldots . \epsilon_{2m+1}$ are the fictits of the former and $\epsilon_1^2 = \ldots . = -1$; if we put

$$\iota_1 = \epsilon_1\epsilon_{2m+1}, \ \iota_2 = \epsilon_2\epsilon_{2m+1}, \ldots . \ \iota_{2m} = \epsilon_{2m}\epsilon_{2m+1},$$

and therefore

$$\iota_1\iota_2 \ldots . \iota_{2m} = \epsilon_1\epsilon_2 \ldots . \epsilon_{2m}$$

the identification is obviously effected and $\iota_1^2 = \ldots . = -1$.

If $S, S_1, S_2 \ldots .$ have their usual meanings with regard to $\iota_1, \iota_2, \ldots .$ and $S_{(0)}, S_{(1)}, S_{(2)} \ldots .$ the corresponding meanings with regard to $\epsilon_1, \epsilon_2, \ldots .$, and if I be the operator negativing ϵ_{2m+1}, it is easy to show that

$$S = S_{(0)}, \ S_1 = \tfrac{1}{2}(1 - I)S_{(2)}, \ S_2 = \tfrac{1}{2}(1 + I)S_{(2)} \ldots .$$
$$S_{2c-1} = \tfrac{1}{2}(1 - I)S_{(2c)}, \ S_2 = \tfrac{1}{2}(1 + I)S_{(2c)}$$

so that

$$S_{(0)} = S, \ S_{(2)} = S_1 + S_2, \ S_{(4)} = S_3 + S_4 \ldots .$$

The following convention should, I think, be adopted not only in the present method but in all allied methods. It is derived from the i, j, k of Hamilton and the $e_1, e_2, \ldots .$ of Grassmann. Whenever primitive units are to satisfy the condition $\iota^2=-1$ consider $i_1, i_2 \ldots ,$ as permissible symbols to denote them; whenever they are to satisfy $\iota^2=+1$ consider $e_1, e_2, \ldots .$ as permissible; but *never* consider the converse permissible. Thus

$$-1 = i_1^2 = i_2^2 = \ldots .$$
$$+1 = e_1^2 = e_2^2 = \ldots .$$

When we want $\iota^2=\pm 1$ we can use $\iota_1, \iota_2, \ldots .$

It is just as well that in this first paper the sign of ι^2 has been left ambiguous, because it was desirable to test fully what assumption seemed best.

I would also recommend another convention, namely, utterly to ignore the concept of a continent multiplex. The multiplicative combinations of the $(n-1)^{\text{th}}$ order multits, when n is odd, are all of even order; and the multenions formed from them have precisely the same properties as those of what I have called the continent multiplex; so that there is absolutely no reason why we should ever impose artificially the condition that the product of n independent fictits is a scalar. The product of the $(n-1)^{\text{th}}$ order multits just mentioned is of necessity a scalar.

I did not realise these simplicities when writing the paper; but with the light that has come from later reflections, I have no hesitation in recommending these two important restrictions, namely: a real fictit shall be one for which $\iota^2=-1$; and (2) there shall be no continent multiplex.

I am inclined also to restrict the meaning of replacement to what is above called a unireplacement. What is called above the fictorlinity replacement may very fairly be called a strain; $e^{\omega}(\;)e^{-\omega}$ a fictor rotation; $q(\;)q^{-1}$ a multenion rotation; and what is above called a complementary replacement may be called a complementary substitution.

Geometrical applications are not our object here, but I may just state that I find there are interesting Euclidian-geometry applications of quadriquaternions.

A quaternion may be looked upon both geometrically and algebraically as a degenerate octonion, an octonion as a degenerate Combebiac triquaternion, a triquaternion as a degenerate quadriquaternion. A quadriquaternion is defined as

$$\omega' p+q+\mu r+\omega s,$$

where p, q, r, s are four independent quaternions; ω', μ, ω are commutative with quaternions and satisfy the equations

$$\mu^2=1,\ \omega^2=\omega'^2=0,$$
$$\omega\omega'=\tfrac{1}{2}(1-\mu),\ \omega'\omega=\tfrac{1}{2}(1+\mu),$$
$$\omega\mu=\omega=-\mu\omega,\ -\omega'\mu=\omega'=\mu\omega'.$$

A triquaternion requires $p=0$, an octonion requires $p=r=0$, and a quaternion requires $p=r=s=0$.

A quadriquaternion may be regarded as a multenion of a fourth order multiplex in many simple ways. The following is one. Let

$$\iota_1^2=\iota_2^2=\iota_3^2=\iota_4^2=-1,$$

and let the i, j, k of quaternions be given by

$$i=\iota_2\iota_3,\ j=\iota_3\iota_1,\ k=\iota_1\iota_2.$$

Then we may put

$$\mu=\iota_1\iota_2\iota_3\iota_4,$$
$$\omega=\tfrac{1}{2}(1-\mu)\iota_1\iota_2\iota_3=\tfrac{1}{2}\iota_1\iota_2\iota_3(1+\mu),$$
$$\omega'=\tfrac{1}{2}(1+\mu)\iota_1\iota_2\iota_3=\tfrac{1}{2}\iota_1\iota_2\iota_3(1-\mu).$$

TABLE OF CONTENTS.*

* I have prepared this brief epitome of what seem to be the salient points, to facilitate reference.—C. G. KNOTT.

SUPPLEMENT—(continuation of § 11).

(*Issued separately August* 28, 1908.)

XXXV.—The Inca Bone: Its Homology and Nomenclature. By

W. Ramsay Smith, D.Sc., M.B., C.M., F.R.S.E., Permanent Head of the Department of Public Health of South Australia.

(MS. received March 18, 1908. Read May 4, 1908.)

THE inca bone in the human skull is usually regarded as the homologue of the interparietal of some other mammals. The commonest or best-known form of the interparietal occurs in the rabbit as a single somewhat oval bone, its long axis being transverse, filling up a space between the parietals just in front of the line or curve of the occipital.

Carl Vogt (*Lectures on Man*, London Anthropological Society, 1864) makes frequent and detailed reference to a Helvetian skull. From the woodcuts, p. 52, fig. 15; p. 66, fig. 22; p. 70, fig. 26; p. 389, fig. 124; and p. 390, fig. 125, it is clear that a small round undivided bone occurred in this skull in a situation roughly corresponding with the position of the interparietal in the rabbit. I say "roughly," because the posterior border of the bone in the Helvetian skull just touches the occipital at the lambdoid suture, while in the rabbit the interparietal for about half the extent of its perimeter is in contact with the occipital.

Vogt says (p. 390) regarding the adventitious bone: "There seems also in these Swiss skulls to exist a tendency to the separation of the lambdoid suture. The skull from the vicinity of Geneva has that isolated piece of bone at the point of this suture, which was formerly considered as peculiar to Peruvian skulls, and hence called the bone of the Inca (*Os Incæ*). I saw the same thing in some other skulls of Biel and Grenchen, also large Wormian bones in the lateral wings of the suture."

Deniker (*The Races of Man*, London, 1900, p. 66) classes the inca bone as a Wormian bone, and describes it as being found between the parietal bones and the occipital. He gives statistics of its appearance in a perfect and an imperfect state in various races. His illustration (*op. cit.*, p. 88) represents a large crescentic bone in the position of the membranous portion of the occipital.

Ward (*Human Osteology*, 3rd edn., p. 34) gives a very complete description of the occurrence and appearance of Wormian bones, in the course of which he describes what is undoubtedly the separate or ununited membranous portion of the occipital. Béclard's case, to which he refers, appears to have been similar to one that I have to describe: "The whole of the occipital bone above the upper curved lines was formed by two large triangular Wormian bones, united by a vertical suture, continuous with the sagittal." He refers to and criticises Cloquet's statements regarding Wormian bones.

On turning to Cloquet's plates (*Manual d'Anatomie descriptive du Corps humain*) one finds (plate 7, figs. 1, 4) Wormian bones depicted in the right fronto-parietal suture, and a small quadrilateral one at the posterior end of the sagittal suture. In fig. 2 of the same plate, two Wormian bones are shown in a very common position—in the limb of the lambdoid suture.

From these authors, who seem to identify the inca bone with Wormian bones, we pass to some who make a distinction more or less completely. Students of human anatomy are familiar with modern descriptions of a separate interparietal, such as are found in Quain's *Osteology* and Morris's *Anatomy*. In these descriptions the separate interparietal corresponds in position and extent with the "membranous" portion of the occipital which, in course of development, has remained distinct from the "cartilaginous" portion of the same bone. Bland Sutton (*A Treatise on Anatomy*, edited by H. Morris, 1893, p. 31) figures it as a large crescentic bone, concavity downwards, capping the occipital. He does not call it the inca bone; but it corresponds with the bone to which some other writers have applied that name. He says: "Not infrequently the interparietal portion remains separate throughout life, and may be even represented by numerous detached ossicles or Wormian bones." It is to be noted that Thane in *Quain's Anatomy* discusses this subject apart altogether from his description of Wormian bones.

Leidy (*Human Anatomy*, London, 1889, p. 137) appears to make a distinction between a "sutural bone," or a pair, which replaces to a variable extent the summit of the occipital, and the condition of permanent separation of the upper division of the supraoccipital of the embryo, which "appears to have been a more frequent one in some of the tribes of the South American people." This "isolated portion of the supraoccipital," he says, "corresponds with the interparietal bone of some of the lower animals, as in the rabbit."

Macalister (*A Textbook of Human Anatomy*, London, 1899, p. 245) also distinguishes between Wormian bones and such a bone as the true os inca. He says: "Detached ossicles, known as Wormian bones, are often to be seen in the lambdoid suture, and more rarely in other sutures. These result from the presence of supernumerary ossific centres, and are not to be confounded with the separate bones caused by want of union of the normal centres.

The persistence of the separation of the upper part of the supraoccipital and of the lower part of the malar are examples of the latter, and are not Wormian bones."

In view of these different statements, one naturally wishes to have some evidence regarding the particular structure to which the name inca bone was originally applied. I think I can supply this, even if the information is somewhat second-hand. Some time ago I became possessed of a copy of Lieut.-Colonel Charles Hamilton Smith's work on *The Natural History of the Human Species* (Edinburgh, W. H. Lizars, 3 St James' Square, 1848). In it (on plate i.) there are two small lithographs of the skull of a Taticaca child. The occipital view shows a large undivided inca bone, posterior to the lambdoid suture. In the text (p. 145) Colonel Hamilton Smith says: "Dr Tschudi, describing this form, in his paper on the ancient Peruvians, remarks on the flattened occiput of the cranium, and observes, 'that there is found, in children, a bone between the two parietals, below the lambdoidal suture, separating the latter from the inferior margin of the squamous part of the afterhead; this bone is of a triangular shape, the upper angle between the ossa parietalia, and its horizontal diameter, being twice that of the vertical. This bone coalesces at very different periods with the occipital bones, sometimes not till after six or seven years. In one child of the last-mentioned age, having a very flat occiput, the line of separation was marked by a most perfect suture from the squamous part, and was four inches in breadth by two in height.' In remembrance of the nation where this conformation is alone found, the learned doctor denominated this bone *Os Incæ*; and he further remarks, that it corresponds to the *Os interparietalis* of Rodentia and Marsupiata."

Perhaps the homology in Tschudi's description halts a little, like the Latinity. But it is clear from the quotation that the inca bone of Tschudi bore little if any resemblance to the bone in the Helvetian skull figured by Vogt and described by him as a "well-developed os incæ." It appears to me that the round bone in the Helvetian skull is nothing else than a Wormian bone in a somewhat unusual position, viz., in the sagittal suture. And yet the position may be found on investigation not to be so very unusual, seeing that Wormian bones occur in every suture. I have seen a large triangular bone in the position of the anterior fontanelle of an Australian aboriginal. Had this *os antiepileptium* occurred in a similar condition at the posterior extremity of the sagittal suture, it might have been called by some an inca bone or an interparietal

The "separate interparietal" figured by Morris and referred to in Quain, is a bone that belongs undoubtedly to a different category. It is the inca bone as originally described and named by Tschudi and figured in Hamilton Smith. I think a good deal of light is cast upon the appearances and the homology of this bone, and of bones closely resembling it, by some specimens I have now to describe.

A very fine example of "unpaired" bone occurs in a skull (No. 970*) of an Australian aboriginal (fig. 1). This is undoubtedly a true inca bone. The subject in whom it occurred was a remarkable character. He was the lowest and most animal-like in habits of any of the aboriginals ever seen by the police here, and he spent his latter days in confinement as a criminal lunatic. It is interesting to know that he was the fellow-countryman of

Fig. 1.

the aboriginal whose head is the subject of that most interesting monograph by Professor Cunningham in the *Journal of the Royal Anthropological Institute*, vol. xxxvii., 1907.

Wilhelm Krause describes about 200 skulls of Australian aboriginals in the *Verhandlungen der Berliner anthropologischen Gesellschaft*, 20th November 1897. He says (p. 516): "The lambdoid suture was found in 82 of 185 skulls, or in 42·9 per cent.; also one or more ossa intercalaria up to 21 (No. 170). A real os incæ could not be found; but an os interparietale forming the point of the squama occipitalis occurred five times (Nos. 18, 27,

* These numbers, except where otherwise stated, refer to the catalogue of my series of pathological and anthropological specimens.

104, 131, and 175)." Three of the numbers quoted by Krause refer to skulls in the South Australian Museum. I have examined the skulls, and by permission of the Director, Professor E. C. Stirling, F.R.S., I am able to describe and illustrate the conditions occurring in them and in some other skulls not referred to by Krause.

The first (Krause's list, No. 104) has a small irregular bone at the posterior end of the sagittal suture. This is the sort of bone usually described as the inca bone or interparietal. Numerous Wormian bones are present in the outer half of each limb of the lambdoid suture, and in each temporal region there is an epipteric bone.

FIG. 2.

The second (Krause's list, No. 131) has a small bone very much resembling the first in size and position. In the inner half of the right limb of the lambdoid suture there is a larger Wormian bone, and in a corresponding position on the left side there is a smaller one.

The third (Krause's list, No. 175) shows a remarkable condition of the lambdoid suture as regards Wormian bones, viz., two series of bones, one in each limb of the lambdoid suture, irregularly oblong, lanceolate, or linear, the long axis of each bone being sagittal in direction (fig. 2). A small Wormian bone is placed almost on the vertex in the sagittal suture, and there are two in the right limb of the fronto-parietal suture.

Among the skulls of other races than Australian aboriginals in the

South Australian Museum is one from Torres Strait, presented by the Chief Justice, the Right Honourable Sir Samuel Way. The skull had been prepared after the native fashion, and is evidently the skull of a native. It has a specimen of an inca bone, closely resembling the one in the aboriginal skull I have just described. It possesses an added interest in having several Wormian bones in the true lambdoid suture (*i.e.*, the suture between the parietals and the inca bone), and in having a small bone, or two small bones, at the termination of the sagittal suture corresponding to what often passes for the interparietal or inca bone.

From the fact that the membranous portion of the occipital bone is

Fig. 3.

developed from two centres, one would naturally expect that the inca bone might be found presenting the appearance of being bilaterally double, as in Béclard's case quoted by Ward. In a skull that I received recently from New Caledonia from Dr Carter through Mr Maning, the Acting British Consul, the double character of the true inca bone is very apparent, although the various sutures of the skull are either very firmly knit or obliterated (fig. 3). The median suture of the inca bone in this kanaka skull is well marked; the whole lambdoid suture (by lambdoid I mean the suture immediately posterior to the parietals) is fairly well outlined; while the inner part of the suture between the right half of this bone and the occipital is almost obliterated.

I think there can be no doubt about the homology in the examples illustrated in figs. 1 and 3, and in the skull from Torres Strait. The bone in each case is the portion of the occipital that develops in membrane—the inca bone of Tschudi. This bone in the human skull was said by him and is said by most others to correspond with or represent the "interparietal" (Parker and Bettany, Claus, etc.). One wishes, however, for details of the evidence, embryological or other, on which the statement is based.

By a strange coincidence I received, about the same time, another New Caledonian kanaka skull from Captain Perny of the Messageries Maritimes Company. In it Wormian bones abound. It presents also a very striking

Fig. 4.

feature in the occipital region. Behind the left limb of the lambdoid suture there is interpolated a large bone which one might properly describe as a huge Wormian bone (fig. 4). Curiously enough, the lower margin resembles closely in position and appearance the corresponding suture in the other kanaka skull (fig. 3). Had the inner margin of this bone coincided with the median line of the skull, one would probably have classed the bone as the left half of an inca bone.

In the South Australian Museum there is an Ancient British skull from a round barrow presenting a condition very similar to this, but the bone is not so large. The adventitious bone in this case is no doubt a Wormian bone.

In the same collection there is also the skull of a Malay who was murdered in the Northern Territory by means of a tomahawk. The condition it exhibits is unusual (fig. 5). Four bones combine to fill up the space usually occupied by the membranous portion of the occipital. Three sutures running in the sagittal direction divide into four portions what would otherwise be the representative of a single inca bone. The central suture, however, does not appear to correspond closely in position and character with the main sagittal suture of the skull; and the question arises whether after all the "subdivided inca" is not really made up of Wormian bones, presenting a regular and fairly symmetrical appearance.

Another skull (No. 35 in the South Australian Museum Collection),

FIG. 5.

one of an Australian aboriginal, shows a condition somewhat similar, but less in extent and presenting greater irregularity. Another aboriginal skull (No. 31 in the Museum Collection) shows two large Wormian bones separated by a "tongue" of the occipital (fig. 6). A similar condition occurred in the skull of a man here of European descent, twenty-seven years old. This skull presented also a persistent frontal suture. My attention was directed to it by Professor Watson of the Adelaide University.

The British skull referred to above, like several other Ancient British skulls in the same collection, has a persistent frontal suture. The association of such a suture with the existence of an "interparietal" has been noted by Vogt and others, although in many instances the adventitious bone has not been a true inca bone.

Other writers have pointed out that the true inca bone occurs in races in which a persistent frontal suture frequently occurs. It is of some interest to know that about the same time that I received the New Caledonian kanaka skull with the double inca bone, I also received the skull (No. 1027) of a Malekula kanaka woman with a well-marked persistent frontal suture.

Conclusions.—The term "inca bone" was used primarily to designate a bone corresponding in position and extent with the ununited membranous

FIG. 6.

portion of the occipital. It would be well to restrict its application accordingly. The "inca bone" would thus be distinguished from other bones occurring in this region to which the name "interparietal" has been loosely and sometimes indiscriminately applied. Some adventitious bones occurring in the region of the junction of the sagittal and lambdoid sutures are simply Wormian bones; others are doubtful as regards homology, and may be Wormian bones modified, in respect of some of their margins, by the normal "embryological sutures."

(*Issued separately August* 29, 1908.)

XXXVI.—The Middle Cells of the Grey Matter of the Spinal Cord.

By J. H. Harvey Pirie, B.Sc., M.D., M.R.C.P. Ed. Abstract of Thesis for the Degree of M.D. Edin. Univ., 1907. (*From the Pathological Department of the Royal Infirmary, Edinburgh.*) *Communicated by* Dr Alexander Bruce. (With Two Plates.)

(MS. received February 6, 1908. Read February 17, 1908.)

These cells were first described and recognised as a separate and distinct group by Waldeyer (1). From his paper the following notes are taken:—"The middle cells are found at the junction of the anterior and posterior cornua, but especially in the former. They are small and medium-sized polygonal cells, arranged sometimes in a fairly compact group, sometimes more loosely scattered over a wider area. They are never so closely aggregated as are the cells of Clarke's column or those of the intermedio-lateral tract (Seitenhorn), but they are close enough to be regarded, especially in the upper cervical region, as a special nucleus of cells. Their situation varies somewhat in the different regions of the cord. As a group, they are most distinct in the upper cervical region. They lie here closely compacted to the outer side of, or a little in front of, Clarke's column. In the lower cervical region they form a less distinct group near Clarke's column, but quite internal to the postero-lateral motor group. In the upper dorsal region they are also abundant. They are situated laterally, and even somewhat posteriorly, to Clarke's column, extending into the posterior horn and into the scattered cells (Zerstreutenzellen) of the anterior cornu without a distinct limit. They are always quite distinctly internal to the margin of the grey matter. With the growth of Clarke's column in the dorsal cord the middle cells become fewer. In the lower dorsal region they again become more numerous, no longer form a distinct group, but are more in their former position on the outer or antero-external aspect of Clarke's column, in great numbers. Similarly in the lumbar cord, though less abundantly developed. In the sacral cord there is a group of cells which, from their character and situation, are probably middle cells, but at this level no characteristic distribution of the various cell groups is recognisable."

Other small cells in the grey matter of the cord named by Waldeyer must also be referred to. His account of the intermedio-lateral tract is sufficiently noticed in Bruce's paper on that tract (2). The cells which he calls "scattered cells" (Zerstreutenzellen) "form no definite group. They are situated—(1) in the anterior horns; (2) in the neighbourhood of the central canal; (3) in the Rolandic substance of the posterior horns; (4) in the white matter."

Those in the anterior horn and in the neighbourhood of the central canal I have not been able definitely to separate from the middle cells, and have included them in my description of that group. But, for clearness, the small cells in the anterior horn are sometimes referred to as the scattered cells; those in the neighbourhood of the central canal as the para-central cells—using topographically a term introduced by Onuf and Collins (3), without meaning to convey that they are the same cells which these authors describe (in the cat) as lying "ventrad of Clarke's column, on each side of the central canal, and showing in longitudinal sections a segmented arrangement." In man, they further state that "this group seems to have lost its individuality and to form part of Clarke's column, except at certain levels (upper dorsal and middle sacral) where a cell group is seen which apparently corresponds to the para-central group, although situated considerably more laterally than in the cat." Elsewhere they speak of the cells of the intermediate zone (apparently the middle cells) as "for the most part small, approaching in shape and structure to the cells of the lateral horn and of the para-central group."

Waldeyer classified the small cells of the posterior horn which "lie posterior to the level of the hindmost part of Clarke's column, but do not form any well-marked group, and are not always present, as (1) Basal, (2) Central, (3) Marginal. They are seldom or never all present together, and are never in large groups, often only single cells, or at most two or three cells. The Basal cells lie immediately behind Clarke's column, middle cells, and lateral horn cells; the Central cells in the posterior horn nucleus; the Marginal cells on the inner and outer borders respectively of the posterior horn, the inner being apparently the more abundant." I have been much puzzled over these cells in the fœtal cord: sometimes they can be distinguished according to Waldeyer's subdivisions, but very often it is impossible to separate them from the reticular group of the intermedio-lateral tract, on the one hand, and from the middle cells, on the other. In appearance and size they are all much alike; and as regards position, the outer marginal cells might be simply a continuation backwards of the reticular group, while the basal, and sometimes even the more central and inner marginal cells, are often directly continuous as a group with the middle cells. I have observed at all levels of the cord, but especially in the lumbar segments in all of the situations of these posterior horn cells, certain large cells, resembling, both in size and appearance, the motor cells of the anterior cornua. They are usually single, but, rarely, two may occur together.

Argutinski's observations (4) must now be referred to. His description of the middle cells of the new-born child is fuller than that of Waldeyer, but differs from his in several points. In particular, he finds a regular and sharply defined segmentation of the middle cell column, but limits the system to the dorsal portion of the cord. After a careful and thorough examination of the spinal cord both in longitudinal and in transverse sections—including serial sections of a complete cord—I am convinced that this segmentation of the middle cells does not exist, and, further, that what Argutinski has described is the reticular group of the intermedio-lateral tract. His description is too long to quote in its entirety; briefly, it is that of a double chain of segmented cell columns (Seitenhornzellen and Mittelzellen), limited to the dorsal portion of the cord, the segmentation of the two columns being parallel, but sharper in the middle cell group. His description is based principally on the examination of longitudinal sections, although, as he himself states, conclusions as to position can only be drawn from transverse sections. That this double chain exists I fully admit, but I hold that it is formed of the two constituent groups of the intermedio-lateral tract, and not of the middle cells and lateral horn cells. Waldeyer, as pointed out by Bruce, includes the reticular group of cells in his Seitenhornzellen. Argutinski does not do so; he only includes the apical group of the intermedio-lateral tract under that heading. If, throughout his paper, we read for "Seitenhornzellen," apical cells or apical group, and for "Mittelzellen," reticular cells or reticular group, we get an excellent description of the complete intermedio-lateral tract. The points of agreement between his "Mittelzellen" and the reticular group of cells as described by Bruce (whose observations I entirely corroborate, except in some small points, probably due solely to the difference in age of the subjects from whom the cords were obtained) are as follows:—

(1) Longitudinally limited almost to the dorsal portion of the cord.

(2) Lie very near to the lateral horn cells (apical group), sometimes even in direct contact with them.

(3) Lie posterior and internal to the lateral horn cells, usually close to the re-entrant reticular angle.

(4) Direction of cells—chiefly inwards and forwards.

(5) Shape of cells—like the lateral horn cells (apical group), but broader.

(6) Parallel segmentation of the apical and reticular groups (Seitenhorn- and Mittelzellen of Argutinski), the segmentation being a much sharper one in the latter of the two groups.

There are two points of non-agreement in the description:—

1. Argutinski's middle cells never approach quite close to the lateral columns—a point which he insists on strongly, as distinguishing them from the Seitenhornzellen. Well, in the cord of the new-born I find that the reticular group not infrequently does lie a little way from the edge of the grey matter, sometimes even a considerable distance (particularly in D 3 and D 12). Moreover, from the shape which the reticular group so often has, it is obvious from the accompanying diagram (fig. 1) that many frontal longitudinal sections would show the reticular group quite internal to the edge of the grey matter. But I cannot agree with nor explain his statement that the group *never* approaches quite to the lateral columns.

2. He describes his cells as sometimes departing from their usual

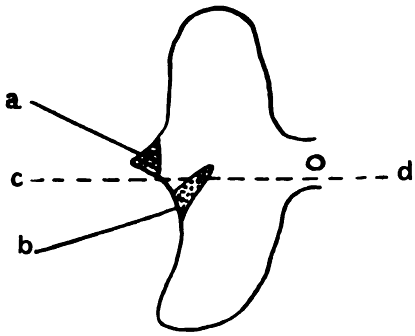


FIG. 1.

a. Apical group of the intermedio-lateral tract.
b. Reticular group of the intermedio-lateral tract.
c d. Line of a frontal longitudinal section which would show the reticular group quite internal to the edge of the grey matter, and in a position more usually occupied by middle cells.

position, and being found more or less mesially, or even in the region of the anterior horn cells or Clarke's column cells, and that all transitions may be found between these positions. It is obvious that here he must be referring to the real middle cells of Waldeyer, and I am in complete agreement with this statement. But when he says, further, that if the cell group is seen lying closer to the middle line, his middle cell group (reticular group) is absent from the usual place, I must part company from him, and can only differ without explaining. But certainly I have often observed a reticular group in its usual situation and a group of middle cells present in another part of the same section.

So far as I am aware, no other writers have given any further account of the middle cells of the spinal cord.

The present investigation was made on the spinal cord of a full-time healthy fœtus. The cord was hardened in absolute alcohol and divided into root segments by cutting transversely just below the lowest fibre of each nerve root. The segments were embedded in paraffin and each cut into serial sections; the sections were then stained with Giemsa's stain (azure blue and eosin). Special means were taken to see that each segment was cut and mounted so that the sections were numbered serially from above downwards, and that there was no confusion between right and left sides. There were in all nearly 16,000 sections. At first, when the possibility of the middle cells having a segmented arrangement was before me, I attempted to enumerate them, with the idea of representing the numbers graphically, but it soon became evident that no such segmented

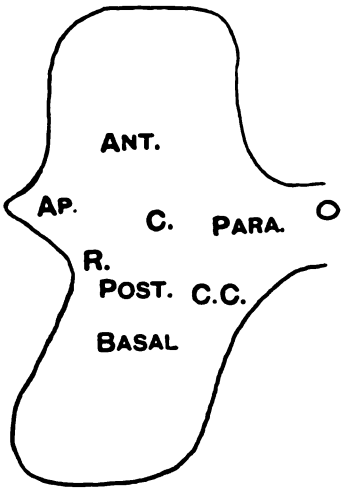


Fig. 2.

C. = Central area.
Ant. Anterior central area in the base of the anterior horn.
Para. Para-central area.
Ap. Apical or lateral horn area.
C.C. Clarke's column area.
Post. Post-central area.
R. Reticular area and area of the formatio reticularis.
Basal. Posterior basal area.

character existed; and further, that the task of counting all the cells was too herculean to attempt. I therefore contented myself with noting the position and arrangement of the middle cells throughout the whole series of sections, summing up the results for each 25 as I went along, and again for each segment. By this method, I consider, a very accurate idea of their distribution has been obtained.

Some difficulty was felt with regard to nomenclature and division of the grey matter into areas. Waldeyer recognises three regions of the grey matter: a free anterior horn, a middle region, and a free posterior horn. The middle region is defined as including the central canal with transverse commissures, the lateral horn, the region of the middle cells, and Clarke's column. Everything in front is the free anterior horn; everything behind, the free posterior horn. But it is evident that to describe in detail the position of the middle cells, some other plan must be adopted; and the terms which I have used in the description are best explained by reference to the accompanying diagram (fig. 2).

At most levels these different areas or regions can be recognised, but they are quite arbitrary, and only employed for topographical convenience, and it seems futile to attempt to define any exact boundaries for them. The middle cells are certainly not in the least regardful of boundary-line between these regions. In the cervical and lumbo-sacral enlargements there is, of course, no apical region; Clarke's column, though not always represented by cells of that column, still has its "area" there; in addition, an external central area might perhaps be added. I shall now go on to the description of the middle cells as found in the various segments of the cord.

C 1.—See fig. 3.—Vary considerably in number, on the whole fairly abundant, either scattered diffusely or in loose groups. In the base of the anterior horn always some scattered cells. The commonest group is a central one, tending to extend into the basal area, and especially into the para-central area; may be 30 or more cells, forming a not very compact group. Frequently a small group in the position corresponding to that of the lateral horn. Merely a few scattered cells in the reticular formation, post-central area, and region of Clarke's column. The cells vary in size in all areas; those in anterior horn on the whole larger than the others, and sometimes with difficulty distinguished from motor cells, but they are never in such compact groups. They are all of polygonal or rounded polygonal outline, and their mode of occurrence is typical, *i.e.* irregular and casual—any group that may be found can never be traced through more than three or four sections at most; there is never any sign of regular segmentation such as is seen in the intermedio-lateral tract of the dorsal region.

C 2.—Cells fewer in number than in C 1. A central group of from 12–15 cells is a very common feature of the segment; in fact, for long stretches it seems to be almost constant, particularly on the right side. Sometimes it is more para-central than central. A small group at or just in front of the apex—such as it is—of the lateral horn is also common. In the other areas their appearances are much as in C 1.

C 3.—See fig. 11, Plate I.—Cells not very abundant, characteristically scattered, sometimes thickly, more often thinly. Small aggregations may be found in the area of Clarke's column, the post-central region, and in the base of the anterior horn, either near the outer edge or nearer the centre, but none of them are very compact or very large. A central or central-para-central group is the most common. In other regions only scattered cells occur. In size they appear to be of mixed sizes in all areas; their distribution lengthwise is typical of the middle cells—here one section, gone the next.

C 4.—The description given of C 3 applies practically unchanged to this segment, although, with the growth of the lateral motor group, the middle cells disappear from the outer portion of the base of the anterior horn, but they may still be seen between the motor nuclei. On the whole, the cells are both rather more numerous and rather larger than in C 3,

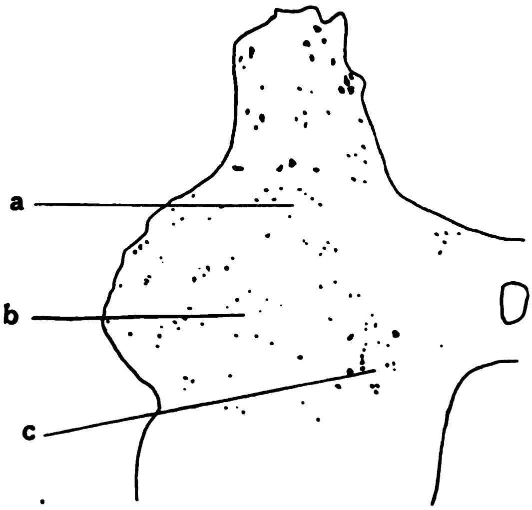


Fig. 3.*

C 1—162 L (× 33).
a. Scattered cells in the base of the anterior horn.
b. Middle cells scattered over the area of the lateral horn and the reticular formation.
c. Group of middle cells in the posterior part of the central-para-central area, including a few of larger size than elsewhere, all the others being small.

particularly the more anterior cells, though here also the difference in size between these and the others is not at all marked.

C 5.—Cells increasing in number, descending the segment, also tending to be more grouped. Anterior to the frontal plane of the central canal few and scattered, but often of larger size than elsewhere, sometimes not much smaller than motor cells. In the reticular formation, scattered at first and sometimes outlying in the white matter, oftener small groups in the lower part. A central group more often in upper part than in lower. In lower part of the segment frequently a well-marked band of cells stretching

* These figures in the text are traced from photographs, and give exact position of cells and approximately their relative sizes and shapes. The numbering is as follows:—C 1—162 L means the 162nd serial section from the top of the first cervical segment, the left side.

through the reticular and post-central areas to the central or even para-central region. Along this band, aggregations of cells may be present in any part, most often post-centrally. Very few in the region of Clarke's column. Inconstancy of their appearance a typical feature, as always. Medium and small-sized cells mixed, the more anterior being generally larger than the others.

C 6.—See fig. 12, Plate I.—Cells, on the whole, abundant. In individual sections they may be few or many, scattered or grouped. Most numerous about junction of central and post-central areas (these two regions are not very distinct in the cervical enlargement). A small reticular group also common. More often, simply scattered cells in the other areas. There is a distinctly large proportion of big cells in this segment, not only anteriorly, but also more posteriorly, although here they are mingled with others, smaller and more rounded than polygonal in outline.

C 7.—Cells abundant, especially along a broad band from the reticular angle to the central canal, but particularly, as in lower C 6, in the post-central areas. More often, just scattered cells anteriorly and in Clarke's column areas, although there are often cells lying rather behind the region of Clarke's column (posterior basal ?). No trace of any regular segmentation. In size and shape the anterior cells are fairly large and polygonal, the central and post-central of large and medium sizes, the reticular and posterior basal medium-sized and more rounded.

C 8.—See fig. 4.—Upper part as in C 7. In the lower, cells diminish in number and are more scattered, although small groups may be seen in any of the usual positions. In the lower, also, middle cells are more often seen lying between the anterior and lateral motor nuclei. In size they are much as in C 7. Cells are not more common in the reticular formation, nor is there any sign of segmentation or special grouping here. Very few cells in region of Clarke's column.

D 1.—Cells of the bigger type in base of anterior horn and (in upper part) between median and lateral motor groups. Medium-sized cells in central, para-central, and post-central areas, also in the formatio reticularis, also a few at the base of the posterior horn. Most abundant, and often persisting through a number of sections, just internal and a little anterior to the reticular angle, but never so closely packed as the cells of the intermedio-lateral tract. The medium-sized cells individually closely resemble those of that tract, and sometimes it is impossible to say to which series any particular cell is to be allotted. Many of the post-central cells behind and external to Clarke's column are of a smaller, more rounded type.

D 2.—See fig. 5.—Anterior cells, sometimes in small groups, but usually scattered. On the whole, larger than the others, but cannot be definitely separated from the central middle cells either by this or by another feature which helps to distinguish them, viz. the fact that they are often elongated along one axis. The long axis may have any orientation. The other middle cells may be scattered thickly or thinly, but often small, but not very compact, groups of from 6 to 18 cells are seen. Groups, when traced through two or three sections, often disappear or shift in position—there is no constancy. A central group is the most common, next most a post-central,

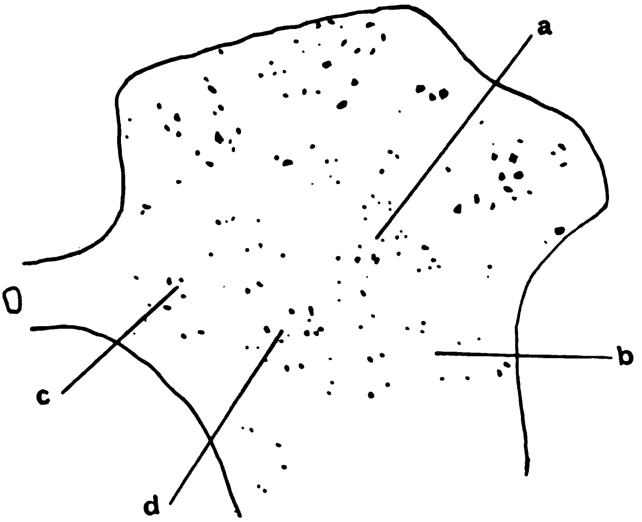


Fig. 4.

C 8—88 R (× 32).

a. An external central group of middle cells, extending between the motor groups.
b. A few cells in the reticular area.
c. Small para-central group.
d. Central post-central group.

A few small cells in area of Clarke's column and anterior "scattered" cells.

on the outer side of and behind Clarke's column. In the upper part more especially there are often cells in the reticular formation which, from their size, character, and casual occurrence, are probably not cells of the intermedio-lateral tract, but middle cells. In appearance and size, however, the individual cells are very much alike.

D 3.—Cells erratically distributed as usual, sometimes scattered, sometimes in small groups. Chiefly polygonal, about the same size as those of the apical group of the intermedio-lateral tract, but staining rather more darkly; others smaller, and rather more rounded in central and post-central areas. In base of anterior horn they are not markedly larger than elsewhere, but are frequently elongated in one direction. Both central and post-central cells are sometimes with difficulty separated from the reticular group of the intermedio-lateral tract, especially when there are isolated cells of the latter group left stranded in the central area, a feature which is not uncommon in this segment.

D 4.—See fig. 13, Plate II.—Appearances very much as in D 3. Central grouplets are the commonest, but para-central and post-central are not uncommon. In other regions they are chiefly just loosely scattered cells.

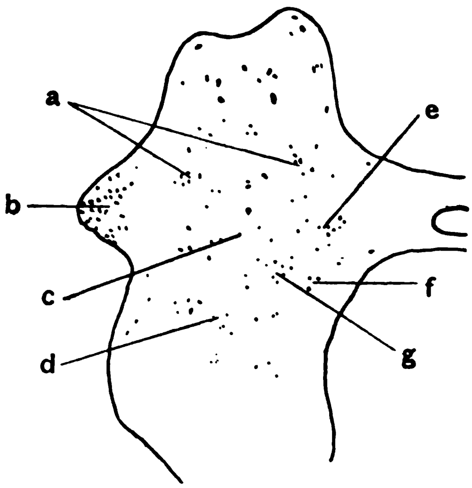


FIG. 5.

D 2—173 L (× 32).
a. Two small groups of anterior middle cells.
b. Apical group of intermedio-lateral tract.
c. Scattered middle cells in central area.
d. Posterior basal cells, indistinguishable from middle cells.
e. Para-central group of middle cells.
f. Three cells of Clarke's column.
g. Post-central group of middle cells.

D 5.—Anterior scattered cells, not abundant; they vary considerably in size, and at times it is difficult to distinguish some of them from motor cells, on the one hand, and from reticular group (intermedio-lateral tract) cells, on the other. In other regions, also, not abundant; may be disposed anywhere around the whorl of fibres surrounding Clarke's column, in front as a para-central group, to the outer side or even behind as a central or post-central group. The occurrence of any group is, as usual, short-lived. Most cells are distinctly polygonal; the anterior often have one long axis, while some of the central and post-central are almost rounded.

D 6.—Cells nowhere numerous. Usual erratic distribution, scattered or grouped; small groups in base of anterior horn or centrally are probably the most frequent. With the still further relative increase in size of

Clarke's column, the post-central cells are often crowded into the form of a band on the outer aspect of Clarke's column.

D 7.—See fig. 6.—Middle cells appear to be relatively, though not absolutely, abundant in this segment. In appearance, they are practically indistinguishable from apical group cells of the intermedio-lateral tract. The post-central cells, which may extend to quite behind Clarke's column, or into the formatio reticularis in the absence of the reticular group cells, are frequently of the smaller, more rounded type.

D 8.—Similar to D 7. Cells moderately abundant, and with all the usual characteristics—erratic, irregular distribution, scattered thickly or thinly, or aggregated into loose groups in any of the possible sites.

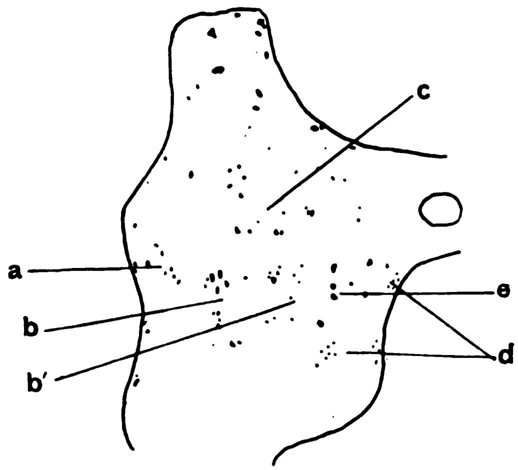


Fig. 6.

D 7—438 L (× 40).
a. Intermedio-lateral tract—a few cells only.
b. Large post-central middle cells.
b′. Small post-central middle cells.
c. Scattered middle cells in anterior and central areas.
d. Small cells behind and to inner side of Clarke's column.
e. Clarke's column.

D 9.—As before. The post-central cells, though sometimes of medium size and definite polygonal shape, are, more than in any other segment, mainly of the small, rounded type.

D 10.—In the lower part of the segment the cells are increasing somewhat both in number and size. The post-central cells (in contrast to D 9) tend to be large, and are often hard to distinguish from the reticular group, which is not very sharply circumscribed on its inner aspect.

D 11.—Anterior and central cells more numerous, either scattered or in small groups, very similar individually to cells of the apical group, although the anterior ones are sometimes more elongated. The post-central cells may also be similar, and difficult to separate from the innermost cells of the reticular group, but many are smaller and more rounded. They may form a band round Clarke's column, and may lie directly behind it. The middle cell distribution is everywhere casual and erratic.

D 12.—See fig. 7.—Very similar to D 11. With the great increase of Clarke's column, the post-central cells are crowded characteristically into band form.

L 1.—See fig. 8.—Increase in size and number along with the other cell-

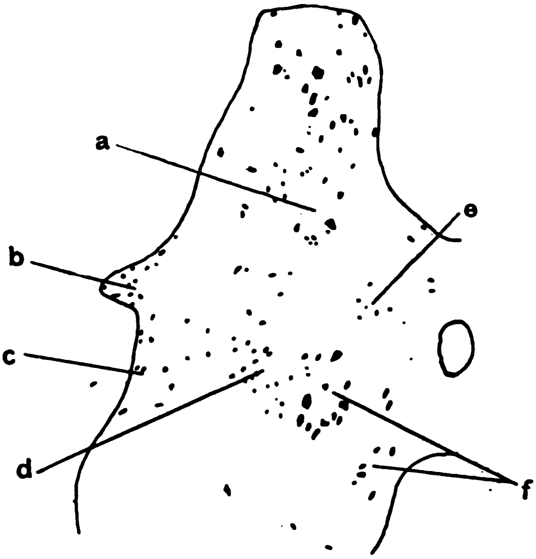


FIG. 7.

D 12—104 L (× 40).

a. Abundant anterior middle cells, large and small.
b. Apical group, intermedio-lateral tract.
c. A few cells representing the reticular group of the intermedio-lateral tract.
d. Large group of post-central middle cells.
e. Small para-central group.
f. Clarke's column.

There are only a few scattered middle cells in the central area.

systems. Anterior cells particularly numerous, either as groups in various parts of the base of the anterior horn, or very commonly scattered thickly over it. Central cells also common, of typical middle cell character as regards their occurrence. Both anterior and central cells are polygonal, very like the intermedio-lateral tract cells individually, and not always readily separable from that tract when lying near its boundaries. Post-centrally, they are also mostly of the same type, but both here and occasionally in the reticular formation there are some of the smaller, more rounded cells.

L 2.—A large increase in the number of middle cells, *pari passu* with that of the grey matter. Many of the individual cells are also larger. The cells may be in groups or scattered irregularly over the basal and inner portions of the anterior horn, in the area corresponding to the position of the now dwindling apical group, or extending right to the margin of the grey matter between the apical and reticular groups of the intermedio-lateral tract. They are also present in the region sometimes occupied by the tip of the triangular-shaped reticular group, and are not always readily distinguished from it. The post-central cells are, on the whole, smaller than the others, but most are polygonal in outline, although some are rounded; they usually form a band on the outer side of Clarke's column.

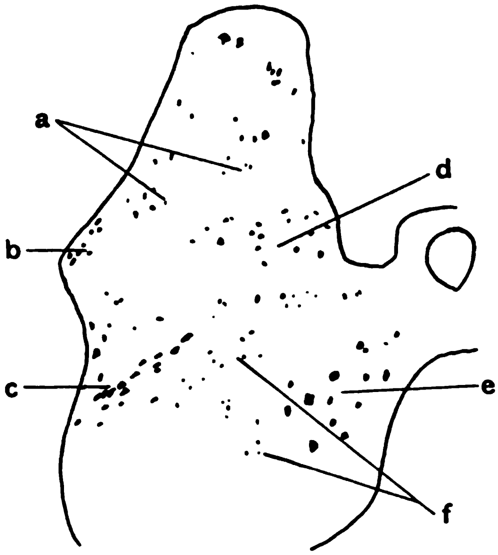


Fig. 8.

L 1—433 L (×40).
a. Anterior middle cells.
b. Apical group, intermedio-lateral tract.
c. Reticular group, intermedio-lateral tract, of large cells, extending well into central area.
d. Group of fairly large middle cells.
e. Clarke's column.
f. Small, rounded, post-central middle cells.

L 3.—See fig. 14, Plate II.—The middle cells now form a striking feature of the section. They are very numerous, and may be present anywhere in the posterior part of the anterior horn, central area, and base of the posterior horn. They have also invaded the areas occupied at higher level by the reticular group and by Clarke's column, but in the latter position they are never very numerous. They may be scattered or forming loose groups in any part of this wide field, but, as at other levels, these groups never persist through more than three or four consecutive sections. Groups may be present in the reticular formation, but they have none of the characters of the reticular group of the intermedio-lateral tract (which dies out in the upper fourth of this segment) and all those of the middle cell system. In size and shape there is great variation. Many are large, polygonal cells, larger than intermedio-lateral tract cells; the post-central ones are very often smaller and more rounded in outline, but quite a fair proportion are as large and distinctly multipolar as the central and anterior cells.

L 4.—Much as in L 3. In the lower part the middle cells are extending rather further forwards on the median aspect of the anterior horn. In the reticular formation groups are rare, but a few cells are generally to be seen.

L 5.—[Note, segment destroyed; description taken from examination of

FIG. 9.

S 1—357 R (× 32).
a. Motor cells.

The middle cells are scattered thickly over whole area not occupied by motor cells, the only other cells present. They are perhaps most abundant and of largest size in the central and reticular areas.

a segment obtained from another cord of practically identical age.]—Very similar to L 3 and L 4. Individual sections may show few or many cells, and scattered or aggregated in any part of the base of anterior horn, central area, or base of posterior horn. They are extending further forwards now in the outer part of the lateral enlargement. The same variations in size.

S 1.—See fig. 9.—The distribution is much as in the lower lumbar segments. Usually abundant, and anywhere within the large area shown in the accompanying figure. In the Clarke's column area they are fairly common, though not so abundant as in other parts. In the lower part of the segment, with the disappearance of the median motor group, the middle cells extend forwards almost to the antero-internal angle of the anterior horn, and may be very abundant in the anterior area.

S 2.—Area of distribution proportionately larger than in S 1, extending further forwards and relatively further outwards in the anterior horn. Although individual sections may be picked out with few cells, they are on the whole very numerous. Vary much in size. Scattered diffusely or forming loose groups in almost any part, but anterior, central, and reticular

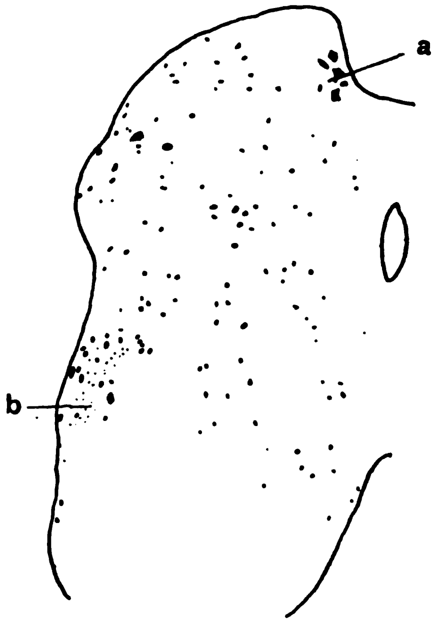


FIG. 10.

S 4—210 L (× 33).

a. Anterior median motor group.

b. Small round cells in posterior part of reticular formation, apparently the representatives of the sacral intermedio-lateral tract.

The middle cells occur as fairly large cells over the whole of the anterior horn except where group *a* is; also in the central and post-central areas and area of Clarke's column. In the reticular area they are mingled with group *b*.

are perhaps the most common. Have much the same variations in size and irregularity of occurrence in all areas.

S 3.—Area of distribution as in S 2. Cells diminishing in number in lower end, where they are only found thickly in the base of the anterior horn. Distinct groups are now rather rare; at places they may be thickly scattered rather than grouped.

S 4.—See fig. 10.—With the gradual dying out of the motor group, the middle cell area gradually comes to take in the whole anterior cornua, as well as its more usual limits. At the same time the cells become progressively fewer in number and also smaller in size. More scattered than forming definite groups; may be in any part, but most abundant in the anterior horn, and here also, on the whole, larger than in central and post-central regions. In the reticular area are cells indistinguishable in every way from other middle cells, but mingled irregularly with others which are quite different, sometimes the one set, sometimes the other predominating. [Those cells, probably representing the intermedio-lateral tract, are much smaller, pale, and almost circular in outline. They seem to consist almost entirely of nucleus. They appear first in the extreme lower end of S 3.] In character and irregularity of occurrence the middle cells are as in other segments.

S 5.—Cells gradually diminishing in number. At the lower end only occasional in their occurrence—one or two cells only. May be found in any part of the grey matter, but chiefly in two situations: firstly, the extreme anterior part of the anterior horn; secondly, near the reticular angle, along with cells of the intermedio-lateral tract, as in S 4. In the first-named area they are usually larger in size.

Coccygeal Segment.—At its upper end there are some straggling middle cells both anteriorly and near the reticular formation; they gradually become rarer and rarer. Some of the intermedio-lateral tract cells seem also to be present.

Summary.

The middle cells are present throughout the whole length of the spinal cord. They are situated in the middle region of the grey matter, between the free anterior and free posterior cornua; but they sometimes also extend into the regions usually occupied by the anterior cornual cells, by the intermedio-lateral tract or by Clarke's column. The small cells in the base of the anterior horn (scattered cells) cannot be sharply separated from the middle cells, nor can most of the small cells about the base of the free posterior horn.

Although some of the middle cells may be found in all this wide area of distribution at practically any level of the cord, there are certain arrangements of cells which may be looked upon as typical of each segment, or at least of each region of the cord. In the upper cervical region they are not on the whole very abundant (this differs from Waldeyer's account), but are best developed in the central and para-central fields. In the cervical enlargement they are much more numerous, particularly within a broad band extending from the formatio reticularis to about the anterior grey commissure. Throughout the dorsal region they are again com-

paratively few in number, and may be scattered irregularly; but small groups are often found, most commonly in the central area, about on a level with the central canal, and in the post-central area, between Clarke's column and the reticular group of the intermedio-lateral tract. In the lumbar segments the middle cells are abundant, particularly centrally and in the base of the anterior horn. Their field extends anteriorly, until in the lower sacral region they come to be found over the whole area of the anterior cornua in addition to their more usual situations.

At all levels, as studied in serial sections, the distribution of the middle cells is seemingly erratic and casual. No regular plan of arrangement can be made out, and there is most certainly no segmentation (as Argutinski described) like that so well seen in the intermedio-lateral tract. The cells are sometimes just dotted here and there singly; sometimes they are scattered fairly thickly and evenly over the whole or part of the regions they are to be found in; or, again, they may occur more thickly in one part, or be aggregated into a distinct little cell group or nest, but even then these cell groups are seldom so closely packed as are the cell groups of the intermedio-lateral tract. The duration of any one of these types of cell arrangement is inconstant; and although there are levels where cells seem for a bit to be almost persistent in one place, this much can be stated as a general rule, that no middle cell group lasts through more than a very few serial sections. If traced further, the group is found either to shift to some other area or to die out altogether. Occasionally there appears to be a variation in number of cells parallel with the oscillations of the intermedio-lateral tract, but closer study shows that this is by no means absolute, and is probably only a local accidental variation.

Without expressing any opinion as regards function, I am inclined to divide the cells I have described collectively as middle cells into three groups, basing this division merely on the distribution and arrangement of the cells and on the microscopic appearances of the cell bodies. These divisions are, however, not very sharply defined either as regards the character of the individual cells or in the cell distribution. Still they seem to warrant such a division being made, and to suggest at least that the cells of the three groups may be functionally different. The groups I would make are—

(1) The middle cells proper, or central cells, occurring chiefly in the central area of the grey matter. These cells are of medium size and very similar to those of the apical group of the intermedio-lateral tract, multipolar, polygonal or rounded polygonal in outline, with a comparatively large nucleus and a few chromatic granules round it in the cell substance.

Sometimes scattered, but more often present as a small clump of cells. With them may be included the cells in the para-central area, which, although sometimes forming a distinct and separate aggregation of cells, can mostly not be separated off from the central cells. Many of the cells occurring in the reticular area at levels where the reticular group of the intermedio-lateral tract is not present (especially in the lower cervical region) may probably also be included here.

(2) Anterior central cells in the base of the anterior horn—one of Waldeyer's "scattered" cell groups. As this name implies, these are often simply scattered over the area in question, but sometimes they are gathered into small cell-nests, but rarely very compact ones. Although there is no sharp boundary between them posteriorly and the central cells, they are in the main larger cells, and are further distinguished by their shape. They are not so often definitely multipolar and of approximately equal diameter in different directions, but more frequently *appear* to be bipolar, with long-drawn-out processes. The axis of elongation may be variously oriented. With these would fall to be included the middle cells in the external central area of the lateral enlargements, particularly in the lumbo-sacral cord. Possibly also some of the para-central cells should be classed with this group, and not with the previous one.

(3) Post-central cells. These lie in the area between Clarke's column and the formatio-reticularis, or in the corresponding region of the grey matter at level where Clarke's column is unrepresented. As with the anterior central cell, there is no sharp boundary between this series and the central middle cells. They are often continuous, or the cells may lie betwixt and between the two areas. But although many of the posterior cells may be as large, they are distinguished on the whole by being smaller in size than the central cells and less definitely polygonal, more rounded in outline. They are generally present either as a small clump or as a band of cells on the outer and posterior aspects of Clarke's column. Posteriorly, it is very difficult to separate this group from the posterior-basal and posterior-marginal cells of Waldeyer. With this group may be included cells present in the area of Clarke's column, particularly when that column is absent or only represented by occasional cells. Also some of the small cells found in the reticular formation, especially those lying between (vertically) the nuclei of the reticular group of the intermedio-lateral tract.

These subdivisions may require modification from subsequent investigations, particularly by tracing the course and termination of the cell processes. I have attempted to do this by means of Cajal's silver impregnation methods, but, so far, have not succeeded. All I can affirm is that the fibres arising from the middle cells run in a variety of directions to begin with; but as some could be traced far enough to be seen doubling more or less sharply upon themselves, this gives no real clue to their destination. In no case could I follow any one to a termination. In some cells there was observed an endo-cellular fibrillary network, similar to that in the large motor cells of the anterior horn. In this particular cord the following figures give approximate average diameter of the cells of the different groups:—

Anterior motor cells	·023–·035 mm.	
Anterior central middle cells	·018–·023 mm.	
Central middle cells	·015–·018 mm.	
Post-central middle cells	·011–·015 mm.	
Apical group } intermedio-lateral	·016 mm.	
Reticular group } tract	·018 mm.	In lower sacral ·010 mm.
Clarke's column	·020–·027 mm.	

In conclusion, I have to express my thanks to Dr Alexander Bruce for suggesting this subject for a thesis; to Dr Theodore Shennan, in whose laboratory the preparation of the sections was carried out; to Mr Henry Wade for the use of micro-photographic apparatus; and lastly to the Carnegie Trust, since this thesis is part of work done under the terms of a grant from the Trust for original research.

7th *July* 1908.—Since this paper was read before the Society, Jacobsohn has published a paper dealing with the cells of the spinal cord ("Über die Kerne des menschlichen Rückenmarks," *Anhang z. d. Abhandl. d. königl. Preuss. Akad. der Wissenschaft*, 1908).

He describes the middle cells as the *Tractus Cellularum*, scattered over the whole grey matter, and forming no very definite groups or nuclei, but three series may be distinguished —(*a*) an antero-median group, lying along inner edge of the anterior horn; (*b*) a postero-median group, the smallest of the three, and also composed of the smallest cells, in the position of, amongst, or surrounding Clarke's column; (*c*) a lateral intercornual series, the largest, near the formatio-reticularis, and in the outer part of the base of the posterior horn. All three series are indefinitely bounded towards the centre of the grey matter, and may meet there.

It must be noted, however, that this author, in addition to a thoracic and a sacral sympathetic nucleus (intermedio-lateral tract), describes a third or lumbo-sacral median sympathetic nucleus, extending from L 4 to the coccygeal segment, which would include practically all the cells which I have described as an extension of the middle cells into the lateral enlargement of the anterior horn as the motor groups die out.

REFERENCES.

(1) Waldeyer, "Das Gorilla-Rückenmark," *Abhand. d. könig. Akad. d. Wiss. zu Berlin*, 1888.

(2) Bruce, "Distribution of the Cells in the Intermedio-lateral Tract of the Spinal Cord," *Trans. Roy. Soc. Edin.*, vol. xlv., pt. i., No. 5, 1906.

(3) Onuf and Collins, *The Sympathetic Nervous System*, 1900.

(4) Argutinski, "On a regular segmentation in the grey matter of the spinal cord in the new-born and on the Middle Cells," *Arch. f. micros. Anat.*, xlviii., 1897, p. 496.

DESCRIPTIONS OF FIGURES.

[*Photographs untouched save outlining of the grey matter.*]

Plate I.

Fig. 11. C 3—200 L (× 33).

The middle cells are scattered in the base of the anterior horn and in the areas of the lateral horn and reticular formation, those in the anterior horn being of larger size. They form a fairly compact group in the central area, and there is also a group of smaller-sized cells in Clarke's column area.

Fig. 12. C 6—274 L (× 32).

The middle cells occur in the manner most characteristic of the cervical enlargement, viz. as a broad band of cells stretching from the reticular angle towards the central canal. There are also some scattered cells anterior to this, and a very few small cells in the area of Clarke's column and base of posterior horn.

Plate II.

Fig. 13. D 4—23 R (× 40).

Shows apical group of intermedio-lateral tract occupying most of the lateral horn, and with some outlying cells in the white matter. The reticular group cells are larger in size, tend to be elongated along a line running inwards and forwards from the reticular angle, and the group lies some way internal to the edge of the grey matter. There are a very few scattered cells in the base of the anterior horn; some centrally, near the tip of the reticular group, and a small group of smaller, more rounded, post-central middle cells.

Fig. 14. L 3—259 L (× 32).

The middle cells form a large central aggregation. There are small groups post-centrally and behind the anterior median motor group.

A few cells in the reticular area and base of posterior horn.

(*Issued separately September 2, 1908.*)

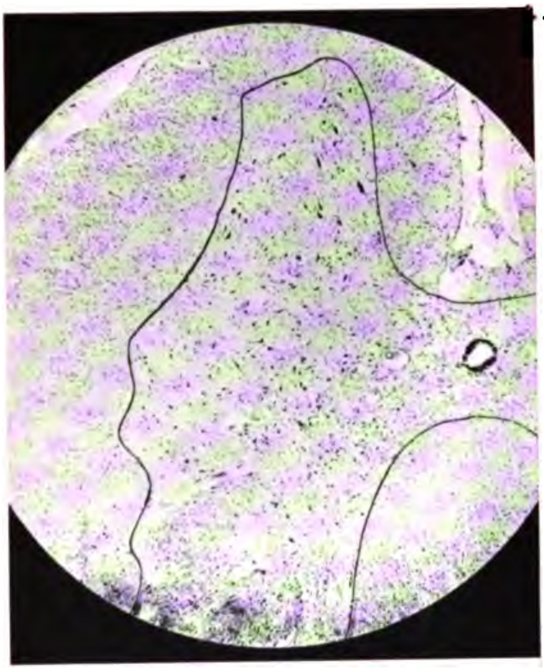


Fig. 11.

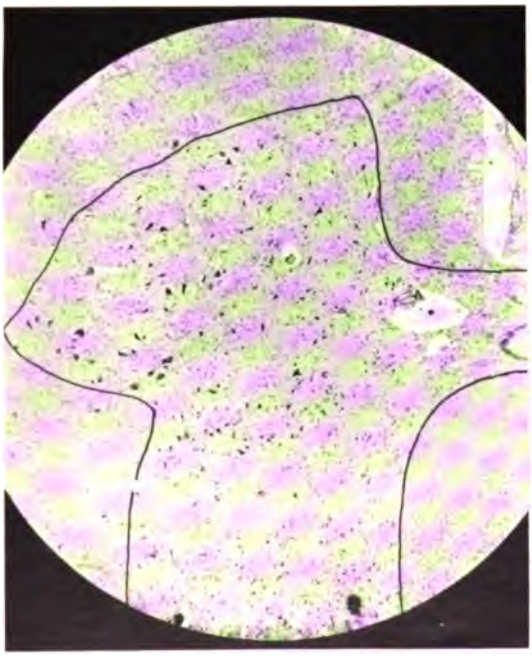


Fig. 12.

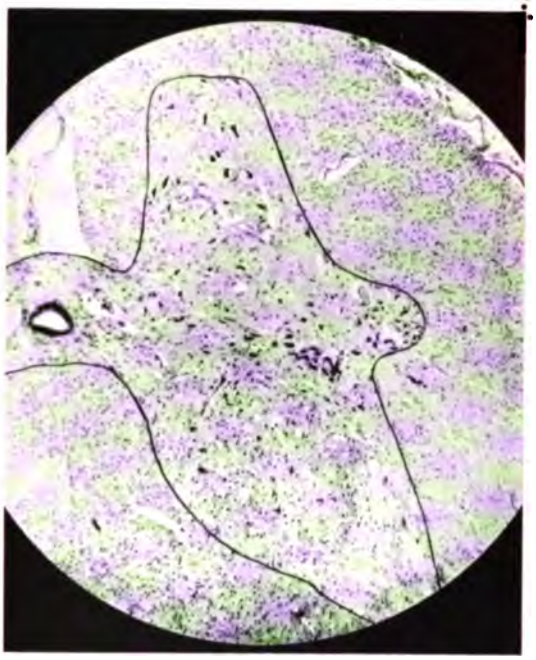


Fig. 13.

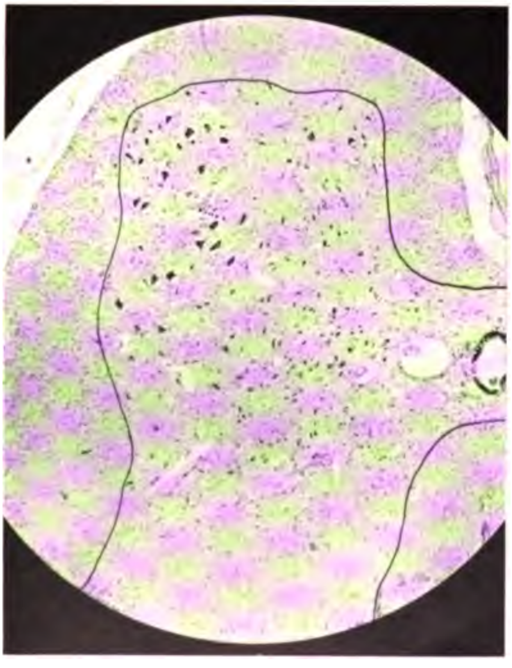


Fig. 14.

XXXVII.—On a Sensitive State induced in Magnetic Materials by Thermal Treatment.

By **James G. Gray**, B.Sc., Lecturer on Physics in the University of Glasgow, and **Alexander D. Ross**, M.A., B.Sc., Assistant to the Professor of Natural Philosophy in the University of Glasgow. *Communicated by* Professor A. Gray, F.R.S.

(MS. received June 23, 1908. Read July 20, 1908.)

Part II.

It has been shown by the authors in Part I. * of the present paper that most magnetic materials after having been heated to even a moderate temperature are in a peculiar magnetic condition. The magnetic quality of a test specimen is then superior to that of the specimen in its normal condition, to which it may be reduced by the simple process of demagnetising by reversals. For low fields the increase in susceptibility was found to amount to as much as 40 per cent. in the case of some varieties of steel, 24 per cent. in the case of cast iron, and 15 per cent. in the case of cobalt. Similar tests have now been carried out on specimens of nickel and of the Heusler alloy. It was found that in the case of the former metal the thermal treatment results in an increase of the susceptibility, as tested for a field strength of 8 c.g.s. units, of somewhat less than 2 per cent. Similar treatment applied to the Heusler alloy results in an increase of about 5 per cent.

Composition of the Steels.—Table I. exhibits the chemical composition of the various steels employed in the experiments described in the present and the previous paper.

Table I.—Composition of the Steels employed.

Description of material.	Percentage Composition.					
	Carbon.	Manganese.	Silicon.	Sulphur.	Phosphorus.	Tungsten.
Mild steel . . .	0·162	0·218	0·041	0·015	0·040	...
Spindle steel . .	0·896	0·286	0·094	0·020	0·043	...
Steel wire . . .	0·755	0·660	0·066	0·017	0·027	...
Magnet steel . .	0·853	0·462	0·072	Trace	0·038	2·745
Special hard steel .	1·321	0·339	0·143	0·023	2·745	...

* *Proc. Roy. Soc. Edin.*, vol. xxviii., part iii., p. 239 *et seq.*

Tensile Strengths of the Steels.—With a view to obtaining complete information relating to the steels, it was decided to determine their tensile strengths in the annealed condition. Specimens were accordingly prepared and pulled out on a Wicksteed ten-ton testing machine. The results are given in Table II.

TABLE II.—TENSILE STRENGTHS OF THE STEELS EMPLOYED.

Description of material.	Breaking stress in tons per square inch.	Ultimate percentage extension on a length of 4 inches.	Percentage contraction in sectional area at fracture.
Mild steel	26·3	24·0	67·0
Spindle steel	53·4	8·0	18·0
Steel wire	70·0	9·0	10·0
Magnet steel	57·6	6·4	11·0
Special hard steel	70·8	7·5	4·0

The above figures are the means of tests carried out on two specimens of each variety of steel. The results agreed closely with one another.

Effect of Vibration.—In the previous part of this paper it was mentioned that the "sensitive state" could be greatly reduced by jarring the specimens. This effect has now been much more fully investigated. The method adopted in the preliminary experiments for vibrating the specimens has been in the main adhered to, but important alterations have been introduced in the minor details. It was found in the case of those specimens which were dropped vertically several times on the stone slab that a considerable amount of permanent magnetism was built up if the same end was lowest in each fall. The amount so induced was sufficient to affect the subsequent tests, which were confined to small magnetising forces. Dropping the specimen on either end alternately would have avoided this gradual augmentation of permanent magnetism. The procedure is, however, objectionable, in that it results in the specimen being subjected to an alternating magnetic field which, though of small intensity, would diminish the "sensitive state" for much larger values of H, as has previously been shown by the authors. The following method was therefore devised and adhered to. A solenoid of insulated copper wire was wound over a brass tube about 130 cms. in length, and the arrangement mounted vertically, with its lower end resting on the stone slab. With a view to avoid jarring of the specimens subsequent to their fall, the tube was lined with felt. A current was passed through the solenoid of such strength as to exactly neutralise the vertical component of the earth's field. The specimen was then dropped in the tube through a distance of 1 metre on to the stone slab. As there was no magnetising force to influence the "sensitive" condition of the test rods, they were dropped on either end alternately, so as to affect the material as uniformly as possible throughout its entire mass.

A large number of similar specimens of thick steel wire were employed in the experiments. All were annealed at 900° C. Six were tested without further treatment. A further six were given one fall each in the manner described above, and like numbers were subjected to 3, 5, 10, 25, and 50 falls respectively. The rods were thereafter tested by the magnetometer, complete hysteresis cycles being taken. It was found that the enhancement of the intensity of magnetisation, the retentivity, and the coercive force, induced by the heating had been reduced by the jarring, the magnitude of the reduction increasing with the amount of vibration applied. Table III. shows the general nature of the results obtained for the six bars.

TABLE III.—EFFECT OF VIBRATION.

Number of Falls.	0	1	3	5	10	25	50
Percentage "Sensitive State."	35·4	26·1	20·9	18·0	13·8	12·5	10·2
	32·9	22·4	17·5	15·1	13·1	13·8	12·0
	41·8	24·9	20·1	16·5	14·6	10·8	11·3
	39·5	26·7	20·8	18·5	15·9	10·1	9·1
	42·1	22·5	16·7	16·0	16·0	16·3	9·7
	35·5	21·4	19·3	16·2	13·7	12·1	10·2
Mean	37·9	24·0	19·2	16·7	14·5	12·6	10·4

The figures given correspond to a magnetising field of 10 C.G.S. units. The "sensitive state" is expressed by the percentage by which the enhanced values of I exceed the normal magnetic intensities—that is, the intensities obtained in tests after the specimen has been subjected to an alternating magnetic field gradually diminishing from a large value to zero. The mean values show a steady reduction of the "sensitive state" with continued jarring of the specimens. The variation of the individual values from the means is not larger than one would expect considering the sensitiveness of the specimens to vibration, and the unavoidable shaking they must experience in being removed from the gas furnace in which they were annealed, and in being placed within the coil of the magnetometer.

Similar figures were obtained for other values of the magnetising field, and for the residual magnetism and coercive force. In the latter cases, as the quantities to be measured were small, the experimental error caused a somewhat greater variation of the results from the mean, but their general character was the same.

In Table IV. the "sensitive states" remaining, after varying amounts

TABLE IV.—EFFECT OF VIBRATION.

Number of Falls .	0	1	3	5	10	25	50
Sensitive State .	100	63	51	44	38	33	27

of vibration, are expressed as a percentage of the effect shown by a specimen tested directly after annealing.

A specimen annealed at 900° C. and subjected to an exceedingly large

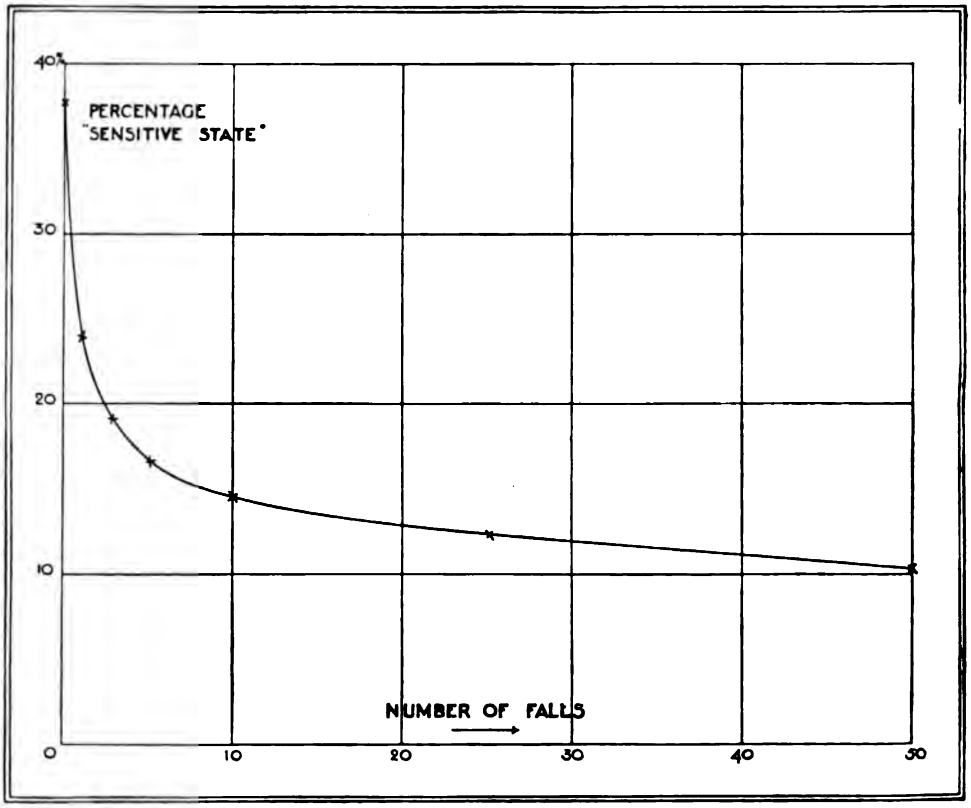


FIG. 1.

number of falls still showed distinct evidence of the "sensitive state." This result was to be expected from the previous observations, as will be at once evident from fig. 1. In this diagram the percentage residual "sensitive state" is plotted against the number of falls which the specimen had experienced. It will be observed that while the effect of the first fall is very marked, succeeding falls are of rapidly diminishing influence. Thus the curve lies considerably above the axis for any moderate number of falls of the specimen, and possibly does not meet the axis even when the number of falls is increased indefinitely.

Fatigue Effect.—In carrying out the foregoing investigation of the effect of vibration on material in the sensitive condition, it was at first intended to use only seven specimens. These were to be employed in the following manner. Having been annealed at 900° C., they were to be given 0, 1, 3, 5, 10, 25, and 50 falls respectively, and then tested by the magnetometer. Afterwards they were to be reannealed at the same temperature, and —taken in the same order—to be given 50, 0, 1, 3, 5, 10, and 25 falls respectively and tested. This procedure was to be repeated until all the specimens had been subjected to each of the various numbers of falls. It was found, however, that the percentage "sensitive states" for each treatment showed a general tendency to diminish; that is to say, the second and subsequent annealings had not restored the initial conditions. In fact, the specimens showed a fatigue effect. This gradual narrowing of the limits between magnetic intensities corresponding to the "sensitive" and "normal" conditions of the steels was made the subject of a separate examination. Seven specimens of the thick steel wire were annealed, tested, demagnetised, retested, and then reannealed, and the procedure repeated. This was continued until the specimens had undergone seventeen successive heatings. Table V. gives the mean values obtained.

TABLE V.—FATIGUE EFFECT.

Annealing.	"Sensitive State."	
	Actual Percentage.	Relative Percentage.
1st	38·0	100·
2nd	33·1	87·
3rd	29·3	77·
4th	25·8	68·
5th	22·8	60·
6th	20·5	54·
7th	18·2	48·
8th	16·7	44·
9th	15·2	40·
10th	14·1	37·
11th	12·5	33·
12th	11·4	30·
13th	10·3	27·
14th	9·5	25·
15th	8·7	23·
16th	8·0	21·
17th	7·2	19·

In the second column are entered the percentage "sensitive states" induced by the successive annealings, while in the third column these are expressed as a percentage of the effect produced by the first annealing. The figures in all cases correspond to a magnetising field of 10 C.G.S. units. Similar reductions took place in the enhancement of the residual magnetism and the coercive force. These results are not appreciably affected by any "ageing" of the material through the repeated exposures to the high temperatures employed in the annealing process. A few similar measurements have been made with the hard steel mentioned in the analyses at the commencement of this paper. These indicate that the fatigue effect is decidedly less marked in the case of this high carbon steel than in that of the steel wire.

Several specimens which had been reannealed and tested from twelve to sixteen times were laid aside for fifty-four days, and then once more annealed and tested. They exhibited no indication of recovery from the fatigued condition.

Repeated Annealing.—Fresh specimens of the steel were annealed at 900° C., some twice, some four times, and some six times, without any intermediate testing or application of any magnetic field whatever. They showed no difference in percentage "sensitive state" beyond what could be accounted for by experimental error and accidental jarring. In particular, they gave no sign of an increased "sensitive state" induced by repeated annealings, and showed that the fatigue effect referred to above only comes into play when a specimen is changed from the "sensitive" into the normal magnetic condition.

"Sensitive States" induced by low temperatures.—So far, the "sensitive state" had been induced in the specimens by heating them up to temperatures varying from 100° C. to 900° C., and leaving them to slowly cool. It appeared probable that an effect might be obtained by starting with a specimen at an extremely low temperature, allowing it to rise to room temperature, then cooling again to the initial condition, and there testing it. This was tried in the following manner.

A specimen of the stout steel wire A (fig. 2) was placed in a glass tube B C D, closed at one end B, and open at the other D. The portion C D was bent up at right angles to the main length B C. The magnetising solenoid E E was fitted with cork bungs F F F. The glass tube passed through two of these at such a height as to make the axes of the specimen and coil coincident. Liquid air was poured in at D, and the specimen thus cooled down to about −190° C. As the corks were bad conductors of heat and the air inside the solenoid was at rest, it was possible to keep the temperature of the specimen perfectly steady. An alternating current was sent through the solenoid, and gradually reduced from a large value to zero. The specimen in this demagnetised condition was allowed to rise gradually to about 10° C., and then once more cooled to – 190 ° C. with liquid air. When it had assumed this temperature throughout its mass, a cycle was taken, the specimen de-

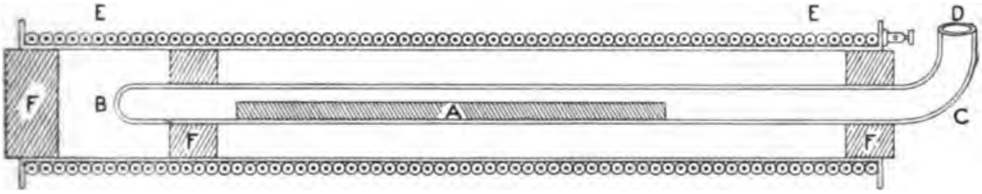


Fig. 2.

magnetised by reversals, and the magnetometric test repeated. For a field of 12 c.g.s. units the intensity of magnetisation in the first test exceeded that in the second by nearly 4 per cent. Three specimens in all were tested in this manner, and gave results in very close agreement. Similar tests were made with the hard steel and showed only a very feeble indication of a "sensitive state"—about 0·5 per cent. for a field of 10 c.g.s. units.

This production of a "sensitive state" by employment of low temperatures pointed to a very important fact. Since specimens which have been demagnetised do not improve in quality with time at ordinary room temperature, it follows that the "sensitive state" induced by raising the specimen from – 190° C. to 10° C. and lowering it again was acquired whilst the temperature was changing, and not during the period when the temperature was constant at 10°. Tests were therefore made to measure the enhancement of magnetic quality produced (1) by heating the specimen from – 190° C. to 10° C., and (2) by cooling from 10° C. to – 190° C., testing, it is to be noticed, after each variation of temperature, and wiping out the sensitive state before imposing the next variation. The results obtained are given in Table VI., and are contrasted with those obtained in the previous test.

Table VI.—Sensitive State with Liquid Air.

Thermal Treatment.	Percentage Sensitive State.	
	Steel wire H = 12.	Hard steel H = 10.
From – 190° to 10°, and then to – 190° C. .	3·5	0·5
From – 190° C. to 10° C.	15·	11·
From 10° C. to – 190° C.	24·	6·
From 10° to – 190°, and then to 10° C. .	3·	1·5

The values of the "sensitive states" are expressed in the usual manner, as the percentage by which the magnetic intensity for the specimen in the "sensitive" condition exceeds the normal value. The figures for the steel wire and the hard steel are taken for fields of 12 and 10 C.G.S. units respectively, those being the magnetic forces for which the effect is about a maximum. The results are the means of three tests, which are in excellent agreement. They show that the "sensitive state" produced by a certain rise or fall of temperature may greatly exceed that produced by a cycle consisting of the rise and the fall imposed in succession without intermediate testing and wiping out.

Specimens kept steady for an hour at $-190°$ C. showed no alteration.

"*Sensitive State*" *at high temperatures.*—The authors now proceeded to carry out a similar series of experiments at high temperatures. For this

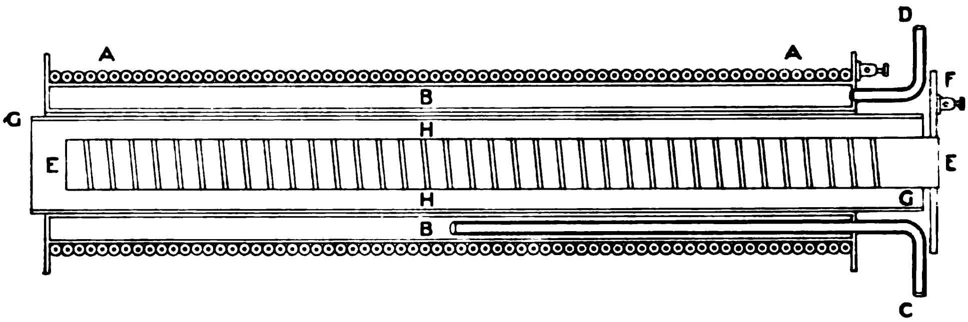


FIG. 3.

purpose a special arrangement for heating the specimens was adopted. In the preliminary experiments the bars had been annealed in a Fletcher gas furnace, and were thereafter removed to the solenoid of the magnetometer. Although they were handled with the greatest care, occasional slight jarring was inevitable. It was therefore decided to subject the specimens to the heating process while in their position in the magnetising coil. For this purpose an electric furnace was fitted as shown in section in fig. 3.

A A represents the magnetising solenoid which was wound in four layers, of which only one is shown in the sketch. B B is a water jacket, with an inlet tube at C and at outlet at D. E E is a porcelain tube, on which a length of fine platinum wire is wound non-inductively, and the terminals are mounted on a slate frame at F. This furnace is placed within a tube of Jena glass G G, the intervening space being packed with kaolin clay H H, which prevents the coils of the platinum wire from altering their position on the porcelain tube when expanded by the heating. A thick sheet of copper folded into cylindrical form was placed within the porcelain tube, and assisted in keeping the temperature uniform within the furnace. As in the earlier experiments, the temperatures were measured with a platinum, platinum-iridium pyrometer.

A steel specimen whose critical temperature was about 845° C. was heated from 15° to 800° C. and slowly cooled. When tested at 15° C. it gave for a field of 12 C.G.S. units a "sensitive state" of 29 per cent. The specimen was then demagnetised and heated up to 800° C., and tested when the temperature had become uniform. During the process of heating it was found to have acquired a "sensitive state" of 8 per cent. for the same magnetising field. After demagnetisation it was slowly cooled to 15° C. and tested thereat. A "sensitive state" of 12 per cent. had been brought on by the cooling. Similar tests were made, taking the specimen through the smaller range from 15° to 430° C. The heating and cooling then gave 10 per cent., the heating alone 13·5 per cent., and the cooling alone 25 per cent. These results are grouped in Table VII. for the sake of comparison.

TABLE VII.—SENSITIVE STATES BY HEATING AND COOLING.

Thermal Treatment.	Percentage Sensitive State.
From 15° to 800° C., and then to 15° C. .	29·
From 15° to 800° C.	8·
From 800° to 15° C.	12·
From 15° to 430° C., and then to 15° C. .	10·
From 15° to 430° C.	13·5
From 430° to 15° C.	25·

The "sensitive states" induced by equal augmentations or reductions of temperature are of widely different amounts, depending on the position of the temperature ranges on the temperature scale and on the composition of the steel. Even a small increment, say of only 25° C., in the neighbourhood of 180° C., produces in most steels a relatively large "sensitive state," in some cases as much as 10 per cent. The authors are engaged on an extensive investigation of these changes in magnetic quality in various steels for different ranges of temperature between 0° C. and the critical points. The results of the research they hope to lay before this Society in a subsequent paper.

Rods of steel of varying composition were tested at 100° C., 200° C., 400° C., and just below the critical point, to ascertain whether any "sensitive state" was induced by prolonged heating at constant temperature. As was to be expected from the results of the liquid air tests, no such improvement in magnetic quality was detected. The effect, therefore, is one which is associated only with temperature variation, and not with continued exposure to a constant temperature, either high or low.

A specimen of hard steel was demagnetised by reversals, heated at an almost uniform rate of 20° C. per minute to 840° C., cooled at a rate of about 12° C. per minute, and the amount of "sensitive state" induced measured. The experiment was repeated with the heating and cooling carried out at about a fifth of the previous rates. The variation in the temperature gradient was found to be without appreciable influence.

"Sensitive State" and various physical constants of the material.—It occurred to the authors that, starting with a specimen of steel in the "sensitive" condition, the change brought about in the permeability by the process of demagnetising by reversals might be accompanied by changes in the other physical constants of the material. Experiments were accordingly undertaken with a view to arriving at a conclusion on this point, the constants investigated being the electrical conductivity and the moduli of elasticity.

Specific Electrical Resistance.—For the resistance measurements, the following procedure was adopted. A helix having a length of about 150 cms. was set up with its length east and west, and connected up in series by way of a reversing key with a battery and a variable resistance. The specimen, in the form of a wire about 140 cms. long, was carefully annealed from 900° C. in the furnace, placed within a glass tube having a length somewhat greater than that of the helix, and the whole fitted in the coil. The wire was connected up in series with a rheostat and a large cell of constant e.m.f. Two potential leads from a sensitive galvanometer of high resistance were connected to two points on the wire near its ends, but within the terminal binding screws. On passing a current through the wire, a deflection of the galvanometer took place, the deflection being proportional to the resistance of that part of the wire lying between the potential terminals. By means of the rheostat, this deflection was adjusted to a suitable value and kept under observation. It was found to remain perfectly steady, showing that practically no heating took place in the wire. The specimen was now demagnetised by reversals with the variable resistance and commutator. No alteration took place in the deflection of the galvanometer, showing that the change from the "sensitive" to the "normal" condition was accompanied by no change in the specific resistance of the material. Further experiments carried out with different specimens only served to confirm this result.

Rigidity Modulus.—An examination of the elastic constants of the material in the two conditions was carried out in the following manner. A wire about 150 cms. in length, which had been annealed from 900° C., was attached to a rigid wall-bracket, and carried at its lower end a cylindrical lead vibrator. The wire was surrounded by a long helix, so that it might be put through the process of demagnetising by reversals without the end connections being interfered with. The solenoid, which was wound on a brass tube, served the additional purpose of protecting the suspended system from air currents. The temperature of the room was kept as nearly as possible constant at 19·5° C. by means of several small gas jets. Thermometers placed at the upper and lower ends of the wire did not vary beyond the limits 19·3° C. and 19·7° C. The wire was placed in position five hours previous to readings being taken, a precaution which ensured that it had assumed a uniform temperature throughout. Determinations of the period of vibration before and after the demagnetising process showed no indication of change. If any alteration does take place, it is certainly less than 1 part in 4000. The slight variations in temperature did not affect the readings, as the consequent change in the rigidity modulus would not exceed 0·01 per cent., according to the observations of Katzenelsohn and Pisati.

As it seemed possible that the molecular strains taking place at each swing of the vibrating system might have to a considerable degree removed the "sensitive state" early in the experiment, a static method of measuring the modulus was afterwards employed. A freshly annealed wire was placed as in the previous experiment, the vibrator having a surrounding zinc case of large radius. To the circumference of this case a finely graduated scale was attached, which could be observed by a reading telescope provided with cross wires, and placed at a distance of several feet from the wire. To the vibrator was attached a cylindrical brass rod of about 8 mms. diameter, the axis of which coincided with that of the test wire. Fine threads of equal weight were attached to this rod, given a few turns round it, passed over pulleys mounted on ball bearings, and attached at their free ends to 50-gramme weights. By this means a torsional couple was applied to the wire, and the amount of the resulting twist was measured by means of the telescope and scale. After demagnetisation by reversals, no variation could be detected in the angle of twist produced by the same couple. Several other varieties of steel wire were tried with like result, although in some cases the sensitive state wiped out amounted to as much as 35 per cent. for a field strength of 8 to 12 C.G.S. units.

Young's Modulus.—Measurements of Young's modulus by the bending of a steel strip laid on two horizontal knife-edges showed that this constant had the same value for the material when in the two states.

SUMMARY.

1. Nickel and the Heusler alloy give "sensitive states" of nearly 2 and about 5 per cent. respectively for a magnetising field of 8 C.G.S. units.

2. Steel wire specimens dropped vertically on a stone slab from a height of 1 metre showed a reduction of 37 per cent. in the "sensitive state" for a single fall, 49 per cent. for three falls, 62 per cent. for ten falls, and 73 per cent. for fifty falls.

3. After the "sensitive state" has been removed from a specimen by the process of demagnetising by reversals, it cannot be completely restored by reannealing. That is, the specimens exhibit a fatigue effect.

4. In the case of one variety of steel, the "sensitive state" had been reduced to less than one-half its original value after seven annealings, and to one-fifth after seventeen.

5. No recovery from the fatigue condition was observed in specimens which had been laid aside for fifty-four days.

6. Repeated annealings without intermediate magnetic testing showed neither an augmentation of the "sensitive state" nor a fatigue effect.

7. Specimens demagnetised at $-190°$ C., heated to room temperature, and cooled again to $-190°$ C., showed a small "sensitive state" at that temperature.

8. Larger effects were induced by heating from $-190°$ C. to 15° C., or by cooling from 15° C. to $-190°$ C.

9. A "sensitive state" could be induced by any variation of temperature, but not by exposure to a steady temperature, either high or low. The effect is associated solely with change of temperature.

10. The amount of "sensitive state" induced by equal temperature alterations varies with the position of the range on the temperature scale and with the material.

11. The change from the "sensitive" to the normal condition is unaccompanied by any appreciable change in the specific electrical resistance or elastic constants of the material.

(Issued separately September 1, 1908.)

XXXVIII.—The Preparation of a Glass to conduct Electricity.

By **Charles E. S. Phillips.**

(MS. received May 18, 1908. Read same date.)

THE electrical conductivity of most glasses at a temperature of about 100° C. is barely a measurable quantity.

For this reason, and on account of the fact that glass conforms to the general rule applicable to non-conductors in showing an increase of resistivity for a fall of temperature, this material has come to be regarded as practically incapable when cold of allowing the passage of an electric current.

With a view to the further study of electrical conduction in glass, and also because of certain experimental advantages which a conducting glass would possess, I have endeavoured to produce a transparent vitreous substance having that property.

The first attempt consisted in the preparation of a hard, conducting varnish, and for this purpose a small quantity of commercial sodium or potassium silicate was thinned in hot water, filtered, and evenly laid upon a strip of warm ordinary glass. It was then found that when quite dry and hard the varnish conducted electricity very readily.

Owing to the hydroscopic character of these substances, however, their brilliant surfaces soon became "tacky" in the moist atmosphere of a room. The interior of a glass electroscope may nevertheless be safely coated with a thin layer of the silicates, which has the additional advantage that, beyond screening the gold leaves from external electrical action, it may be easily washed off in warm water and replaced after experiments in which emanations from a radio-active body have settled upon the interior of the instrument.

The sodium silicate was then fused in order to see if it retained the property of electrical conduction, or whether that was merely due to the absorbed water.

Some small beads were therefore melted upon a platinum loop in the blow-pipe, and these, when welded together, formed a short transparent rod through which an electroscope was instantly discharged.

The potassium silicate gave a non-conducting glass, and was therefore not further used.

Two platinum wires were then sealed into one of the sodium silicate beads, and the whole arrangement coated with paraffin wax. Placed in series with a sensitive reflecting galvanometer and twelve accumulators, a well-marked deflection (50 mms.) was obtained, which varied greatly as the temperature was slightly altered. It was seen that the resistivity diminished with rise of temperature, so that the substance was behaving in one sense as a (so-called) non-conducting solid, while at the same time conducting electricity comparatively well. But sodium silicate when fused to a glass is soluble in boiling water, and in addition the material has a high fusing point, is very brittle and difficult to work. Attempts were therefore made to combine some substance with it which would increase its stability and improve it in other ways without interfering with its ability to conduct electricity.

In a paper read before the Royal Society of London in 1898, Professors Andrew Gray and Dobbie * pointed out that "a glass which approaches in composition to a definite chemical compound has a high resistance." It therefore appeared necessary, in the first place, to test the silicate in order to find whether its conductivity was due to the presence of impurities. The sample used was found to be free from iron, and a quantity of chemically pure silicate when fused conducted as well as the commercial product. It does not appear, therefore, that the foregoing generalisation can be upheld.

The addition of a small quantity of lead, bismuth, or lime seriously affected the resistivity of the glass, although these substances helped to prevent the attack of hot water. Of all the numerous materials tried borax gave the best result; it can be added to the extent of 25 per cent. without appreciably diminishing the conductivity, while at the same time rendering the glass far less soluble in hot water.

The addition of a small quantity of good lead glass was also found to still further improve matters, provided that the amount did not exceed one part in thirty-two of the silicate. Otherwise the conductivity was seriously affected. In order to ensure uniformity of this added substance the flint glass made by Messrs Powell of London was chosen.

The composition was therefore as follows:—

Sodium silicate	32 parts
Borax	8 „
Powell's flint glass	1 part

It is certainly surprising that a glass made of these things and with such a preponderance of alkali can remain sufficiently permanent, when it is

* *Proc. Roy. Soc.*, vol. lxiii., Feb. 1898.

remembered that borax is very hydroscopic and that the sodium silicate itself is readily attacked by water. But I will exhibit plates of this glass which have been lying exposed in my laboratory for three years without deterioration. Their durability is largely dependent, however, upon a process of maturing to be described later.

The addition of borax so far lowers the fusion point of the mixture that both in the preparation and the working special methods have had to be developed.

And again, on account of the instability of the sodium silicate compound at high temperatures, no ordinary crucible is of use.

Messrs Powell very kindly made several attempts to prepare plates of this glass, but complained that the mixture, when molten, ate its way through their crucibles. I am also indebted to the Morgan Crucible Company for their efforts to supply a clay sufficiently refractory to resist the action of the mixture, but in the end the only possible crucible was made for me of platinum by Messrs Johnson & Mathey. It stands 7·5 cms. high and has a capacity of 156 c.c. This crucible shows no deterioration after innumerable firings in contact with the mixture at a white heat.

It was desirable to produce a glass which, while conducting electricity to a sufficient extent, should be transparent, capable of being re-heated without devitrification so as to be workable in the blow-pipe, able to sufficiently resist the action of water or acid, of a suitable hardness without excessive brittleness, to enable it to withstand ordinary usage, and finally, so prepared that fine annealing was possible in a reasonably short time.

In the manufacture of ordinary glass the molten mass is kept white hot for twenty-four hours or so in order to drive off the myriads of air-bubbles. Owing to the low fusion point of the conducting glass it is found practicable to make use of a vacuum for this purpose. It is evident that in the preparation of special glasses prolonged heating may produce serious loss of some of the ingredients through evaporation. By means of a vacuum it is possible to reduce the time required to remove the air-bubbles to about two and a half hours. The evaporation of the borax after three hours generally results in the glass being left too brittle to work. Plates cast in brass moulds fly to pieces before they can be placed in the annealing muffle, and with such explosive force as to be exceedingly dangerous. A suitable method for the rapid production of special glasses on a small scale consists in placing the crucible with its white-hot contents into a vacuum vessel until all the air-bubbles disappear. In this way a block of conducting glass measuring $10 \times 3 \times 1$ cms. can be made perfectly clear and cast in a time not exceeding two and a half hours.

The apparatus for performing this operation is shown in fig. 1.

A Fletcher blast gas furnace is used for fusing the mixture, the wind pressure being maintained by a motor-driven blower running at 500 revolutions per minute. The platinum crucible is wedged into a clay one by means of small pieces of platinum foil, and a hole bored near the bottom of the outer crucible prevents distortion of the inner one, due to pressure differences when in the vacuum pot. There is considerable efflorescence when the mixture is first heated, so that to prevent foaming over the edge of the platinum crucible it was found best to make a supply of roughly fused glass in a clay pot and to break up the slab formed by pouring this out upon a brass plate, and to use it in the platinum crucible.

The heat is then increased till the whole mass fuses and becomes very fluid. At first the contents appear cloudy with air-bubbles, which soon begin to rise and burst. After twenty minutes at a white heat, and before the liquid glass has lost its cloudy appearance, the crucible is removed with tongs from the furnace and placed in the vacuum pot V. This is a gunmetal casting 7 inches high and 6 inches in diameter, jacketed as shown, and kept cool by a flow of water. The gunmetal lid L rests upon a flat ring of rubber R, and is provided with a window, made preferably of quartz, and cemented in position with any suitable heat-resisting material.

The platinum crucible retains its heat well on account of its clay covering, and the time taken to place it within the vacuum pot and close the lid need not exceed three seconds. The tap T_1 being closed, T_2 is opened carefully, thus placing the pot in communication with the reservoir S (previously pumped out by means of a mechanical air pump). The mercury gauge M shows the pressure in the pot V when T_2 is wide open. The capacity of S should be large enough to maintain the vacuum practically constant with the pump working, in spite of the gas given off by the molten glass. At first the bubbles rush out so rapidly that some practice is required to prevent the contents of the crucible rising over the lip. For fully two minutes they continue to stream off, and then the tap T_2 is closed and T_1 opened, so that clean air may enter through the cotton-wool in C.

The lid can now be raised and the crucible replaced in the furnace for a second heating.

After four times in the vacuum vessel the fluid glass is so clear that the bottom of the platinum crucible may be easily seen through it.

The process of stirring is then begun. A rod of fused silica or clay was rapidly attacked, and a platinum stirrer became necessary. In order to minimise the cost of this, advantage was taken of the recent improvements in the manufacture of fused silica tubing.

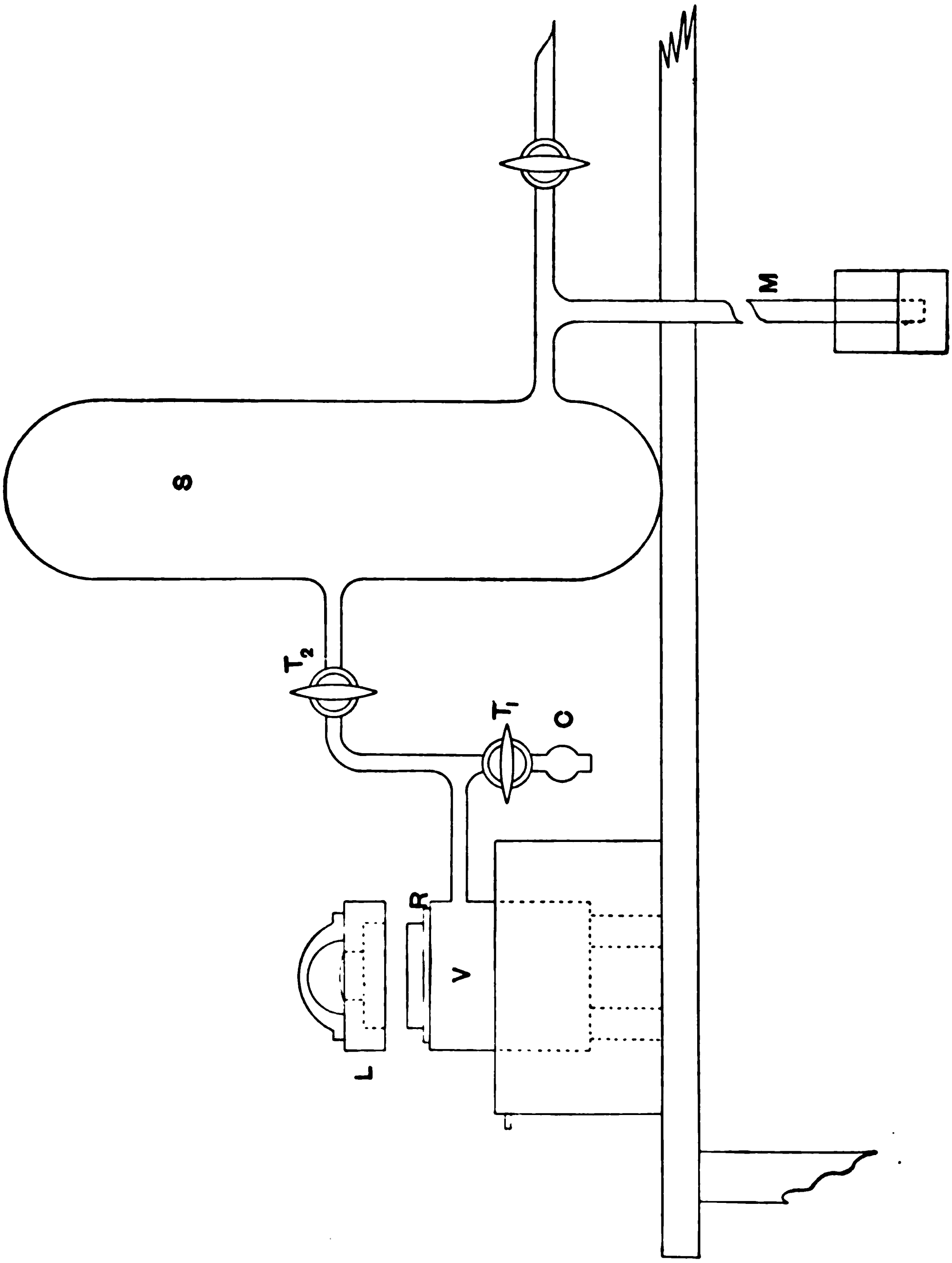


FIG. 1.

Messrs Johnson & Mathey made for me a slightly tapered silica tube two feet long, and having at its thick end a thin platinum sleeve welded on so as to extend eight inches up the tube. A cap of platinum closes the lower end, and the upper narrow one is left open to allow air to escape when the rod is lowered into the molten glass. By this means a light, handy stirrer is obtained at a moderate cost, and having even greater rigidity than a solid platinum bar of the same dimensions. An asbestos screen protects the hand from the heat of the furnace, and the mixture may be stirred for thirty minutes or more, continuously, without discomfort.

Since stirring introduces a number of fresh bubbles, the crucible has to go into the vacuum pot another four times to remove them, and care must then be taken not to let the temperature of the mass become too high. At a full white heat the evaporation of the borax is greatly accelerated *in vacuo*, and striæ are abundant in the finished glass in spite of the additional mixing which the vacuum undoubtedly also produces. After eight times the bubbles have all disappeared, so that the contents may be poured out after about two hours of full heat. The glass is then cast or pressed into sheets as required, and removed to the annealing muffle. It is necessary to have this at a dull red heat, and to ensure that its temperature shall fall very slowly. A prism of conducting glass made last April was annealed for only ten hours, and showed afterwards no trace of stress whatever.

A plate of glass so prepared indicated, however, by a milky appearance of its surface after exposure to moist air, that it was not sufficiently stable. It was also seen that the clay crucible used for roughly melting up the mixture before transferring it to the platinum pot had given a bluish-green colour to the plate. This was traced to impurities in the clay. A drop of HCl (1 in 5) rapidly attacked the surface, and left a white crust of crystals above a layer of fine cracks.

The problem of improving the glass as regards this attack by acid was partially solved in a new series by adding more lead and less borax. The Powell glass was left out and the following mixture—

32 parts of sodium silicate,
7 parts of borax,
·7 part of lead oxide (red lead),
·1 part of sodium antimoniate,

gave a material better in some respects than the original, but easily devitrifying on re-heating.

A glass of the composition—

32 parts of sodium silicate,
5 parts of borax,
·8 part of lead oxide,
·2 part of sodium antimoniate,

showed less action by acid, and besides conducting well it was colourless. This will be referred to later as glass No. 50. About twenty glasses were then made containing small quantities of cæsium, lithium, rubidium, zinc sulphide, strontium, magnesium carbonate, barium chloride, and calcium phosphate; but there seemed no possibility of being able to add enough of any of these substances to reduce the action of acids without also seriously diminishing the conductivity. It was incidentally noticed that as much as 20 per cent. of strontium could be added without appreciably affecting the conductivity. The addition of pure silica, and also various proportions of lead, gave no more hopeful results. A trace of sodium antimoniate effectively removed all colour from the glass. Plates containing a large proportion of alumina were very brilliant, but too brittle to be worked. In the course of the research it was seen that glass plates made of the original mixture resisted water and acid far better after they had stood exposed to the air for some months.

Many of the earliest pieces were so good in this respect that at first it was thought to be due to the alumina or silica absorbed from the clay of the pots first used. A new slab made of the original mixture and in a clay pot was nevertheless strongly attacked by water and acid, especially the latter, so that a change in the surface of the glass with age was looked for. It seemed that a freshly made plate attracted moisture from the atmosphere to a greater extent than was the case in an old plate, and a delicate means of detecting the presence of this film of water was devised for the purpose of watching the alteration of the surface with time. A rod of pure cadmium when dipped into water resting upon an earth-connected sheet of platinum foil becomes negatively electrified, and if it be connected to two quadrants of an electrometer while the other two are to earth, a wide deflection may be obtained. Now if the cadmium rest upon a piece of dry ordinary glass moistened only at the point of contact, the electrical charges cancel, and there is of course no deflection. The conducting property of the new glass, however, makes it possible to detect the least moisture upon its surface by this means.

The needle of the Dolezalek electrometer was charged to 200 volts and a quartz suspension chosen which gave 571 mms. deflection for one volt, so that the instrument was by no means in its most sensitive adjustment. The delicacy of the above method of detecting moisture is, however, so great, that a surface of conducting glass, even when warm, and after careful polishing with a cloth, may give a deflection of as much as 100 mms. in three seconds.

A freshly made plate, tested directly it had cooled sufficiently, gave no deflection at first.

After a few hours it was again examined, and a large deflection obtained. Old plates were found to give a much smaller action than new ones which had been exposed to air and dust for a few hours.

In addition to providing a means of numerically expressing the changes in the hydroscopic property of the surface, the electrical test served to eliminate many of the glasses which for one reason or another had seemed sufficiently well constituted. It was soon found, for instance, that glass No. 50, although resisting acid comparatively well, and being satisfactory in other ways, would not improve with age, and was always greedy for moisture. A plate of it left exposed over night would generally give a deflection of 150 mms. in two seconds the next morning.

The experiments of Kohlrausch, Mylius and Foerster strongly support the view that the surface of certain glasses may, as regards its resistance to water and acids, be improved by contact with those liquids.

It may be pointed out, for instance, that lead glass is chosen for holding wine because long contact with dilute acid is found to increase the power of the glass to resist further attack, and that water boiled in a flask dissolves out more alkali during the first than during subsequent hours. A plate of glass No. 50 was therefore boiled for five minutes to see whether the surface would be improved thereby.

It was placed afterwards in a warm muffle to dry, and unfortunately came out opaque with fine scratches. The previous electrical tests of the surface should have served as a warning. It remained, therefore, to revert to the original mixture and to add a trace of sodium antimoniate in order to remove colour. A plate newly made was broken in two. One half was boiled for five minutes, and both halves were then placed in the muffle for two hours. Their surfaces were unaffected by this treatment and the electrical test showed that there was no moisture whatever upon either piece. Both halves were then laid side by side, and exposed upon the bench to air and dust. After three hours the unboiled piece A gave a deflection of 20 mms. in 100 seconds, while the boiled portion B gave no action. The next morning A gave 105 mms. in 20 seconds (fig. 2), and from that time varied with the saturation of the atmosphere, while B produced no deflection whatever for two days. After the fourth day it gave a deflection of only 10 mms. in 100 seconds. It was very clearly shown, therefore, that the surface had been enormously improved.

The curves Nos. 1, 2, 3, 4, 5 in fig. 2 show the relative degree of moisture upon the surface of the unboiled plate from time to time, and No. 6 represents the greatest deflection produced by the boiled piece during the same total interval.

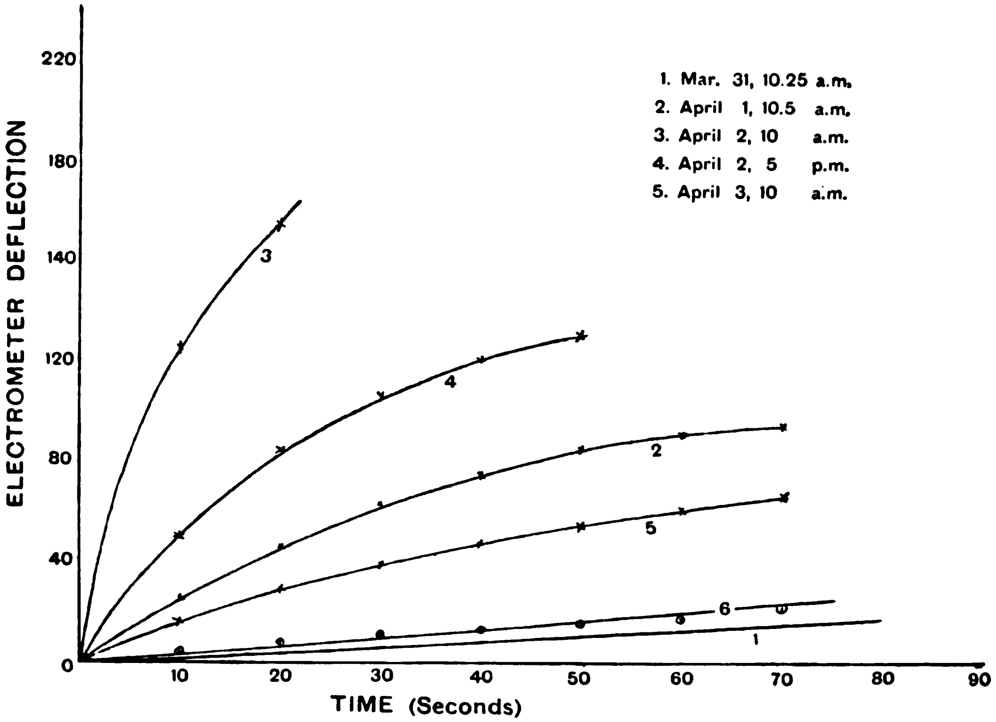


Fig. 2.

A drop of HCl (1 in 5) was then placed upon either half. The surface condition of the unboiled piece after one hour is shown in fig. 3; the surface of the boiled half was quite unaffected. It was clear, therefore, that the slow improvement with time detected by the electrical method was due to the removal of alkali from the surface, and that the subsequent increased power to resist water and acids was due to the preponderance of silica. It is not claimed, of course, that this glass so prepared and treated can compare with ordinary glass as regards withstanding the attack of moist air or acid, but experience shows that in the moderately dry atmosphere of a room the plates may be regarded as permanent.

ELECTRICAL RESISTANCE.

The passage of electricity through the glass is always accompanied by polarisation, showing that decomposition occurs.

To measure the resistivity a thin bulb about 2 cms. in diameter was blown upon the end of a tube of ordinary soda glass. This was then boiled in water, carefully cleaned before filling with pure dry mercury, and connected by means of a platinum-tipped wire to a Wheatstone's bridge. The bulb stood in a cup of clean dry mercury which was similarly connected to the bridge, and a key was provided whereby the mercury within and without

FIG. 3.

the bulb could be joined direct to a sensitive mirror galvanometer. In spite of every care to dry and clean all parts of the arrangement a deflection was, under these conditions, always obtained—the outer surface being invariably positive to the other. A trace of metallic sodium added to the mercury in the cup increased the deflection. It appears, therefore, that the composition of the inner and outer layers of a glass bulb may be different, and that being in a state of thrust the latter is tending always to squeeze out sodium.

The current * set up in the glass, due to its initial difference of richness in sodium, gradually carried more of this material to the inner surface.

All the usual methods of measurement where polarisation is concerned

* It should be noticed that this current appears to be due to the oxidation of the different amounts of sodium upon the two surfaces and is not to be regarded as a voltaic action between sodium and glass.—*July* 20, 1908.

having proved unsuitable on account of this variable P.D. produced in the glass itself, it was found best to momentarily close the battery key and look for a kick of the galvanometer needle at the moment the normal P.D. between the inner and outer faces of the bulb was zero. By reversing the battery current and waiting till the bulb when joined direct to the galvanometer gave no deflection, reliable readings of the resistance were obtained by balancing the bridge until a momentary current gave no sudden kick of the needle. It was easy, in this way, to distinguish between the deflection due to want of balance and that produced by the rapid rise of a back P.D. brought about by polarisation.

To ensure that this momentary contact be made with regularity, a heavy brass cylinder (amalgamated) was rolled down two inclined ebonite rails, portions of which were removed and filled in with strips of brass connected in series with the battery and bridge. The roller thus completed the circuit between the brass strips, and the time of contact could be adjusted by tilting the rails. The smoothness of the two curves shown in fig. 4 is good evidence that the readings may be relied upon. The sets of observations plotted there were obtained with two different bulbs, and show the variation of conductivity for a rise of temperature. It is seen that the resistivity rapidly falls at first, but that at about 90° C. a change occurs, and beyond that a further rise of temperature produces comparatively little alteration in the conductivity. From measurements of the bulb the specific resistivity was calculated and found to be 5×10^8 ohms at 20° C. Although this value is high, it is about a thousand times less than that of ordinary glass at 100° C.

Index of Refraction.

The value of μ for this glass was calculated by means of Fraunhofer's formula, from data obtained by measurements upon a 60° prism. At 17° C. $\mu = 1{\cdot}510$.

Coefficient of Expansion.

The coefficient of expansion is large enough to admit of a direct measurement. It is approximately equal to that of brass. The mean of two measurements upon a rod 292 mms. long was ·00016. The remarkable expansive property is due to the large proportion of sodium present in the glass.

Its relative contraction on cooling may be well shown by allowing two beads, one of conducting and the other of, say, ordinary soda glass, to slightly weld together in a flame before drawing them out into a thread. The curl produced is strongly curved, and forms a sensitive detecter of heat. The shearing force between the two glasses composing the thread may be so

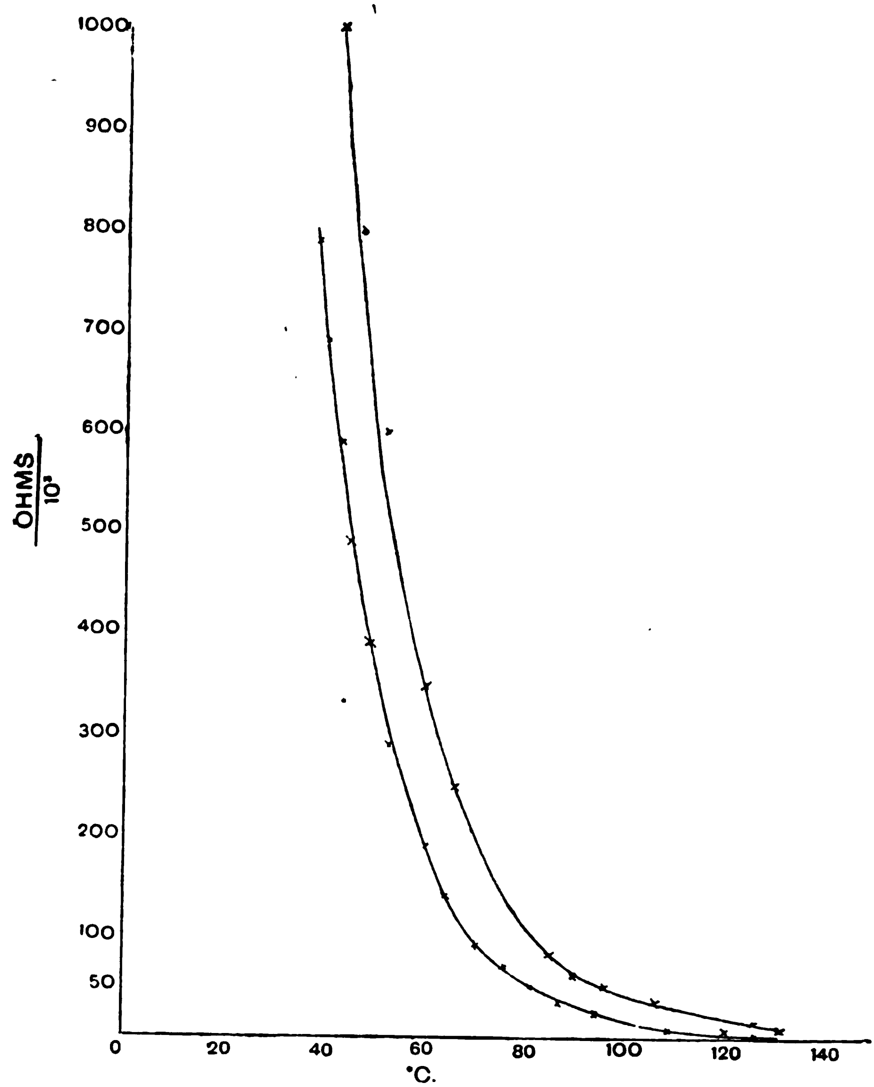


Fig. 4.

great that the halves sometimes come asunder, and thus produce fine fibres with a semicircular cross-section.

Fluorescence, Density, and the Action of Radium Rays.

The glass glows faintly, a pale blue colour, under the impact of cathode rays. It is worth noticing that in spite of the large amount of sodium present, the minute trace of lead in the glass prevents the particles vibrating in the manner usual with soda glasses. There was not the least suggestion of the green tint commonly met with. I am indebted to Mr J. H. Gardiner for a series of tests * as to transparency of the glass to ultra-violet light and to X-rays. He finds that a plate ·5 mm. thick absorbs all radiations shorter than $\lambda = 3000$, so that the glass must be regarded as opaque to ultra-violet light. With regard to X-rays it was found to be slightly more transparent than ordinary soda glass.

It would therefore appear that very little energy is required to produce this vivid fluorescence, since the photographic effect of the rays after traversing equal thicknesses of the two glasses is approximately similar in each case. The transparency to X-rays might have been predicted from the lightness of the glass, its density being 2·609. There is faint phosphorescence under the influence of the rays from radium, and after a three days' continuous exposure the glass is coloured a slate blue.

Softening Point.

The softening point was measured by supporting a rod at one end and horizontally in a muffle while the temperature was varied and read upon a Callender & Griffith's electric thermometer. A number of experiments with rods of diameters varying from one to three millimetres gave a mean temperature of 551° C. as the softening point.

Fine Annealing.

In order to prepare bars and prisms for some optical work now in progress, all the stress in the glass was removed by allowing it to cool very slowly. The apparatus used for this purpose is illustrated in fig. 5.

A brass tube suitably lagged with asbestos, plugged at one end and provided with a flue, was arranged so that it could be heated by means of a fish-tail bunsen burner.

The glass to be annealed was placed in this tube and the temperature slowly raised. On reaching a dull red heat the automatic regulating device was put into action and the flame slowly turned down. The supply of gas from the main, after passing through a regulator, enters the vessel A and bubbles up through water there on its way to the burner.

A constant head of water in the tank B feeds a dropper C, which supplies water to the vessel A through the funnel. Thus with the tube C packed closely with cotton-wool the water may be made to rise so slowly

* *Journal Röntgen Soc.*, vol. iv. No. 14, p. 13.

in A that a week or more can elapse before the temperature in the furnace falls 50° C. The action of the apparatus may also be reversed, for by allowing the water in A to slowly pass the tap D the gas supply will be gradually increased. An additional thermo-regulator may be used, if necessary, to prevent the initial temperature exceeding 500° C.

Working at the Blowpipe.

The usual operations of glass-blowing apply in this case. The low fusion point will at first seem to be a drawback, but with practice bulbs of a few inches in diameter may be easily produced. The glass remains plastic over a wide range of temperature. In order to weld a bulb upon ordinary glass, however, two or sometimes three intermediate glasses are required, so

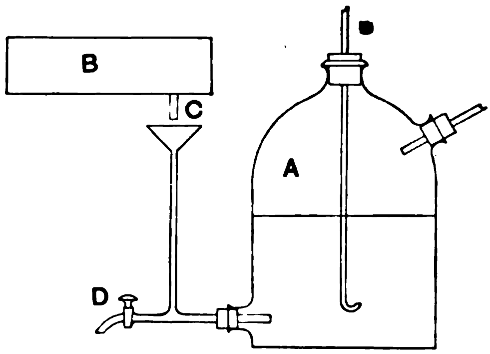


Fig. 5.

as to graduate the stress produced by unequal contraction. These may be made up as follows:—

A. Enameller's copper flux.
B. 2 parts of copper flux plus ·5 part of conducting glass.
C. 2 parts of copper flux plus 1 part of conducting glass.

In order to blow a conducting bulb upon the end of ordinary glass a piece of A is melted into the end of the tube, blown to a small sphere, and burst at a point exactly in the axis of the tube. The thin irregular glass having been removed with a file, a piece of glass B is made use of in the same manner. Having in this way built up three small bulbs upon the end of the tube, the conducting glass is melted on and blown to the required size. During the whole operation care must be taken to prevent the work cooling, and directly it is completed the whole must be annealed for at least half an hour.

Pressing Plates.

Great difficulty was met with at first in making thin plates of the glass. The cast slabs were far too thick for most purposes. It was ultimately found possible, however, to produce good results by pouring a quantity of the molten glass upon a hot brass plate and lowering a heavy flat disc of the same metal (also heated) slowly down upon the plastic mass.

Plates may, in this way, be readily obtained as thin as two mms. after polishing. The action of the press sets up considerable stress in the plates, especially at their edge, so that they require to be annealed for a few hours before grinding.

Conclusion.

Although there is no intention to do more in this paper than describe the preparations and some of the chief properties of the glass, it may, however, be pointed out that this substance is not without practical utility.

Plates of it are already in use for the windows of electrometers and electroscopes, and fibres have been found to be sufficiently conducting over their surface to justify their replacing the gold leaves in the latter instrument. A length of glass fibre fixed in position with a small hinge of gold leaf thus affords an extremely sensitive index whose position may be accurately read by means of a telescope; instruments of this kind are of especial use in the study of radio-activity.

The conducting property enables the contact potential difference between a metal and glass to be further studied—the glass behaving as a solid electrolyte. The effect of a current of electricity through the glass upon its optical properties or optical behaviour in a magnetic field can now, I hope, be ascertained. Incidentally, since the material is, relatively to ordinary glass, somewhat easily attacked by strong HCl, the process of disintegration may be fully examined.

The actual structure of the substance when cold which permits conduction of electricity, and the rapid change of that condition at about 90° C., are matters of interest. That the atomic aggregates existing in glass are capable of re-arrangement under the influence of heat is shown by the absorption of certain wave-lengths of light as the temperature changes. A trace of iron oxide, for instance, turns the conducting glass blue-green while plastic and the colour disappears when cold, whereas a small amount of copper oxide gives a clear molten glass which becomes a dark blue-green on cooling. Although the actual resistivity of the glass is not likely to be affected by minute traces of these materials, their presence may influence the shape of the temperature curve (fig. 4). Experiments upon this point will be shortly begun.

The available data relating to the molecular structure of glass are very limited, and it is hoped that the property of conduction will afford a fresh starting-point for further work in this direction.

I desire to express my indebtedness to the Managers of the Royal Institution for kindly allowing me to make the optical and softening point measurements at the Davy-Faraday Laboratory, and to acknowledge the help given me by my assistant, Mr Threadgold.

(*Issued separately October* 12, 1908.)

XXXIX.—Magnetic Quality in the most open Cubic Arrangement of Molecular Magnets. By Professor W. Peddie.

(Read in part July 6, 1908. MS. received July 21, 1908.)

1. In a former paper (*Proc. R.S.E.*, 1905) an investigation was given of the magnetic properties of the closest packed homogeneous cubic arrangement of molecular magnets, and it was found that the results were in good agreement with the observed properties of crystals of magnetite. It was also suggested that a parallel investigation of other cubic arrangements might lead to a discrimination of molecular arrangement, so far as the magnetic constituents are concerned, in actual crystals. To settle this point, if possible, the present investigation was undertaken.

2. The distances between pairs of centres of magnets in the most open arrangement are given by the values of r in the formula

$$\gamma^2 = \rho^2(\lambda^2 + \mu^2 + \nu^2),$$

where ρ is the least distance between centres, and λ, μ, ν are positive or negative whole numbers.

The direction cosines of the axes of the magnets being a, β, γ, the component of the internal force, exerted by surrounding magnets on the pole of one, taken parallel to the common direction of the axes, is

$$\frac{\mathrm{M}a^2}{4\rho^5}\left[\frac{245}{2}\left\{(a^4+\beta^4+\gamma^4)\left(5\Sigma.\mathrm{N}\lambda^4p^{-\frac{9}{2}} - \Sigma.\mathrm{N}p^{-\frac{5}{2}}\right) - \left(3\Sigma.\mathrm{N}\lambda^4p^{-\frac{9}{2}} - \Sigma.\mathrm{N}p^{-\frac{5}{2}}\right)\right\} - 44\Sigma.\mathrm{N}p^{-\frac{5}{2}}\right],$$

where M is the magnetic moment of a magnet, and a is the semi-axis, while $p = \lambda^2 + \mu^2 + \nu^2$, and N is the number of times of occurrence of given numerical values of λ *or* μ with a given p and ν, and the summation extends from $\nu = 0$ to $\nu = \infty$. The proof of the formula is exactly as in the previous paper.

The transverse component of the internal force is

$$-\frac{245}{}\frac{\mathrm{M}a^2}{8\rho^5}\left[a^6 + \beta^6 + \gamma^6 - (a^4 + \beta^4 + \gamma^4)^2\right]^{-\frac{1}{2}}\left(\Sigma.\mathrm{N}\lambda^4p^{-\frac{9}{2}} - \Sigma.\mathrm{N}p^{-\frac{5}{2}}\right),$$

with direction cosines $a\ (a^4 + p^4 + \gamma^4 - a^2)[a^6 + p^6 + \gamma^6 - (a^4 + \beta^4 + \gamma^4)^2]^{\frac{1}{2}}$, etc. The requisite data for the evaluation of the sums are given in the appended tables.

3. The transverse force is, in the neighbourhood of the quaternary

axes, directed towards these axes. In the neighbourhood of the ternary axes, it is directed from the axes. In the neighbourhood of the binary axes, the force acts from them towards the quaternary axes, and towards them from the ternary axes. Therefore the ternary, binary, and quaternary axes are directions of unstable, unstable-stable, and stable equilibrium respectively. This condition is the reverse of that which held in the case of the least open arrangement of magnets. Therefore, while the least open arrangement gave results agreeing well with the observed magnetic properties of magnetite, it appears that the most open homogeneous arrangement cannot be the one which exists in that substance.

The curves exhibiting the transverse component are similar to those given in the former paper, but positive loops become negative, and conversely.

4. The longitudinal component of the internal force has its maximum positive value when the magnetisation is along the quaternary axes, is very small parallel to the binary axes, and has its maximum negative value along the ternary axes. The values are in the ratios 240 : 5·5 : −127. Absolute values are got by multiplying these numbers by $Ma^2/4\rho^5$.

If we assume that the true value of this component along a binary axis is zero, the curve representing the longitudinal component in all directions (see fig. 5 of the former paper) in a plane parallel to a face of a cubic crystal would consist of four symmetrical lobes having their maxima coincident with the quaternary axes. If the component has really a small positive value along a binary axis, the lobes would not be completely divided along the lines of the binary axes. In either case the curve has its maxima where minima existed in the case of the closest packed arrangement, and conversely.

In a plane parallel to a diagonal plane of a cubic crystal, again on the assumption of zero longitudinal component along a binary axis, the curve would have six lobes,—two positive, having maxima on the quaternary axis, and four negative, with smaller maxima on the ternary axes. If the longitudinal component has really a small positive value along a binary axis, the two positive lobes become slightly wider, the four negative lobes become slightly narrower, and two other small positive lobes appear with maxima on the binary axis.

Again, in either case (fig. 6 of the former paper) there is no correspondence with the results which hold in the closest packed arrangement.

5. Since the internal force has a large positive longitudinal component along a quaternary axis, a small component along a binary axis, and a large negative component along a ternary axis, it follows that the

permeabilities along these axes are in descending order of magnitude. The reverse occurred in the closest packed arrangement.

6. At the surface, supposed to be a plane perpendicular to a quaternary axis, evaluation of the sums from $\nu=0$ to $\nu=10$ gives, for the longitudinal components of the internal force along the quaternary, binary, and ternary axes respectively, the proportionate values 306, -19, -127, the same units as in § 4 being used. The transverse components also are larger by about one-third than they are in the interior, and their directions are unaltered. Thus the quaternary axes are lines of more stable magnetisation at the surface than they are in the interior.

Evaluation of the terms $M/4\rho^3.\sum_0^{10}(3N\nu^2p^{-5/2}-Np^{-3/2})$, which give the longitudinal effect due to the so-called "surface" magnetisation, shows that it is equal to 2·1 I, where $I=M/\rho^3$ is the intensity of magnetisation. It must be remembered that this is merely the value of the force, due to these terms, at a definite set of points in the surface plane. It increases the stability of the surface magnets.

In the closest packed arrangement the corresponding force is 2·84 I. In the former paper the values of the terms $3\Sigma.\nu^2(2p)^{-5/2}$ and $\Sigma.N(2p)^{-3/2}$ were given as 19·75 and 14·54 respectively, the summation being taken from $\nu=-10$ to $\nu=+10$. When the summation is from $-\infty$ to $+\infty$ the sums should be identical, so their approximate equality was regarded as indicating that the limited summation was sufficiently accurate to give a correct general description of the facts. The correspondence is actually much closer, for, in Table I. of that paper, the value $N=2$ should be $N=1$, and consequently the numbers in the first rows of Table V. and of the second half of Table IV., under the headings $\nu=2$, $\nu=4$, . . . , $\nu=10$, require to be halved. Also the decimal point is misplaced in the second rows of these tables under the heading $\nu=0$. The numbers should read 0·125 and 0·5 respectively. These errors do not essentially affect any condition dealt with in the paper; but when they are corrected, the above sums are found to be 14·69 and 14·73 respectively. This gives ample verification of the sufficiency of the partial summation. In the most open cubic arrangement now dealt with the sums are 32·70 and 32·74 respectively.

7. From Weiss's experimental observations on magnetite, it results that the internal magnetic structure of the substance possesses, like its crystalline structure, cubic symmetry. It appears now that the internal magnetic arrangement cannot be that of most open cubic packing, though it may be that of least open homogeneous cubic packing. Exhaustion of the possibilities must discriminate the real arrangement. In this connec-

tion it has to be observed that, while in these investigations the magnetic unit of the cubic arrangement has been regarded as a single magnet of constant moment, the results of the investigations are not altered in their general nature if the unit be complex with a variable moment so long as the moment does not change sign. In the latter case effects analogous to the one discussed below would occur.

In the most open arrangement, alignment of the magnets parallel to a ternary axis is unstable so far as the internal forces are concerned. Therefore such alignment could only take place if the external applied field exceeded the internal field. If the internal field is greatly in excess of the external, the arrangement would be strictly almost unmagnetisable along the ternary axes, although it would apparently be magnetisable in these directions, because of their inclination to the quaternary axes, to one-third of the extent to which it is magnetisable along a quaternary axis.

If the lines of stable magnetisation were at right angles to a line of unstable magnetisation, the substance would seem to be almost unmagnetisable in the latter direction if the internal field were relatively strong enough. An investigation, similar to the above, of the properties of a hexagonal arrangement would almost certainly give an explanation of the magnetic properties observed by Streng and Weiss in pyrrhotine, which is almost unmagnetisable along its axis.

There can, I think, be little doubt that in the question of magnetic arrangement lies the explanation of the non-magnetic quality of certain alloys containing one or more normally magnetic constituents, and of the magnetic quality of others containing normally non-magnetic constituents.

[TABLE

TABLE OF VALUES OF $\lambda^2+\mu^2+\nu^2$, ETC.

λ.	μ.	N.	$\Sigma.N\lambda^4$.	$\nu=0$.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
0	0	1	0	[0]	1	4	9	16	25	36	49	64	81	100
0	1	4	2	1	2	5	10	17	26	37	50	65	82	
1	1	4	4	2	3	6	11	18	27	38	51	66	83	
0	2	4	32	4	5	8	13	20	29	40	53	68	85	
1	2	8	68	5	6	9	14	21	30	41	54	69	86	
2	2	4	64	8	9	12	17	24	33	44	57	72	89	
0	3	4	162	9	10	13	18	25	34	45	58	73	90	
1	3	8	328	10	11	14	19	26	35	46	59	74	91	
2	3	8	388	13	14	17	22	29	38	49	62	77	94	
0	4	4	512	16	17	20	25	32	41	52	65	80	97	
1	4	8	1028	17	18	21	26	33	42	53	66	81	98	
3	3	4	324	18	19	22	27	34	43	54	67	82	99	
2	4	8	1088	20	21	24	29	36	45	56	69	84		
3	4	8	1348	25	26	29	34	41	50	61	74	89		
0	5	4	1250	25	26	29	34	41	50	61	74	89		
1	5	8	2504	26	27	30	35	42	51	62	75	90		
2	5	8	2564	29	30	33	38	45	54	65	78	93		
4	4	4	1024	32	33	36	41	48	57	68	81	96		
3	5	8	2824	34	35	38	43	50	59	70	83	98		
0	6	4	2592	36	37	40	45	52	61	72	85	100		
1	6	8	5188	37	38	41	46	53	62	73	86			
2	6	8	5248	40	41	44	49	56	65	76	89			
4	5	8	3524	41	42	45	50	57	66	77	90			
3	6	8	5508	45	46	49	54	61	70	81	94			
0	7	4	4802	49	50	53	58	65	74	85	98			
1	7	8	9608	50	51	54	59	66	75	86	99			
5	5	4	2500	50	51	54	59	66	75	86	99			
4	6	8	6208	52	53	56	61	68	77	88				
2	7	8	9668	53	54	57	62	69	78	89				
3	7	8	9928	58	59	62	67	74	83	94				
5	6	8	7684	61	62	65	70	77	86	97				
0	8	4	8192	64	65	68	73	80	89	100				
1	8	8	16388	65	66	69	74	81	90					
4	7	8	10628	65	66	69	74	81	90					
2	8	8	16448	68	69	72	77	84	93					
6	6	4	5184	72	73	76	81	88	97					
3	8	8	16708	73	74	77	82	89	98					
5	7	8	12104	74	75	78	83	90	99					
4	8	8	17408	80	81	84	89	96						
0	9	4	13122	81	82	85	90	97						
1	9	8	26248	82	83	86	91	98						
2	9	8	26308	85	86	89	94							
6	7	8	14788	85	86	89	94							
5	8	8	18884	89	90	93	98							
3	9	8	26568	90	91	94	99							
4	9	8	27268	97	98									
7	7	4	9604	98	99									
6	8	8	21568	100										
0	10	4	40000	100										

N must be made zero in the special case $\nu=0$, $p=0$.

TABLE OF CONSTITUENTS OF $\Sigma.N\lambda^4 p^{-\frac{1}{2}}$.

ν=0.	ν=1.	ν=2.	ν=3.	ν=4.	ν=5.	ν=6.	ν=7.	ν=8.	ν=9.	ν=10.
2·00000	0·08839	0·00143	0·00063	0·000006	0·000001	0·000000	0·000000	0·000000	0·000000	0·000000
·17678	2851	126	82	9	1	0	0	0	0	
·06250	2290	276	311	45	8	2	1	0	0	
4866	2142	345	473	76	15	4	1	0	0	
552	3252	89	186	39	9	3	1	0	0	
823	512	157	364	83	21	6	2	1	0	
1037	675	228	577	141	37	11	4	1	1	
377	270	113	353	102	30	10	3	1	1	
195	149	72	262	86	28	10	4	1	1	
299	231	115	441	151	51	18	7	3	1	
73	57	29	117	42	14	5	2	1	0	
152	122	67	286	108	40	15	6	2		
69	58	35	173	74	31	12	5	2	0·000000	
64	54	33	160	69	28	12	5	2		
107	91	56	282	124	52	22	9	4		
67	58	38	199	93	41	18	8	4		
17	15	10	57	28	13	6	3	1		
36	32	22	126	64	30	14	7	3		
26	23	16	94	49	24	11	5	3		
46	40	29	171	90	45	21	10			
32	29	21	130	71	36	18	9	0·00003		
19	17	13	80	44	23	11	6			
20	18	14	88	51	27	14	7			
12	11	8	56	33	19	10	5			
22	20	15	103	62	35	19	10			
6	5	4	27	16	9	5	3			
12	11	8	57	35	20	11				
17	15	12	83	51	30	16	0·00012			
12	11	9	60	38	23	13				
7	7	5	38	25	15	9				
6	6	5	34	22	14	8				
11	1	9	63	42	26					
7	7	6	41	27	17	0·00033				
9	9	7	53	36	23					
2	2	2	13	9	6					
7	6	5	41	28	18					
5	4	4	28	19	13					
5	4	4	29	21						
3	4	3	21	15	0·00087					
6	6	5	40	29						
5	5	4	35							
3	3	2	20	0·00215						
3	3	3	21							
4	3	4	28							
3	4									
1	3	0·02171	0·00597							
2	1									
4										
	0·21983									
2·32979										

TABLE OF CONSTITUENTS OF $\Sigma.Np^{-\frac{5}{2}}$.

$\nu=0$.	$\nu=1$	$\nu=2$.	$\nu=3$.	$\nu=4$.	$\nu=5$.	$\nu=6$.	$\nu=7$.	$\nu=8$.	$\nu=9$.	$\nu=10$.
4·00000	1·00000	0·03125	0·00412	0·00098	0·00032	0·00013	0·00006	0·00003	0·00002	0·00001
·70711	70711	7155	1265	336	116	48	23	12	7	
·12500	25660	4536	997	291	106	45	22	11	6	
·14311	7155	2210	656	224	88	40	20	10	6	
2210	9072	3292	1091	396	162	74	37	20	12	
1646	1646	802	336	142	64	31	16	9	5	
2530	1265	656	291	128	59	29	16	9	5	
1313	1993	1091	508	232	110	56	30	17	10	
391	1091	671	352	177	90	48	26	15	9	
671	336	224	128	69	37	21	12	7	4	
291	582	396	232	128	70	39	23	14	8	
447	254	176	106	59	33	19	11	7	4	
256	396	284	177	103	59	34	20	12		
128	232	177	119	74	45	28	17	11	0·00078	
232	116	88	59	37	23	14	8	5		
117	211	162	110	70	43	26	16	10		
69	162	128	90	59	37	23	15	10		
119	64	51	37	25	16	10	7	4		
51	110	90	66	45	30	20	13	8		
96	48	40	29	21	14	9	6	4		
79	90	74	56	39	26	18	12			
74	74	62	48	34	23	16	11	0·00198		
59	70	59	45	33	23	15	10			
24	56	48	37	28	20	14	9			
45	23	20	16	12	8	6	4			
23	43	37	30	23	16	12	8			
41	22	19	15	11	8	6	4			
39	39	34	28	21	15	11				
31	37	33	26	20	15	11	0·00402			
28	30	26	22	17	13	9				
12	26	23	20	15	12	9				
23	12	10	9	7	5	4				
23	23	20	17	14	10					
21	23	20	17	14	10	0·00758				
9	20	18	15	12	10					
18	9	8	7	6	4					
17	17	15	13	11	8					
14	16	15	13	10	8					
7	14	12	11	9						
13	7	6	5	4	0·01468					
12	13	12	10	8						
12	12	11	9							
11	12	11	9	0·02988						
10	10	10	8							
9	10	9	8							
4	8									
8	4	0·25966	0·07555							
4										
	2·21824									
5·08819										

Table of Constituents of $\Sigma N p^{-\frac{1}{2}}$.

$\nu=0.$	$\nu=1.$	$\nu=2.$	$\nu=3.$	$\nu=4.$	$\nu=5.$	$\nu=6.$	$\nu=7.$	$\nu=8.$	$\nu=9.$	$\nu=10.$
4·00000	1·00000	0·12500	0·03704	0·01562	0·00800	0·00463	0·00292	0·00195	0·00137	0·00200
1·41421	1·41421	·35777	·12649	5707	3017	1777	1131	763	539	
0·50000	·76980	·27217	·10964	5238	2851	1707	1098	746	529	
0·71554	·35777	·17678	8534	4472	2561	1581	1037	713	510	
·17678	·54433	·29630	15272	8313	4869	3047	2016	1396	1003	
·14815	·14815	·09623	5707	3402	2110	1370	929	655	476	
·25298	·12649	·08534	5238	3200	2017	1325	906	641	468	
·17068	·21928	·15272	9660	6034	3864	2564	1765	1257	921	
·06250	·15272	·11413	7753	5123	3415	2263	1639	1184	878	
·11413	·05707	4472	3200	2210	1524	1067	763	559	419	
5238	·10476	8313	6034	4220	2939	2073	1492	1097	825	
8944	4830	3877	2851	2017	1418	1008	729	539	406	
6400	8313	6804	5123	3704	2650	1909	1396	1039		
3200	6034	5122	4035	3047	2263	1679	1257	953	0·07111	
6034	3017	2562	2018	1524	1131	839	628	476		
5123	5702	4869	3864	2939	2197	1639	1232	937		
2210	4869	4220	3415	2650	2016	1527	1161	892		
4035	2110	1852	1524	1203	929	713	549	425		
1852	3864	3415	2837	2263	1765	1366	1058	825		
3555	1777	1581	1325	1067	839	655	510	400		
3162	3415	3047	2564	2073	1639	1283	1003			
3047	3047	2741	2332	1909	1527	1208	953	0·15692		
2650	2939	2650	2263	1859	1492	1184	937			
1166	2564	2332	2016	1679	1366	1098	878			
2263	1131	1037	906	763	628	510	412			
1131	2197	2016	1765	1492	1232	1003	812			
2133	1098	1009	880	746	616	501	406			
2073	2073	1909	1679	1427	1184	970				
1811	2016	1859	1639	1396	1161	953	0·26989			
1679	1765	1639	1459	1257	1058	878				
781	1639	1526	1366	1184	1003	837				
1527	763	713	641	559	476	400				
1527	1492	1396	1257	1097	937					
1427	1492	1396	1257	1097	937	0·41397				
655	1396	1309	1184	1039	892					
1283	641	604	549	485	419					
1257	1257	1184	1077	953	825					
1118	1232	1161	1058	937	812					
549	1097	1039	953	851						
1077	539	510	468	419	0·63379					
1021	1058	1003	922	825						
1021	1003	953	878							
953	1003	953	878	0·93943						
937	937	892	825							
837	921	878	812							
412	825									
800	406	2·50487	1·47335							
400										
	5·69920									
8·40785										

TABLE OF CONSTITUENTS OF $\Sigma.\nu^2 Np^{-\frac{4}{3}}$.

ν.	$\nu^2.\Sigma Np^{-\frac{4}{3}}$.	ν.	$\nu^2.\Sigma Np^{-\frac{4}{3}}$.
1	2·21314	6	0·27288
2	1·03864	7	0·19698
3	0·67995	8	0·12416
4	0·50000	9	0·06156
5	0·36700	10	0·00100

(*Issued separately October* 12, 1908.)

XL.—The Variation of Young's Modulus under an Electric Current. By Henry Walker, M.A., B.Sc. *Communicated by* Professor J. G. MacGregor, F.R.S.

Part II.

(MS. received July 13, 1908. Read same date.)

In my first paper on this subject * the behaviour of soft iron, steel, copper, and platinum was examined. In this paper the experiments have been carried a stage further, viz. the reaching of the cyclically steady state with a small load, the effect when the load is increased, and the heating by the ordinary method.

As the behaviour of the wire when heated by the current was somewhat complicated, I considered it necessary, for purposes of comparison, to heat the wire in some other manner. To effect this, a double-walled tube of tinned iron was made, through the inner tube of which the wire was passed, and the ends plugged with cotton-wool. The wire was horizontal, and measurements were made with the microscopes as in the other experiments. As the turning values of the modulus were all found at comparatively low temperatures, it was not sufficient to determine the modulus at the temperature of the air and at 100° C., and then assume that the decrease was uniform between these two temperatures. To get suitable intermediate temperatures, steam, and the vapours of boiling sulphuric ether, ethyl-alcohol, and amyl-alcohol were passed through the annular space between the two tubes. In this way temperatures of about 35° C., 78° C., 100° C., and 130° C. were obtained. Marks were made on the wires as near to the ends of the tube as it was possible to place them, so as to be in the field of the microscope. Except for these small lengths, which did not exceed 1 centimetre at each end, the part of the wire measured was at the temperature of the inner tube. The marks were observed in the microscopes, and no reading was taken until a short time after the wire had ceased expanding, the temperature being taken by a platinum thermometer, as recommended by Gray, Blyth, and Dunlop.† The results of these experiments gave in every case graphs which were straight lines.

* *Proc. R.S.E.*, vol. xxvii., p. 343, read June 1907.

† *Proc. R.S.*, vol. lxvii., p. 180.

Again, in my first paper, the wire was put through only one cycle; but this investigation has been carried out in more detail by putting each wire through a sufficient number of cycles to bring it to the cyclically steady state. The iron and steel had to be carried through several cycles before this was accomplished, the platinum required two cycles, while the copper reached it at the end of the first.

The investigation has also been extended in another direction, on account of the totally different results obtained by Miss Noyes * in a second paper, where another series of experiments on various wires is described. In her first paper the results, generally speaking, were similar to those I described in my first paper, viz. an increase of the modulus to a maximum, and then a decrease. In her second paper, however, the graphs are straight lines when the wire was heated by a current through it, as well as when it was heated by a helix and by a non-inductive current. Now, the only difference in the conditions was that in the second case the load was much greater than in the other. It became necessary, then, to examine this, and experiments were performed on all the wires with much greater loads than in the previous experiments. My results quite confirm those of Miss Noyes, and show that, under a load approaching the elastic limit, the decrease in Young's modulus is uniform.

In those experiments in which the temperature was determined by measuring the resistance of the wire, the method is perhaps open to criticism, and may seem to stand in need of justification. The temperature coefficient of resistance had been determined in the usual way in an oil-bath when the wire was unstretched, whereas in the experiments the resistance was determined under tension. Therefore the assumption is that no appreciable difference is produced in the resistance of the wire by the load. The experiment with which I am most familiar is that described by Kelvin,† where he discusses the electrical resistance of a wire under tension. No exact quantitative results are given, but the effect is small, and, as the curves are wide apart and cut at a large angle, the assumption seems a legitimate one to make.

The results are correct to a unit in the fourth significant figure, that is, the deviation of any individual reading of a set from the mean does not exceed a unit on either the one side or the other.

When the wire was bare there was radiation, and consequently a temperature gradient in the wire. To see if any change was produced when there

* *Phys. Rev.*, vol. iii., p. 452.

† *Math. and Phys. Papers*, vol. ii., p. 298. The fourth paper of the series on "The Electro-Dynamic Qualities of Metals."

was no temperature gradient, the wire was covered with asbestos and put through a cycle of heating and cooling, but there was no difference in the results.

Soft Iron.

A wire from a different coil, but of the same gauge as in the first experiment, was taken. The modulus was determined at the temperature of the room, and found to be $18 \cdot 14 \times 10^{11}$, which was lower than the value in the first wire. A weak current was then passed through it, which raised the temperature slightly, and this was accompanied by a decrease in the modulus. On strengthening the current the modulus rose in value, and continued to rise as the current was increased until the temperature was a little over 50° C. After the maximum had been reached there was a gradual diminution, the rate of which was fairly uniform up to a temperature of about 110° C. Beyond this the rate of fall diminished. On decreasing the current there was at first a further slight fall in the modulus, but as the temperature continued to fall a minimum was reached. and the increase that ensued went on to a temperature a little over 60° C. This maximum with the diminishing current has a lower value, and is reached at a higher temperature, than that with the increasing current. Beyond this point, as the current becomes less, the modulus falls, and finally reaches a value lower than what it had when the current was started. After cooling to the temperature of the room, the modulus had a higher value than before any current was passed through the wire. This was the behaviour of the wire during the first cycle, and these results are quite the same as those described in my former paper.

As the wire had not returned to the same state as when the current was started, the cycle was repeated and readings taken as before. There were slight changes, the value being always higher than at the corresponding temperature in the first cycle. The value at a temperature only slightly above that of the room was still lower than what it was with a commencing current at the same temperature, and it was not till the fifth cycle had been completed that the steady state was reached.

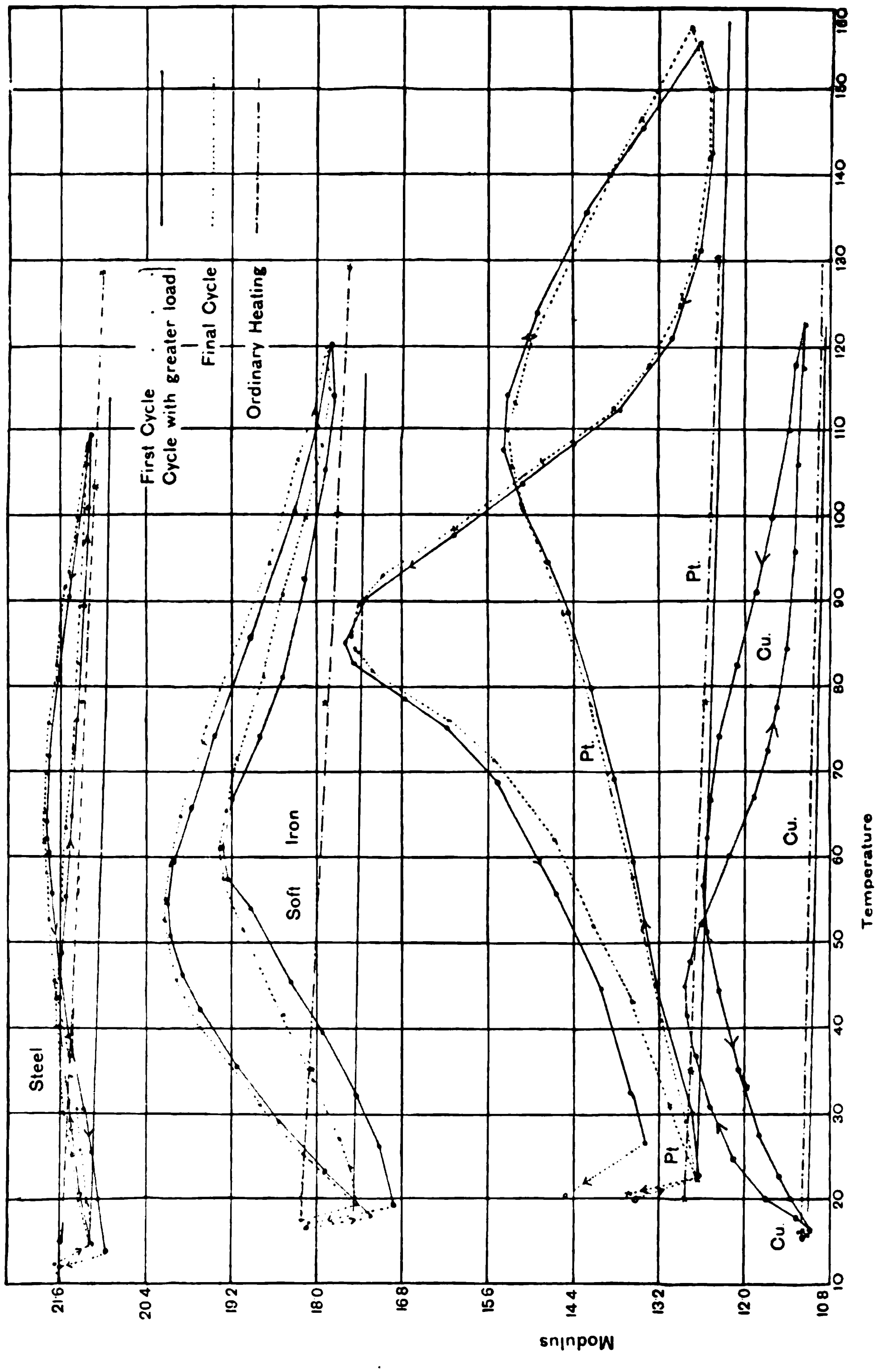
After the steady state was reached, the value was higher than at the same temperature in any of the preceding cycles. It is to be noted that the change is greatest with a diminishing current when the temperature is below 50° C. The same effect, viz. an increase in the modulus, was found when no current was being carried by the wire. The value was higher after the cyclically steady state had been reached than under the same conditions in any of the preceding cycles.

As I have already stated, in consequence of the results obtained by Miss Noyes, it was deemed necessary to examine the behaviour of the wire when it was subjected to a greater load. In the first experiment the load was small, viz. ·8 kilo, which was equal to nearly 14·62 kilos per sq. mm. The load was then increased to 2 kilos, that is, 36·54 kilos per sq. mm. In this case my results were quite in agreement with those of Miss Noyes, and these results are shown in the graph.

The wire was next heated in the ordinary way, and readings taken at various temperatures. These results are shown on the graph, and it will be seen that by this method of heating the modulus undergoes a uniform decrease. Usually, when Young's modulus has been determined at different temperatures, readings were taken at the temperature of the room and then at about 100° C., no intermediate temperatures being used. This was the case in the experiments described in Shakespear's * paper on Young's modulus, in which the extension of the wire is measured by the method of interference. The same temperatures were employed by Gray, Blyth, and Dunlop in their paper already referred to. The only investigations with which I am familiar in which intermediate temperatures are employed are those of Miss Noyes in her two papers. In her experiments the wire was heated both by a magnetising coil and by a non-inductive coil, and the graph of the results was always a straight line. My results are the same as hers, the only difference being in the value of the coefficient. That, however, is a minor point, for it varies widely in different specimens. In three determinations of the temperature coefficient for soft iron wires in Gray, Blyth, and Dunlop's paper there was a difference of 33 per cent. between the extreme values. A still greater difference was found by them in copper, as the coefficient for electro hard-drawn copper was fully three times larger than that for commercial copper. It seems, then, to be beyond doubt that, when a wire is heated in the ordinary way, Young's modulus undergoes a uniform decrease.

When the wire is heated by the current and carries the greater load, the modulus is lower than when it is heated in the ordinary way, but the coefficient is smaller, so that the two lines converge, and if produced they would intersect at about 215° C. Now, the two graphs may not be absolutely straight lines, so that a slight alteration in their rates of fall may produce such an effect as to make them after meeting coincide with one another. Further, the graph for electric heating with a load of ·8 kilo slopes down to these two lines in such a way as would seem to make them

* *Phil. Mag.*, 1899, p. 539.


First Cycle
Cycle with greater load
Final Cycle
Ordinary Heating
Steel
Soft Iron
Pt.
Cu.
Temperature
Modulus
10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160
216 204 192 180 168 156 144 132 120 108

meet; it, too, therefore, may coincide with the other graphs, in which case all three would ultimately coincide, and there would be only one value for the modulus, no matter what the method of heating be, or the load within the limit of elasticity. It would be of interest to find this out, but the current at my disposal was not powerful enough to heat the wire to the necessary temperature.

Soft Iron Wire.

Length = 97·92 cms.
Area of cross-section = ·0005474 sq. cms.
Elongation weight = 500 grams.
Total load on wire = 800 „
Load per sq. mm. = 14·62 kilos.

Table I.—First Cycle.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	16°·5 C.	·04839 cm.	6	$18·14 \times 10^{11}$
2	18·8	·05090	8	17·25
3	23·4	·04905	8	17·90
4	29·1	·04736	8	18·54
5	35·3	·04592	8	19·12
6	42·0	·04465	9	19·66
7	45·9	·04412	8	19·90
8	50·6	·04377	8	20·06
9	54·2	·04368	8	20·10
10	59·5	·04387	9	20·01
11	65·7	·04438	7	19·78
12	74·4	·04519	6	19·43
13	85·6	·04638	8	18·93
14	100·4	·04795	8	18·31
15	110·3	·04880	8	17·99
16	120·3	·04932	8	17·80
17	114·1	·04938	8	17·78
18	105·2	·04909	7	17·89
19	92·5	·04838	8	18·15
20	80·9	·04754	8	18·47
21	74·0	·04670	9	18·80
22	66·8	·04576	8	19·19
23	61·2	·04544	6	19·32
24	57·5	·04566	7	19·23
25	53·0	·04635	8	18·94
26	45·2	·04780	8	18·37
27	39·3	·04899	8	17·92
28	32·0	·05034	5	17·44
29	25·9	·05125	8	17·13
30	19·5	·05186	8	16·93
31	16·7	·04828	8	18·18

TABLE II.—FINAL CYCLE.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	19°·2 C.	·05043 cm.	8	17·41 × 10^{11}
2	25·3	·04819	8	18·22
3	30·9	·04670	8	18·80
4	40·1	·04468	8	19·65
5	45·4	·04396	8	19·97
6	49·3	·04373	8	20·08
7	52·5	·04359	8	20·14
8	55·0	·04361	10	20·13
9	58·6	·04371	8	20·09
10	64·7	·04403	8	19·94
11	73·4	·04479	8	19·60
12	83·1	·04487	9	19·12
13	94·2	·04700	8	18·68
14	104·5	·04801	8	18·29
15	112·0	·04863	8	18·04
16	119·7	·04910	8	17·88
17	114·9	·04918	6	17·86
18	108·3	·04891	8	17·95
19	99·8	·04832	8	18·17
20	90·6	·04751	8	18·48
21	81·1	·04674	8	18·78
22	71·5	·04594	9	19·11
23	65·2	·04554	8	19·28
24	61·0	·04537	8	19·35
25	57·4	·04547	8	19·31
26	50·9	·04614	6	19·03
27	41·5	·04751	9	18·48
28	33·8	·04870	8	18·03
29	27·1	·04962	8	17·69
30	23·7	·05011	8	17·52
31	19·2	·05043	8	17·41
32	17·6	·04815	8	18·23

Total load = 2·0 kilos.
= 36·54 kilos per sq. mm.

TABLE III.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	18°·8 C.	·05020 cm.	8	$17·49 \times 10^{11}$
2	26·3	·05029	8	17·46
3	35·5	·05037	8	17·43
4	42·9	·05034	8	17·44
5	50·8	·05052	7	17·38
6	57·4	·05043	8	17·41
7	65·7	·05052	8	17·38
8	76·2	·05058	6	17·36
9	89·6	·05082	6	17·28
10	100·3	·05070	8	17·32
11	116·5	·05091	8	17·25
12	108·5	·05094	8	17·24
13	97·7	·05076	8	17·30
14	84·1	·05064	10	17·34
15	71·4	·05064	8	17·34
16	60·3	·05043	8	17·41
17	51·9	·05049	9	17·39
18	43·2	·05037	8	17·43
19	36·8	·05026	8	17·47
20	29·5	·05034	8	17·44
21	24·1	·05031	8	17·45
22	19·0	·05023	8	17·48

ORDINARY HEATING.

Length = 94·60 cms.
Load = 1·2 kilos.

TABLE IV.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	17°·5 C.	·04644 cm.	6	$18·22 \times 10^{11}$
2	35·0	·04680	5	18·08
3	78·0	·04734	6	17·87
4	100·0	·04777	7	17·71
5	129·0	·04829	6	17·56

Steel Wire.

With this wire the method of procedure was the same as with the soft iron. The modulus was first determined at the temperature of the room before any current had been passed through the wire, and the value was found to be $21{\cdot}60 \times 10^{11}$. On passing a weak current the modulus was lower than before any current was passed, but on increasing the current slightly the modulus rose in value. With a further increase in the current it continued also to rise until a maximum was attained at about 45° C., and then fell regularly as the temperature rose.

When the current was diminished the modulus increased, and that, too, more rapidly than it had fallen with the increasing current. The maximum was reached at about 62° C., and this was the highest value throughout all the experiments on this wire. As the temperature continued to fall the modulus diminished, and came finally to a lower value than it had when the current was started. On allowing the wire to cool to the temperature of the room, the value was a little higher than it was initially.

Since the modulus towards the end of the cycle did not have the same values as it had at these temperatures at the beginning, the cycle of operations was repeated, and the values were found to be a little higher than in the first. This increase obtained all through, and there was also a slight increase in the value after the wire had cooled to the temperature of the room, compared with that at the corresponding stage of the first cycle. In all, four cycles had to be completed before the cyclically steady state was reached. When allowed to cool to the temperature of the room, the modulus had a higher value than in any of the previous determinations without a current. The effect, then, is to produce a permanent increase.

In accordance with the plan on which the experiments were carried out, the wire was next loaded with 2·2 kilos, that is, about 46·6 kilos per sq. mm., and heated by passing a current through it, with the results as shown in the graph. With this load on the wire the decrease is uniform, for the graph is a straight line.

The same wire was then heated in the ordinary way in the double-walled tube, the graph being again a straight line.

We see that the results for steel are very much the same as those for soft iron, the most important respect in which they differ being that with the decreasing current the soft iron has a lower modulus than with the increasing, whereas in steel it is at first higher with the decreasing current. Again, as with the soft iron, the temperature coefficient for ordinary heating is greater than that for heating by the current when the load is 2·2 kilos, and if these lines were produced they would intersect at about 240° C. In this case, also, the graph for the electric heating with a load of ·8 kilo approaches the other two graphs, and it is again possible that the three may ultimately all coincide.

STEEL WIRE.

Length = 98·45 cms.
Area of cross-section = ·0004714 sq. cm.
Elongation weight = 500 grams.
Total load on wire = 800 „
Load per sq. mm. = 16·96 kilos.

TABLE I.—FIRST CYCLE.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	12·0° C.	·04745 cm.	8	$21·60 \times 10^{11}$
2	14·1	·04839	8	21·18
3	20·3	·04803	8	21·34
4	28·4	·04776	8	21·46
5	33·5	·04760	8	21·53
6	40·0	·04747	8	21·59
7	43·2	·04745	10	21·60
8	48·8	·04749	8	21·58
9	55·4	·04760	8	21·53
10	64·7	·04778	6	21·45
11	76·1	·04798	8	21·36
12	89·5	·04818	8	21·27
13	100·9	·04830	7	21·22
14	109·2	·04837	8	21·19
15	106·0	·04830	7	21·22
16	97·3	·04800	8	21·35
17	90·4	·04773	6	21·47
18	80·6	·04740	8	21·62
19	71·7	·04716	10	21·75
20	64·0	·04701	8	21·80
21	62·1	·04699	9	21·81
22	60·2	·04702	8	21·79
23	55·4	·04710	8	21·76
24	45·7	·04742	8	21·61
25	38·9	·04760	8	21·48
26	30·5	·04818	8	21·27
27	24·8	·04848	6	21·15
28	20·0	·04867	7	21·06
29	14·0	·04880	8	20·98
30	12·5	·04740	8	21·62

TABLE II.—FINAL CYCLE.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	12°·6 C.	·04729 cm.	8	$21·67 \times 10^{11}$
2	14·3	·04835	8	21·20
3	19·8	·04800	8	21·35
4	25·1	·04778	8	21·45
5	31·5	·04758	8	21·54
6	35·0	·04749	8	21·58
7	37·7	·04745	8	21·60
8	40·2	·04743	8	21·61
9	43·8	·04743	8	21·61
10	47·3	·04745	7	21·60
11	54·9	·04756	8	21·55
12	63·0	·04769	6	21·49
13	72·5	·04784	8	21·42
14	82·7	·04803	9	21·34
15	89·4	·04814	8	21·29
16	96·0	·04821	8	21·26
17	100·9	·04827	6	21·23
18	106·0	·04832	8	21·21
19	109·1	·04835	7	21·20
20	105·3	·04825	9	21·24
21	99·6	·04803	8	21·33
22	91·5	·04771	6	21·48
23	82·2	·04738	7	21·63
24	75·7	·04714	8	21·74
25	69·8	·04702	8	21·79
26	65·4	·04699	9	21·81
27	62·0	·04697	10	21·82
28	58·1	·04699	9	21·81
29	52·5	·04710	8	21·76
30	45·3	·04736	8	21·64
31	36·9	·04767	6	21·50
32	30·2	·04794	7	21·38
33	25·6	·04809	8	21·31
34	19·7	·04825	8	21·24
35	14·3	·04835	8	21·20
36	12·6	·04729	9	21·67

Total load = 2·2 kilos.
= 46·6 kilos per sq. mm.

TABLE III.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	13°·8 C.	·04830 cm.	5	$21·22 \times 10^{11}$
2	22·5	·04834	6	21·20
3	30·9	·04844	6	21·16
4	41·2	·04848	6	21·14
5	52·3	·04856	6	21·10
6	62·7	·04862	7	21·07
7	74·1	·04868	6	21·04
8	83·6	·04876	6	21·00
9	95·9	·04882	6	20·97
10	104·0	·04890	5	20·93
11	112·7	·04894	7	20·91
12	101·2	·04886	7	20·95
13	92·1	·04882	5	20·37
14	80·5	·04874	6	21·01
15	69·8	·04866	6	21·05
16	60·4	·04860	6	21·08
17	49·3	·04856	6	21·10
18	41·1	·04850	7	21·13
19	29·9	·04842	7	21·17
20	23·2	·04836	5	21·19
21	13·8	·04830	6	21·22

ORDINARY HEATING.

Load = 1·3 kilos.
Length = 94·90 cms.

TABLE IV.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	15°·0 C.	·04571 cm.	6	$21·61 \times 10^{11}$
2	35·0	·04591	7	21·52
3	78·0	·04638	6	21·30
4	100·0	·04660	6	21·20
5	129·0	·04703	6	21·06

Copper.

A piece of wire of the same gauge was used as in the first experiment. First, a measurement at the temperature of the room was made, the value being $11{\cdot}21 \times 10^{11}$. On strengthening the current there was a rapid increase in the modulus, the maximum being at about 45° C. There was then a diminution, somewhat rapid at first, but the rate of which gradually fell away, until at about 95° C. it had become fairly uniform. When the current was diminished there was an increase in the modulus, which continued until a maximum was attained at about 57° C. This maximum was lower than that with the increasing current, and the value fell gradually until it was the same as when the current started. In this case, then, the cyclically steady state was reached in the course of the first cycle. On cooling to the temperature of the room, the modulus had a value somewhat higher than its original value before any current was passed through the wire, so that there was a permanent increase produced by the current.

In this experiment the wire was loaded with ·8 kilo, that is, 12·8 kilos per sq. mm. The wire was then loaded with 1·6 kilos, that is, 25·6 kilos per sq. mm., and the graph was a straight line, the values being as shown in the diagram.

The same wire was next heated in the ordinary way, as in the other two cases, and here again the graph was a straight line.

The results for copper are, generally speaking, similar to those for iron and steel, the differences being in degree rather than in kind. For example, the fall in the modulus from the value it had before any current was passed to its value when there was a weak current, also the rise in the modulus at the end of the cycle after the wire had cooled to the temperature of the room, were both smaller than the corresponding changes with iron and steel. Again, the difference between the values obtained by the ordinary heating and the electric heating with the wire under greater tension is less than in the two preceding cases.

When the two straight lines are produced they intersect at about 180° C.; and since the graph for electric heating with a load of ·8 kilo has become at 120° C. nearly a straight line, in which the rate of fall is greater than in the other graphs, they will all finally intersect one another. Therefore it is possible in this case also that the three may ultimately coincide.

Copper.

Length = 97·83 cms.
Area of cross-section = ·0006026 sq. cm.
Elongation weight = 300 grams.
Total load on wire = 800 „
Load per sq. mm. = 12·8 kilos.

Table I.

No.	Temp.	Elongation for 300 grams.	No. of Observations.	M.
1	15°·2 C.	·04264 cm.	8	$11·21 \times 10^{11}$
2	16·3	·04306	8	11·10
3	17·5	·04230	8	11·30
4	20·0	·04082	8	11·71
5	24·8	·03934	8	12·15
6	30·6	·03830	8	12·48
7	36·7	·03766	8	12·69
8	41·4	·03734	8	12·80
9	44·8	·03731	8	12·81
10	47·5	·03746	8	12·76
11	52·0	·03796	7	12·59
12	60·1	·03925	7	12·18
13	65·3	·04020	7	11·89
14	70·9	·04086	6	11·70
15	77·7	·04138	8	11·55
16	84·2	·04178	8	11·44
17	95·6	·04215	8	11·34
18	106·1	·04238	8	11·28
19	115·5	·04267	8	11·22
20	122·3	·04277	9	11·18
21	116·2	·04225	8	11·29
22	110·0	·04193	8	11·40
23	99·8	·04103	8	11·65
24	91·0	·04026	8	11·87
25	82·3	·03944	8	12·12
26	74·1	·03858	8	12·39
27	67·4	·03825	8	12·50
28	62·5	·03812	7	12·54
29	56·7	·03809	7	12·55
30	51·5	·03822	8	12·51
31	44·2	·03861	9	12·38
32	34·9	·03955	7	12·09
33	27·6	·04051	8	11·80
34	22·8	·04150	8	11·52
35	20·0	·04212	8	11·35
36	16·3	·04306	8	11·10
37	15·8	·04249	6	11·25

Total load = 1·6 kilos.
= 25·6 kilos per sq. mm.

TABLE II.

No.	Temp.	Elongation for 300 grams.	No. of Observations.	M.
1	18°·7 C.	·04295 cm.	8	$11·13\times 10^{11}$
2	26·4	·04306	8	11·10
3	35·3	·04312	8	11·09
4	43·9	·04327	6	11·05
5	55·1	·04341	6	11·01
6	64·5	·04341	8	11·01
7	72·2	·04356	7	10·97
8	87·0	·04368	8	10·94
9	99·3	·04377	7	10·92
10	106·1	·04385	8	10·90
11	113·8	·04396	8	10·87
12	122·2	·04405	8	10·85
13	111·4	·04388	8	10·89
14	102·5	·04385	9	10·90
15	96·7	·04385	8	10·90
16	85·3	·04372	8	10·93
17	74·8	·04348	8	10·99
18	63·7	·04345	7	11·00
19	51·2	·04339	6	11·02
20	45·6	·04319	8	11·07
21	36·5	·04315	8	11·08
22	29·1	·04303	8	11·11
23	22·6	·04306	8	11·10
24	18·7	·04295	8	11·13

ORDINARY HEATING.

Length = 94·72 cms.
Load = ·8 kilo.

TABLE III.

No.	Temp.	Elongation for 300 grams.	No. of Observations.	M.
1	15°·7 C.	·04118 cm.	6	$11·24\times 10^{11}$
2	35·0	·04144	6	11·17
3	77·8	·04196	6	11·03
4	100·0	·04217	6	10·97
5	129·6	·04257	6	10·87

Platinum.

In this case the wire was that used in the first experiment, and the first cycle is, with a few trifling exceptions, practically identical with that experiment. When the modulus was determined at the temperature of the room, the value was $13 \cdot 55 \times 10^{11}$, and then when a weak current was passed the value fell. Here there is a difference from the first experiment, for in it the first effect of the current was to increase the modulus. In the two experiments, however, the modulus had the same value when the current was weak. Now, in the first experiment the effect of the cycle was to increase the modulus when the wire had cooled to the temperature of the room, so that some of this effect must have remained in the wire; for, while the value is lower than the final determination at the room temperature in the first experiment, it is higher than the initial value.

The modulus was then determined at various temperatures with a gradually increasing current, with the results as shown in the graph. It will be seen how closely this agrees with the first experiment. The maximum is at about 108° C., and the modulus falls more rapidly than it rose. On decreasing the current there is at first a fall, then a rise, which at about 125° C. increases in rate until a maximum is reached at about 85° C. The subsequent fall is at first fairly rapid, then the rate gradually diminishes down to the lowest temperature at which a reading was taken. After cooling to the temperature of the room, there was an increase in the modulus.

As the wire had not returned to its original state, the cycle was repeated, and the results were as shown in the graph. With an increasing current the value was a little greater than at the corresponding stage in the preceding cycle until a temperature of about 107° C. was reached, at which point the graphs intersect. Beyond this, until a temperature of about 143° C. was reached, the second cycle was lower than the preceding. At this point they crossed again, and the second did not diminish so rapidly as the first. On decreasing the current there was again a fall, but the values with the diminishing current were still higher than those with the preceding cycle. The two graphs then run closely side by side until near the maximum with a decreasing current, when it was found to be lower than that which preceded it. Beyond this point the rate of fall is at first almost the same as previously; then, at about 77° C., the rate of decrease becomes greater than at the same stage, and continues increasing until, on being brought back to the temperature produced by a weak current at the beginning of this cycle, it was found that the cyclically steady state had been reached. On allowing the wire to cool to the temperature of the room, the modulus was higher than at the beginning of these two cycles, but it was lower than at the corresponding stage at the end of the first experiment.

The wire was next loaded with 2·2 kilos, that is, 29 kilos per sq. mm., with the results shown in the graph.

Finally, the wire was heated in the ordinary way, and the results of this experiment are also recorded in the graph.

With the platinum, as with the other three wires, the value with the ordinary heating was higher than when it was heated by the current with the greater load on. When produced, the lines intersect at about 240° C.; and again, from the slope of the curve, with the electric heating and smaller weight, we see that the graphs will all ultimately intersect. It is again possible also to suppose that all the three may finally coincide, and there be at last only one value for the modulus.

[TABLE

PLATINUM.

Length = 62·12 cms.
Area of cross-section = ·0007548 sq. cms.
Elongation weight = 300 grams.
Total load on wire = 800 „
Load per sq. mm. = 10·56 kilos.

TABLE I.—FIRST CYCLE.

No.	Temp.	Elongation for 300 grams.	No. of Observations.	M.
1	20°·0 C.	·01788 cm.	8	$13·55 \times 10^{11}$
2	22·8	·01918	8	12·63
3	30·0	·01897	8	12·74
4	45·1	·01829	6	13·25
5	59·3	·01786	6	13·57
6	68·9	·01749	8	13·85
7	79·6	·01709	8	14·18
8	88·5	·01671	9	14·50
9	94·2	·01642	10	14·76
10	101·5	·01601	8	15·14
11	107·7	·01573	8	15·41
12	114·0	·01583	8	15·31
13	123·8	·01626	8	14·90
14	135·6	·01704	8	14·22
15	145·4	·01802	8	13·45
16	155·2	·01918	8	12·63
17	150·0	·01948	8	12·44
18	142·6	·01943	8	12·47
19	131·1	·01920	8	12·62
20	120·7	·01858	8	13·04
21	112·3	·01759	6	13·79
22	108·2	·01682	9	14·41
23	103·5	·01602	7	15·13
24	97·6	·01505	8	16·10
25	89·9	·01399	8	17·32
26	85·1	·01376	8	17·61
27	82·6	·01386	8	17·48
28	78·2	·01446	8	16·75
29	74·9	·01497	7	16·19
30	68·8	·01563	6	15·50
31	55·7	·01656	8	14·63
32	44·2	·01731	9	14·00
33	32·3	·01780	8	13·61
34	26·5	·01796	8	13·49
35	23·2	·01802	8	13·45
36	20·3	·01669	8	14·52

TABLE II.—FINAL CYCLE.

No.	Temp.	Elongation for 300 grams.	No. of Observations.	M.
1	22°3 C.	·01912 cm.	8	$12·67 \times 10^{11}$
2	28·9	·01890	8	12·82
3	40·5	·01843	8	13·15
4	49·7	·01812	8	13·37
5	57·2	·01784	8	13·58
6	69·8	·01741	8	13·92
7	77·5	·01710	8	14·17
8	88·4	·01663	8	14·57
9	100·3	·01607	8	15·08
10	105·6	·01587	8	15·27
11	110·0	·01577	9	15·37
12	113·4	·01589	8	15·25
13	120·9	·01617	8	14·98
14	126·2	·01654	10	14·65
15	133·5	·01690	9	14·34
16	140·1	·01748	9	13·86
17	146·2	·01798	8	13·48
18	157·0	·01900	8	12·75
19	149·9	·01940	8	12·49
20	141·7	·01938	8	12·50
21	130·4	·01906	10	12·71
22	124·3	·01874	8	12·93
23	117·5	·01819	7	13·32
24	112·0	·01749	8	13·85
25	104·6	·01611	9	15·04
26	98·4	·01502	8	16·13
27	93·1	·01417	8	17·10
28	89·8	·01391	8	17·42
29	87·3	·01380	8	17·56
30	84·1	·01386	8	17·48
31	81·5	·01405	8	17·25
32	75·9	·01493	8	16·16
33	71·2	·01561	9	15·52
34	65·0	·01623	7	14·93
35	56·8	·01679	6	14·43
36	50·3	·01720	6	14·09
37	41·2	·01787	7	13·56
38	30·9	·01854	8	13·07
39	25·6	·01890	8	12·82
40	22·3	·01912	8	12·67
41	20·7	·01779	8	13·62

Total load = 2·2 kilos.
= 29 kilos per sq. mm.

TABLE III.

No.	Temp.	Elongation for 300 grams.	No. of Observations.	M.
1	23°·0 C.	·01914 cm.	8	$12·67 \times 10^{11}$
2	30·2	·01917	8	12·65
3	42·6	·01922	7	12·62
4	49·5	·01928	7	12·58
5	60·1	·01940	8	12·50
6	71·9	·01940	8	12·50
7	85·0	·01944	9	12·48
8	99·7	·01951	10	12·43
9	112·4	·01969	8	12·32
10	126·3	·01974	8	12·29
11	142·6	·01980	9	12·25
12	157·8	·01993	8	12·17
13	151·5	·01983	8	12·23
14	138·4	·01985	8	12·22
15	125·2	·01972	8	12·28
16	110·9	·01970	9	12·31
17	100·6	·01966	8	12·34
18	87·1	·01942	10	12·49
19	71·3	·01942	7	12·49
20	57·4	·01934	8	12·54
21	45·0	·01925	6	12·60
22	31·7	·01922	7	12·62
23	23·5	·01913	8	12·68

ORDINARY HEATING.

Load = 1·2 kilos.
Length = 62·23 cms.

TABLE IV.

No.	Temp.	Elongation for 300 grams.	No. of Observations.	M.
1	20°·1 C.	·01876 cm.	5	$12·83 \times 10^{11}$
2	35·0	·01899	6	12·78
3	78·0	·01929	6	12·58
4	100·0	·01945	6	12·48
5	129·8	·01967	6	12·34

Discussion of Results.

In this paper, as already stated, the new results are—the reaching of the cyclically steady state with a small load, the effect when the load is increased, and the heating by the ordinary method.

With the copper the steady state was reached during the first cycle, but with the other three more than one cycle of operations had to be completed before this was accomplished. In all four, the heating by the ordinary method gave values that were a little higher than those by the electric heating with the greater load on, but in all cases the temperature coefficient of decrease of modulus was greater with the ordinary heating than with the other. This, of course, means that at some temperature, if the coefficients do not alter, the lines will intersect. Again, the modulus with the electric heating and smaller weight had in all cases settled down to a fairly uniform rate of decrease, these rates being such that it seems probable that all the graphs would intersect, but the current at my disposal was not sufficient to heat the wires to the temperature necessary to test this. Now, the straight-line graphs are inclined, in each case, to one another at a small angle; and quite a slight change in the coefficients would be sufficient to produce such a change in the slope of the lines, for the coefficients may not be absolutely constant, that after meeting they coincide, and then the values by the two methods of heating would be the same. In all cases, too, it might be that the same would be true of the modulus when the wire is heated by the current and loaded with the smaller weight.

There has not been much investigation of the effect of a current through a wire, that is, of a circular magnetic field, on elasticity, but there has been a good deal of research on the effect of a longitudinal field.

In a paper by Stevens and Dorsay,* the effect of a longitudinal field on Young's modulus is investigated. The method was that of flexure, a mirror being attached to the middle of the rod, and the deflections read by the movements of interference bands. The apparatus was arranged so that the rod was not heated. It passed through an inner tube, in the annular space surrounding this tube a current of water at a constant temperature was kept flowing, and on the outside of this the coil was wound. As the temperature of the rod did not rise, their results give the effect of the magnetic field alone.

In their experiments, which were performed on steel and wrought iron,

* *Phys. Rev.*, vol. ix., p. 116.

the load was small, being in some cases ·5 kilo, and in others 1 kilo, and the smaller of these is less than the total load used in any of my experiments. They found that in all cases there was an increase in the modulus. Since the loads were small, the results can be compared with my first set of experiments on each wire, and it is to be noted that there is agreement between them, viz. at the beginning of my experiments, and all through theirs, an increase in the modulus was obtained.

In a second paper Stevens * examined the effect of a longitudinal field on rigidity, in which the same apparatus was used. In summing up his results he makes the following statements:—

I. Magnetisation of an iron or steel rod increases the torsional elasticity.

II. The effect is greater in iron than in steel.

III. Increase in elasticity varies with the length of the rod.

IV. Distinct agreement with results of the experiments on the flexure of rods.

It is of importance to note that in the experiments on rigidity the rods were subjected to different couples, and that the increase was greater when the stress was smaller. This result is altogether in accordance with those I obtained, for in all cases I got a decrease in the modulus when the load was increased. Now, Young's modulus depends on the rigidity, and we may assume that the one will vary with the other, and that those conditions which produce a change in the rigidity will also cause a change in the modulus. There is, therefore, distinct agreement between these two sets of experiments and my own.

I. An increase in magnetisation produces an increase in elasticity.

II. When the metal is subjected to various stresses, the modulus is lowest when the stress is greatest.

This agreement, however, does not hold in every respect, for in one point their results differ from mine, viz. that within the range of their experiments there is no appearance of a maximum. This difference, however, can, I think, be easily accounted for if the different conditions be taken into consideration. In these the temperature of the rod was kept constant, whereas in mine it rose. But an increase in temperature produces a diminution in the modulus, and so a point must be reached at which the increase caused by the current is not sufficient to wholly counterbalance the decrease produced by the rise of temperature, and so the modulus falls.

Gray, Blyth, and Dunlop in their paper examine the effect of change of temperature on Torsional Elasticity, and find in all cases that there is a

* *Phys. Rev.*, vol. x., p. 161.

decrease. This, again, is in harmony with my results, for when I heated the wire in the ordinary way the decrease in the modulus was uniform. It also confirms my argument that those conditions which affect the rigidity of a substance produce a similar change in Young's modulus. Since, then, there is a diminution both in the rigidity and in Young's modulus when the substance is heated in the ordinary way, it is to be expected that, since an increase in the magnetic field produces an increase in the rigidity, it will have the same effect on Young's modulus.

There are other papers dealing with the effects of temperature and magnetic field on elasticity, among which may be cited those of Wertheim,* Pisati,† Katzenelsohn,‡ Hopkinson and Roger,§ Day,‖ and Hopkinson.¶ The first three deal with the effect of temperature on the modulus; they all employ the ordinary method of heating. Hopkinson and Roger's paper also discusses the same question, but in their case the temperature was raised to nearly 800° C., the rod being heated in an electric furnace. Day, on the other hand, examines the effect of a magnetic field on rigidity, while Hopkinson deals with the effect of temperature on the magnetic quality of iron.

Turning again to the graphs of the electric heating with the smaller load, it seems possible that the effect may be connected with the Villari reversal. In that phenomenon, when the field is weak, a tension on the wire increases the susceptibility, but reduces it when the field is strong. Now, in my experiments there is an increase in the modulus when the current is weak and the tension moderate, so that the increase in the susceptibility and in Young's modulus seem as if they might be connected with one another. When the wire is subjected to a greater stress my experiments show that there is no such increase in the modulus, and this also is in agreement with a longitudinal stress in a magnetic field, for Ewing,** in describing the effects of longitudinal pull on iron, says, "In the case of a hard metal, where it is possible to apply a stronger pull without permanently altering the characteristics or structure of the piece, it appears that the presence of a sufficiently great amount of stress may be unfavourable to magnetisation, even in the earliest stages of the magnetising process."

On the other hand, it may be that there is no *direct* connection between

* *Ann. Chim. Phys.*, (3) vol. xii., p. 385. 1844.
† *Gaz. Chim. ital.*, vol. vii., p. 1.
‡ *Beiblätter*, xii., p. 307. 1888.
§ *Proc. R.S.*, 1905, p. 419.
‖ *Am. Journ. Sc.*, vol. iii., p. 449.
¶ *Phil. Trans.*, 1889, A., p. 443.
** *Magnetic Induction in Iron*, 3rd ed., p. 209.

the two phenomena, but only a *general* connection, due to a certain similarity of conditions. In both cases a load is put on which alters the internal molecular state, and the wire tends to set into a new arrangement of molecular equilibrium. Variation of field in the one case, variation of temperature in the other, alters this state of equilibrium, and produces corresponding effects on the intensity in the one case, and on Young's modulus in the other. Now, it may be that increase of temperature at first enables the molecules to set into more stable arrangements under the load, and therefore to increase the modulus. Ultimately, however, the increase of temperature must cause it to diminish. Then, when the tension is great enough, the molecular arrangements tend to be the strongest under the conditions, and increase of temperature can only weaken them, and therefore diminish the modulus.

However, before this matter can be satisfactorily decided, it will be necessary to find out how the modulus changes as the load is gradually increased, and I hope shortly to have the honour of laying before the Society the results of experiments dealing with this part of the investigation.

(*Issued separately October* 23, 1908.)

XLI.—The Theory of General Determinants in the Historical Order of Development up to 1860. By Thomas Muir, LL.D.

(MS. received June 15, 1908. Read July 13, 1908.)

My last communication in reference to the history of *general* determinants dealt with the period 1844–1852 (*Proc. Roy. Soc. Edin.*, xxv. pp. 908–947). The present paper continues the history up to 1860, but in addition contains an account of five writings belonging to previous periods, namely, by Bianchi (1839), Chelini (1840), Terquem (1846), Hermite (1849), Salmon (1852).

Bianchi, G. (1839, January).

[Sopra l'analisi lineare per la risoluzione dei problemi di primo grado. *Mem. della Soc. ital. delle Sci.*, xxii. pp. 184–227.]

Bianchi's knowledge of previous work on simultaneous linear equations must have been slight—confined, probably, to an acquaintance with Cramer's rule and with Cauchy's so-called "symbolical" solution as given in the *Cours d'Analyse* of 1821: unless this were so, he would scarcely have referred to the methods given in such text-books as Ruffini's *Elementary Algebra* and Euler's *Elements*. One is thus prepared to find little new in his conscientiously laboured monograph, consisting of an introduction of five pages, a section of twenty-seven pages on the solution of a set of n equations with n unknowns, and a section of twelve pages on n equations with fewer unknowns. The main interest lies in the first fifteen pages (pp. 189–204) of the earlier section, these being devoted to establishing the validity of Cramer's rule. The procedure consists in eliminating one and the same unknown between the first equation and each of the other equations of the set, then in treating in the same way the set of $n-1$ equations thus derived, and so on until a single equation $Nx_n = D$ results. As negligible factors are not struck out in the course of the work, the discovery of the law of formation of the coefficients in the successive sets of equations is made unnecessarily difficult, and N and D are obtained in unwieldy forms. Thus, in the case of the six equations

$$\left.\begin{array}{l} a_1x_1 + b_1x_2 + \dots + f_1x_6 = s_1 \\ a_2x_1 + b_2x_2 + \dots + f_2x_6 = s_2 \\ \dots\dots\dots\dots \end{array}\right\}$$

the expression found for the last coefficient of x_6 is, in later notation,

$$| a_1 b_2 c_3 d_4 e_5 f_6 | \cdot | a_1 b_2 c_3 d_4 | \cdot | a_1 b_2 c_3 |^2 \cdot | a_1 b_2 |^4 \cdot a_1{}^8 ,$$

and for the term independent of x_6

$$| a_1 b_2 c_3 d_4 e_5 s_6 | \cdot | a_1 b_2 c_3 d_4 | \cdot | a_1 b_2 c_3 |^2 \cdot | a_1 b_2 |^4 \cdot a_1{}^8 ,$$

with the result, of course, that

$$x_6 = \frac{| a_1 b_2 c_3 d_4 e_5 s_6 |}{| a_1 b_2 c_3 d_4 e_5 f_6 |} .$$

It will readily be agreed that this procedure, though fresh, is not an improvement on others previously known.

Chelini, D. (1840).

[Formazione e dimostrazione della formula che dà i valori delle incognite nelle equazioni di primo grado. *Giornale Arcadico di Sci.* . . . , lxxxv. pp. 3–12.]

The writings known to Chelini were Terquem's *Manuel d'Algèbre*, Bianchi's paper of 1839, and Molins' of the same year. The paper, however, which his short and clearly written exposition most readily calls to mind is Gergonne's of the year 1813. The "formazione" is essentially Bezout's, and the "dimostrazione" essentially Laplace's.

Terquem, O. (1846).

[Note sur les équations du premier degré en nombre plus grand que celui des inconnues. . . . *Nouv. Annales de Math.*, v. pp. 551–556.]

Knowing from the four equations (Terquem uses n)

$$\left.\begin{array}{l} a_1 x + a_2 y + a_3 z + a_4 w = a_5 \\ b_1 x + b_2 y + b_3 z + b_4 w = b_5 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\}$$

the usual expressions for w, z, . . . Terquem affirms that if w is to be equal to 0 we must have

$$| a_1 b_2 c_3 d_5 | = 0 ,$$

and that therefore this last equation is the equation of condition for the simultaneous existence of four equations between three unknowns. Continuing, he says that if we are to have $z = w = 0$ we must have

$$| a_1 b_2 c_3 d_5 | = | a_1 b_3 c_4 d_5 | = 0,^* \qquad (\beta)$$

and that therefore these two equations are "les deux équations de condition pour que 4 équations entre 2 inconnues puissent être satisfaites par les

* The second determinant is incorrectly printed in the original.

mêmes valeurs." The words "et ainsi de suite" are added to draw attention to the general theorem.

On this we can only remark that the giving of the equations of condition in the form (β) in the second case, even although the real equations of condition

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_5 & b_5 & c_5 & d_5 \end{vmatrix} = 0$$

are thence deducible, seems quite inexcusable, especially in an exposition meant to be elementary.

HERMITE, C. (1849, January).

[Sur une question relative à la théorie des nombres. *Journ. (de Liouville) de Math.*, xiv. pp. 21–30.]

As a lemma in the process of attaining the main purpose of his paper Hermite gives an identity which for the 4th order we should nowadays write in the form

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = -\frac{1}{b_1c_1}\begin{vmatrix} |a_2b_1| & |a_3b_1| & |a_4b_1| \\ |b_2c_1| & |b_3c_1| & |b_4c_1| \\ |c_2d_1| & |c_3d_1| & |c_4d_1| \end{vmatrix}.$$

This he establishes rather circuitously by taking four quantities $\xi_1, \xi_2, \xi_3, \xi_4$ which satisfy the equations

$$\left.\begin{aligned} a_1\xi_1 + b_1\xi_2 + c_1\xi_3 + d_1\xi_4 &= 1 \\ a_2\xi_1 + b_2\xi_2 + c_2\xi_3 + d_2\xi_4 &= 0 \\ a_3\xi_1 + b_3\xi_2 + c_3\xi_3 + d_3\xi_4 &= 0 \\ a_4\xi_1 + b_4\xi_2 + c_4\xi_3 + d_4\xi_4 &= 0 \end{aligned}\right\},$$

and then multiplying the original determinant columnwise by $(-1)^3b_1c_1$ in the form

$$\begin{vmatrix} \xi_1 & b_1 & \cdot & \cdot \\ \xi_2 & -a_1 & c_1 & \cdot \\ \xi_3 & \cdot & -b_1 & d_1 \\ \xi_4 & \cdot & \cdot & -c_1 \end{vmatrix}.$$

SALMON, G. (1852).

[*A Treatise on the Higher Plane Curves*: . . . By the Rev. George Salmon, M.A. . . . xii+316 pp. Dublin, 1852.]

For the convenience of his readers Salmon appended a fifteen-page note on the subject of *Elimination*, and, as was natural, the note opened with a sketch (pp. 285–292) of the theory of determinants. Short and simple as this is, it contains one paragraph (§ 11) worthy of note, namely, in regard to the multiplication-theorem.

"The determinant

$$\begin{vmatrix} A_1a_1+B_1b_1+C_1c_1 & A_2a_1+B_2b_1+C_2c_1 & A_3a_1+B_3b_1+C_3c_1 \\ A_1a_2+B_1b_2+C_1c_2 & A_2a_2+B_2b_2+C_2c_2 & A_3a_2+B_3b_2+C_3c_2 \\ A_1a_3+B_1b_3+C_1c_3 & A_2a_3+B_2b_3+C_2c_3 & A_3a_3+B_3b_3+C_3c_3 \end{vmatrix},$$

he says, is evidently the result of eliminating x, y, z from the equations

$$\left.\begin{aligned} a_1S_1+b_1S_2+c_1S_3 &= 0 \\ a_2S_1+b_2S_2+c_2S_3 &= 0 \\ a_3S_1+b_3S_2+c_3S_3 &= 0 \end{aligned}\right\}$$

when

$$\left.\begin{aligned} S_1 &= A_1x+A_2y+A_3z, \\ S_2 &= B_1x+B_2y+B_3z, \\ S_3 &= C_1x+C_2y+C_3z \end{aligned}\right\}.$$

But this elimination may be effected at once by eliminating S_1, S_2, S_3: consequently $|\, a_1\, b_2\, c_3 \,|$ must be a factor of the resultant. In the second place, since a set of values of x, y, z can be found to satisfy simultaneously the given equations if a set can be found to satisfy simultaneously the equations $S_1=0$, $S_2=0$, $S_3=0$: and since the condition that the latter shall be possible is $|\, A_1\, B_2\, C_3 \,| = 0$, it follows that $|\, A_1\, B_2\, C_3 \,|$ must also be a factor of the result. The remaining factor being manifestly 1, the desired end, in Salmon's opinion, is attained. We only remark in passing that a little careful scrutiny of the reasoning would have suggested the need for additional support.

Salmon also proposes a fresh enunciation of the same theorem, namely, *If any set of linear equations*

$$\left.\begin{aligned} a_1x+b_1y+c_1z+\dots\dots &= 0 \\ a_2x+b_2y+c_2z+\dots\dots &= 0 \\ \dots\dots\dots\dots & \end{aligned}\right\}$$

be transformed by any linear substitution

$$\left.\begin{aligned} x &= A_1\xi+B_1\eta+C_1\zeta+\dots\dots \\ y &= A_2\xi+B_2\eta+C_2\zeta+\dots\dots \\ \dots\dots\dots\dots & \end{aligned}\right\}$$

then the determinant of the new set will be equal to the determinant of the original set multiplied by the determinant of transformation. This new wording will be recognised as a sign of the advent of the "algebra of linear transformation."

CAUCHY, A. (1853, January).

[Sur les clefs algébriques. *Comptes rendus . . . Acad. des Sci.* (Paris), xxxvi. pp. 70–75, 129–136: or *Œuvres complètes* (1), xi. pp. 439–445, xii. pp. 12–20.]

[Sur les avantages que présente, dans un grand nombre de questions, l'emploi des clefs algébriques. *Comptes rendus . . . Acad. des Sci.* (Paris), xxxvi. pp. 161–169: or *Œuvres complètes,* (1) xii. pp. 21–30.]

These papers add nothing of algebraic importance to the contents of Cauchy's memoir of the year 1847: in fact, they may be looked on as short and simply worded abstracts of parts of that memoir. It is worthy of note, however, that even where problems of elimination are being dealt with "sommes alternées" are not now explicitly referred to.

SAINT VENANT, DE (1853, March).

[De l'interprétation géométrique des *clefs algébriques* et des *déterminants. Comptes rendus . . . Acad. des Sci.* (Paris), xxxvi. pp. 582–585.]

De Saint Venant's suggestion is that Cauchy's "algebraic keys" $\alpha, \beta, \gamma, \ldots$ may be viewed as *directed magnitudes,* and this leads up to the so-called geometric interpretation of determinants. "Un déterminant du $n$ième ordre," he says, "me paraît être le produit géométrique de n sommes algébriques de n lignes ayant, chacune à chacune, les mêmes directions dans les diverses sommes: en sorte que l'on a pour celui du troisième ordre, par exemple,

$$xy'z'' - xy''z' + \ldots = \text{le produit } (\bar{x}+\bar{y}+\bar{z})(\bar{x}'+\bar{y}'+\bar{z}')(\bar{x}''+\bar{y}''+\bar{z}'')$$

où x, x', x'' ont un même direction (c'est-à-dire sont parallèles), y, y', y'' une autre direction qui est la même pour toutes trois, et z, z', z'' aussi une même troisième direction."

HESSE, O. (1853, April).

[Ueber Determinanten und ihre Anwendung in der Geometrie, . . . *Crelle's Journal,* xlix. pp. 243–264: or *Werke,* pp. 319–343.]

The product of two determinants A and B being C, Hesse's professed object is to show "wie die partiellen Differentialquotienten der Determinante C nach ihren Elementen c genommen durch die partiellen Differentialquotienten der Factoren A und B nach ihren Elementen genommen sich ausdrücken lassen." We are prepared, therefore, to find his ground already pretty well covered by Joachimthal's paper of November 1849. The latter established the result

$$\frac{\partial C}{\partial c_{\kappa\lambda}} = \frac{\partial A}{\partial a_{o\kappa}} \cdot \frac{\partial B}{\partial b_{o\lambda}} + \frac{\partial A}{\partial a_{1\kappa}} \cdot \frac{\partial B}{\partial b_{1\lambda}} + \ldots + \frac{\partial A}{\partial a_{n\kappa}} \cdot \frac{\partial B}{\partial b_{n\lambda}},$$

and said others could be found: Hesse established one of these others, namely,

$$\frac{\partial^2 C}{\partial c_{\kappa\lambda}\partial c_{\mu\nu}} = \frac{1}{1 \cdot 2} \sum \frac{\partial^2 A}{\partial a_{p\kappa}\partial a_{q\mu}} \cdot \frac{\partial^2 B}{\partial b_{p\lambda}\partial b_{q\nu}},$$

and said that the next would be

$$\frac{\partial^3 C}{\partial c_{\kappa\lambda}\partial c_{\mu\nu}\partial c_{\rho\sigma}} = \frac{1}{1 \cdot 2 \cdot 3} \sum \frac{\partial^3 A}{\partial a_{p\kappa}\partial a_{q\mu}\partial a_{r\rho}} \cdot \frac{\partial^3 B}{\partial b_{p\lambda}\partial b_{q\nu}\partial b_{r\sigma}},$$

where $p, q, \ldots$ have the values $0, 1, 2, \ldots, n$.

We can only remark that the second and third results are not so simple as they ought to have been: for Hesse does not point out that (1) when p and q are identical the term vanishes; (2) putting $p, q = \alpha, \beta$ gives the same term as putting $p, q = \beta, \alpha$; and (3) therefore the second result should be

$$\frac{\partial^2 C}{\partial c_{\kappa\lambda}\partial c_{\mu\nu}} = \sum \frac{\partial^2 A}{\partial a_{p\kappa}\partial a_{q\mu}} \cdot \frac{\partial^2 B}{\partial b_{p\lambda}\partial b_{q\nu}},$$

where p has any of the values $0, 1, 2, \ldots, n-1$, and q any of the values $1, 2, \ldots, n$, subject to the condition that $p<q$. It would then agree with the extended multiplication-theorem of Binet and Cauchy, and especially with the latter's form of it.

Chio, F. (1853, June).

[Mémoire sur les fonctions connues sous le nom de résultantes ou de déterminants. 32 pp. Turin.]

The title here is not sufficiently descriptive, almost the whole of the thirty-two pages being occupied with the consideration of determinants *whose elements are binomial.* Beginning with the "tableau"

$$\begin{array}{cccc} a_0+m_0 & a_1+m_1 & \ldots & a_{\iota-1}+m_{\iota-1} \\ b_0+n_0 & b_1+n_1 & \ldots & b_{\iota-1}+n_{\iota-1} \\ \cdot & \cdot & \cdot & \cdot \\ l_0+t_0 & l_1+t_1 & \ldots & l_{\iota-1}+t_{\iota-1} \end{array}$$

Chio seeks, of course, to express its determinant as a sum of determinants with monomial elements, and thereafter applies his result to particular cases.

The first matter of real interest is reached on p. 11, where the following theorem is given: "*Soient* s *la résultante de l'ordre* i *formée avec les termes du tableau*

$$\begin{matrix} a_0 & a_1 & \dots & a_{i-1} \\ b_0 & b_1 & \dots & b_{i-1} \\ \cdot & \cdot & \cdot & \cdot \\ l_0 & l_1 & \dots & l_{i-1}, \end{matrix}$$

et s″ *la résultante de l'ordre* i − 1 *formée avec les termes compris dans le tableau*

$$\begin{matrix} S(\pm a_0b_1) & S(\pm a_0b_2) & \dots & S(\pm a_0b_{i-1}) \\ S(\pm a_0c_1) & S(\pm a_0c_2) & \dots & S(\pm a_0c_{i-1}) \\ \cdot & \cdot & \cdot & \cdot \\ S(\pm a_0l_1) & S(\pm a_0l_2) & \dots & S(\pm a_0l_{i-1}). \end{matrix}$$

La résultante s″ *sera égale à* s, *au facteur près* a_0^{i-2}, *en sorte qu'on aura* $s'' = a_0^{i-2}\,s$."

This is one form of the theorem afterwards well known as effecting the transformation of any determinant into one of the next lower order. It may be viewed as a case of Hermite's result of the year 1849.

On p. 17 particular cases cease to be considered, and the multiplication of an array of i rows and $2i$ columns by a similar array is taken up, with a result in accordance with that arrived at by Binet and Cauchy in 1812. From this result, by specialisation, the ordinary multiplication-theorem is then deduced, and with it (Chio's "théorème ix.") the first part of the memoir closes.

The second part, which begins on p. 23, concerns the solving of a set of $2n$ equations of a type which will be sufficiently specified by giving the set where $n=3$, namely,

$$\left.\begin{matrix} x+y+z = d_1 \\ x\xi + y\eta + z\zeta = d_2 \\ x\xi^2 + y\eta^2 + z\zeta^2 = d_3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ x\xi^5 + y\eta^5 + z\zeta^5 = d_6 \end{matrix}\right\}.$$

The connection of this with what precedes consists in the fact, arrived at by Sylvester in his solution of the problem of the canonisation of the quintic, that ξ, η, ζ are then the roots of the equation in ω

$$\begin{vmatrix} d_2-\omega d_1 & d_3-\omega d_2 & d_4-\omega d_3 \\ d_3-\omega d_2 & d_4-\omega d_3 & d_5-\omega d_4 \\ d_4-\omega d_3 & d_5-\omega d_4 & d_6-\omega d_5 \end{vmatrix} = 0.$$

SPOTTISWOODE, W. (1853).

[Elementary theorems relating to determinants. Second edition, rewritten and much enlarged by the author. *Crelle's Journal*, li. pp. 209–271, 328–381.]

A more correct description of Spottiswoode's second edition would be *rearranged, partly rewritten, and much enlarged*, the majority of the titles of the old sections or chapters occurring again but in a different order, the majority of the sections being enlarged, and two or three new sections being inserted. Although the total increase of matter is from 71 pages to 117, there is comparatively little to be noted concerning general determinants.

In § 2, which bears the title "Addition and Subtraction of Determinants," the following appears (p. 232) for the first time:—THEOREM ix. *The sum of two determinants in which* i *rows (on a certain level) are respectively equal, is equal to the determinant whose* i[th] *minors on the aforesaid level are identical with the corresponding* i[th] *minors of each of the two given determinants, and whose* (n – i)[th] *complementary minors are respectively the sum of the complementary minors of the given determinants.* No instance is given where the two determinants have more than one row different.

In § 4, which deals with the multiplication of determinants, much space (pp. 238–248) is given to Sylvester's theorem of 1852 (October). Spottiswoode's own mode of treating the subject is to begin apparently with the two factors and arrive at the product, whereas in reality the opposite is the case. For example, his proof that

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} . \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = \begin{vmatrix} a\alpha & a\alpha' & a\alpha'' & b & c \\ a'\alpha & a'\alpha' & a'\alpha'' & b' & c' \\ a''\alpha & a''\alpha' & a''\alpha'' & b'' & c'' \\ \beta & \beta' & \beta'' & \cdot & \cdot \\ \gamma & \gamma' & \gamma'' & \cdot & \cdot \end{vmatrix}$$

essentially consists in expanding the right-hand determinant in terms of minors formed from the first three rows and minors formed from the last two rows. His other fresh proof is dependent on the connection between determinants and simultaneous linear equations. Taking the two sets of equations

$$\left.\begin{array}{l} \alpha x + \alpha' y + \alpha'' z = u_1 \\ \beta x + \beta' y + \beta'' z = u_2 \\ \gamma x + \gamma' y + \gamma'' z = u_3 \end{array}\right\} \qquad \left.\begin{array}{l} a u_1 + b u_2 + c u_3 = v_1 \\ a' u_1 + b' u_2 + c' u_3 = v_2 \\ a'' u_1 + b'' u_2 + c'' u_3 = v_3 \end{array}\right\}$$

and substituting for u_1, u_2, u_3, in the second set and solving, there is obtained for x an expression whose denominator is known to be

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \cdot \begin{vmatrix} a & a' & a'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{vmatrix} .$$

In the second place, by substituting for u_1 only there is obtained

$$\left.\begin{array}{l} aax + aa'y + aa''z + bu_2 + cu_3 = v_1 \\ a'ax + a'a'y + a'a''z + b'u_2 + c'u_3 = v_2 \\ a''ax + a''ay + a''a''z + b''u_2 + c''u_3 = v_3 \\ \beta x + \beta'y + \beta''z - u_2 = 0 \\ \gamma x + \gamma'y + \gamma''z - u_3 = 0 \end{array}\right\} ,$$

whence comes for x an expression whose denominator is

$$\begin{vmatrix} aa & aa' & aa'' & b & c \\ a'a & a'a' & a'a'' & b' & c' \\ a''a & a''a' & a''a'' & b'' & c'' \\ \beta & \beta' & \beta'' & -1 & . \\ \gamma & \gamma' & \gamma'' & . & -1 \end{vmatrix} .$$

A comparison of the two denominators is supposed to establish the desired result; but, although the dropping of the two negative units in the five-line determinant is quite justifiable, no allusion is made to it.

It may be added that Sylvester's umbral notation is used throughout in dealing with the subjects just referred to,

$$\left\{\begin{array}{cccc} 1 & 2 & \ldots & n \\ 1 & 2 & \ldots & n \end{array}\right\} \quad \text{or} \quad \begin{vmatrix} (11) & (12) & \ldots & (1n) \\ (21) & (22) & \ldots & (2n) \\ . & . & . & . \\ (n1) & (n2) & \ldots & (nn) \end{vmatrix}$$

being used for one of the two determinants, and

$$\left\{\begin{array}{cccc} 1' & 2' & \ldots & n' \\ 1' & 2' & \ldots & n' \end{array}\right\} \quad \text{or} \quad \begin{vmatrix} (11)' & (12)' & \ldots & (1n)' \\ (21)' & (22)' & \ldots & (2n)' \\ . & . & . & . \\ (n1)' & (n2)' & \ldots & (nn)' \end{vmatrix}$$

for the other. The reading is thus rendered tiresome, and inaccurate printing exaggerates the trouble.

GRASSMANN [H.] (1854, February, April).

[Sur les différents genres de multiplication. *Crelle's Journal*, xlix. pp. 123–141.]

[Extrait d'un mémoire de M. Grassmann. *Comptes rendus . . . Acad. des Sci.* (Paris), xxxviii. pp. 743–744.]

Grassmann, having become aware of Cauchy's three communications to the French Academy in January of 1853, claims that the principles there established and the results deduced are absolutely the same as those published by himself in 1844. He says (p. 127), "Les clefs algébriques de M. Cauchy ne sont au fond que les unités relatives; et ses facteurs symboliques conviennent, du moins dans un certain rapport, aux quantités extensives telles que je les ai définies. La différence ne consiste qu'en ce que M. Cauchy regarde les clefs algébriques seulement comme un moyen pour résoudre divers problèmes de l'analyse et de la mécanique et qui, les problèmes étant résolus, disparaissent, tandis que d'après les principes établis par moi, on est en état, à chaque pas du procédé, d'attribuer une signification indépendante aux unités relatives et aux quantités qui en sont composées, qu'elle que soit d'ailleurs la marche que l'on suive."

MAJO, L. DE (1854, March).

[Metodi e formole generali per l' eliminazione nelle equazioni di primo grado. *Memorie . . . Accad. delle Sci.* (Napoli), i. pp. 101–116.]

This is a carefully written but curiously belated exposition, the author apparently being quite out of touch with the writers of his own time, and possibly not familiar with any of the older writers save Cramer, Bezout, and Hindenburg. In the first six pages he defines "il polinomio $P_m(a_1b_2c_3 \ldots s_m)$" after the fashion of Bezout (1764), and gives one or two very elementary properties of it. The remaining ten pages are occupied with simultaneous linear equations, and are notable as containing (§§ 15–19) a clear exposition of Bezout's peculiar rule-of-thumb process of 1779. Herein lies the value of the paper, Majo being not only the first since Hindenburg to recall attention to a neglected process of real practical value, but also the first to give (§ 16) a reason for its validity.

CAYLEY, A. (1854, May).

[Remarques sur la notation des fonctions algébriques. *Crelle's Journal*, l. pp. 282–285: or *Collected Math. Papers*, ii. pp. 185–188.]

The notation referred to is that of *matrices*, and is exemplified by

$$\begin{matrix} (& a_1 & a_2 & a_3 &) \\ | & \beta_1 & \beta_2 & \beta_3 & | \\ | & \gamma_1 & \gamma_2 & \gamma_3 & | \end{matrix},$$

a matrix being defined as a system of quantities arranged in the form of a square, but otherwise quite independent. With its help the set of equations

$$\left.\begin{aligned} \xi &= a_1 x + a_2 y + a_3 z \\ \eta &= b_1 x + b_2 y + b_3 z \\ \zeta &= c_1 x + c_2 y + c_2 z \end{aligned}\right\}$$

may, he says, be written in the form

$$\xi\,,\,\eta\,,\,\zeta \;=\; \left(\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right\between x\,,\,y\,,\,z)\,,$$

and consequently the solution of the set in the form

$$x\,,\,y\,,\,z \;=\; \left(\begin{array}{ccc} \frac{A_1}{\Delta} & \frac{B_1}{\Delta} & \frac{C_1}{\Delta} \\ \frac{A_2}{\Delta} & \frac{B_2}{\Delta} & \frac{C_2}{\Delta} \\ \frac{A_3}{\Delta} & \frac{B_3}{\Delta} & \frac{C_3}{\Delta} \end{array}\right\between \xi\,,\,\eta\,,\,\zeta)\,.$$

The latter matrix he calls the *inverse* of the former, and is naturally led to propose that it be denoted by

$$\left(\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right)^{-1}.$$

Next, supposing that along with the original set there exists the set

$$x\,,\,y\,,\,z \;=\; \left(\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{array}\right\between X\,,\,Y\,,\,Z)\,,$$

so that by substitution ξ, η, ζ are expressible in terms of X, Y, Z, Cayley is led by comparison of the old and the new notations to the conception of the *product* of two matrices, and to the use of

$$\left(\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right\between\left.\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{array}\right)$$

for

$$\left(\begin{array}{ccc} (a_1\,a_2\,a_3\between\alpha_1\,\beta_1\,\gamma_1) & (a_1\,a_2\,a_3\between\alpha_2\,\beta_2\,\gamma_2) & (a_1\,a_2\,a_3\between\alpha_3\,\beta_3\,\gamma_3) \\ (b_1\,b_2\,b_3\between\alpha_1\,\beta_1\,\gamma_1) & (b_1\,b_2\,b_3\between\alpha_2\,\beta_2\,\gamma_2) & (b_1\,b_2\,b_3\between\alpha_3\,\beta_3\,\gamma_3) \\ (c_1\,c_2\,c_3\between\alpha_1\,\beta_1\,\gamma_1) & (c_1\,c_2\,c_3\between\alpha_2\,\beta_2\,\gamma_2) & (c_1\,c_2\,c_3\between\alpha_3\,\beta_3\,\gamma_3) \end{array}\right).$$

Lastly, he explains his related notations for lineo-linear functions and quantics.* These we need only exemplify by saying that

* Cayley's first memoir on *quantics* was presented to the Royal Society of London on 20th April, and this paper on the notation of *matrices* is the first of five which appeared together in *Crelle's Journal* with the date 24th May affixed by the author.

$$\left(\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right)\!\!\between \xi, \eta, \zeta \between x, y, z), \qquad \left(\begin{array}{ccc} a & h & g \\ h & b & f \\ g & f & c \end{array}\right)\!\!\between x, y, z)^2,$$

are made to stand for

$$\begin{aligned} &(a_1\xi + a_2\eta + a_3\zeta)x \\ +\, &(b_1\xi + b_2\eta + b_3\zeta)y \\ +\, &(c_1\xi + c_2\eta + c_3\zeta)z \end{aligned} \quad \text{and} \quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

respectively, and that the latter is also denoted by

$$(a, b, c, f, g, h \between x, y, z)^2,$$

and the binary cubics

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3, \quad ax^3 + bx^2y + cxy^2 + dy^3$$

by

$$(a, b, c, d \between x, y)^3, \quad (a, b, c, d \between x, y)^3$$

respectively.

We may suggest for consideration in passing the following order of ideas, as leading up to Cayley's contracted mode of writing a set of linear equations. First, a *row* of separate quantities *e.g.* $(a, b, c, \ldots)$; second, the statement of the identity of two rows, *e.g.* $(a, b, c, \ldots) = (x, y, z, \ldots)$, or simply $a, b, c, \ldots = x, y, z, \ldots$; third, the so-called product of two rows, *e.g.* $(a, b, c, \ldots \between x, y, z, \ldots)$; fourth, a *square* of separate quantities, *i.e.* a matrix; fifth, the result of multiplying a matrix and a row being a row. It is unfortunate that, from the point of view of notation merely, this does not at once suggest, in the sixth place, the result of multiplying two matrices, where, as Cayley is careful to point out, the multiplication is row-by-column and not row-by-row.

Brioschi, Fr. (1854).

[La Teorica dei Determinanti, e le sue principali applicazioni; del Dr Francesco Brioschi; viii+116 pp.; Pavia. Translation into French, by Combescure; ix+216 pp.; Paris, 1856. Translation into German, by Schellbach; vii+102 pp.; Berlin, 1856.]

This, the second separately published text-book on determinants, is mainly on the same lines as the first, but is marked by greater attention to verbal and logical accuracy. It consists of an historical preface and eleven short chapters or sections, seven of the latter being devoted to determinants in general, and the remaining four to special forms.

Sylvester's umbral notation is given in the form

$$\left|\begin{array}{cccc} a_1 & a_2 & \ldots & a_n \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{array}\right|,$$

but is not afterwards employed. The same author's term "minor" (*minore*) is adopted, this being represented in the French translation by "mineur," and in the German by "Unterdeterminante." "Complete" is used as opposed to "minor," and "principal minor" for what nowadays we call "coaxial."

Conspicuously frequent use is made of differentiation in the specification of minors; and it is well to note that, though the work in this way becomes cumbrous, there is a certain effectiveness attained by the usage. Thus, Δ standing for $\Sigma(\pm a_{11}\, a_{22} \ldots a_{nn})$, Brioschi, like Jacobi, obtains

$$\left.\begin{array}{lll} \dfrac{\partial \Delta}{\partial a_{r1}} & = & 0 \quad + a_{s2}\dfrac{\partial^2 \Delta}{\partial a_{r1}\partial a_{s2}} + \ldots\ldots + a_{sn}\dfrac{\partial^2 \Delta}{\partial a_{r1}\partial a_{sn}} \\ \dfrac{\partial \Delta}{\partial a_{r2}} & = & a_{s1}\dfrac{\partial^2 \Delta}{\partial a_{r2}\partial a_{s1}} + \quad 0 \quad + \ldots\ldots + a_{sn}\dfrac{\partial^2 \Delta}{\partial a_{r2}\partial a_{sn}} \\ \cdot & \cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \dfrac{\partial \Delta}{\partial a_{rn}} & = & a_{s1}\dfrac{\partial^2 \Delta}{\partial a_{rn}\partial a_{s1}} + a_{s2}\dfrac{\partial^2 \Delta}{\partial a_{rn}\partial a_{s2}} + \ldots\ldots + \quad 0 \end{array}\right\},$$

and then, by using the multipliers $a_{r1}, a_{r2}, \ldots, a_{rn}$ and adding, finds

$$\Delta = \begin{vmatrix} a_{r1} & a_{r2} \\ a_{s1} & a_{s2} \end{vmatrix} \frac{\partial^2 \Delta}{\partial a_{r1}\partial a_{s2}} + \begin{vmatrix} a_{r1} & a_{r3} \\ a_{s1} & a_{s3} \end{vmatrix} \frac{\partial^2 \Delta}{\partial a_{r1}\partial a_{s3}} + \ldots + \begin{vmatrix} a_{r1} & a_{rn} \\ a_{s1} & a_{sn} \end{vmatrix} \frac{\partial^2 \Delta}{\partial a_{r1}\partial a_{sn}},$$

which is Laplace's expansion-theorem for the case where the minors of one set are of the second order. The remaining cases, he says, can be established in the same way.

Again, having proved the multiplication-theorem (row-by-row)

$$PQ = R,$$

where

$$P = \Sigma(\pm a_{11}\, a_{22} \ldots a_{nn}), \quad Q = \Sigma(\pm b_{11}\, b_{22} \ldots b_{nn}), \quad R = \Sigma(\pm c_{11}\, c_{22} \ldots c_{nn}),$$

he obtains by differentiation with respect to elements of P

$$\frac{\partial P}{\partial a_{rs}} Q = \frac{\partial R}{\partial c_{r1}} b_{1s} + \frac{\partial R}{\partial c_{r2}} b_{2s} + \ldots + \frac{\partial R}{\partial c_{rn}} b_{ns} \tag{1}$$

$$\frac{\partial^2 P}{\partial a_{rs}\partial a_{\rho\sigma}} Q = \sum_x \sum_y \begin{vmatrix} b_{x\sigma} & b_{xs} \\ b_{y\sigma} & b_{ys} \end{vmatrix} \frac{\partial^2 R}{\partial c_{ry}\partial c_{\rho x}}, \tag{2}$$

and by twice differentiating with respect to an element of P and an element of Q

$$\frac{\partial P}{\partial a_{rs}} \frac{\partial Q}{\partial b_{\rho\sigma}} - \frac{\partial P}{\partial a_{r\sigma}} \frac{\partial Q}{\partial b_{\rho s}} = \sum_x \sum_y \begin{vmatrix} a_{x\sigma} & a_{xs} \\ b_{y\sigma} & b_{ys} \end{vmatrix} \frac{\partial^2 R}{\partial c_{ry}\partial c_{\rho x}} \tag{3}$$

Taking $n=4$ and $r, s, \rho, \sigma = 1, 2, 3, 4$, we can best illustrate these by writing them thus:—

$$-|a_{21}a_{33}a_{44}|\cdot|b_{11}b_{22}b_{33}b_{44}| = \begin{vmatrix} b_{12} & b_{22} & b_{32} & b_{42} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{vmatrix} = b_{12}|c_{22}c_{33}c_{44}| - \cdots,$$

$$|a_{21}a_{43}|\cdot|b_{11}b_{22}b_{33}b_{44}| = \begin{vmatrix} b_{12} & b_{22} & b_{32} & b_{42} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ b_{14} & b_{24} & b_{34} & b_{44} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{vmatrix} = -|b_{12}b_{24}|\cdot|c_{23}c_{44}| + \cdots,$$

$$\begin{Vmatrix} |a_{21}a_{33}a_{44}| & |a_{21}a_{32}a_{43}| \\ |b_{11}b_{23}b_{44}| & |b_{11}b_{22}b_{43}| \end{Vmatrix} = \begin{vmatrix} b_{12} & b_{22} & \cdot & b_{42} \\ c_{21} & c_{22} & a_{24} & c_{24} \\ c_{31} & c_{32} & a_{34} & c_{34} \\ c_{41} & c_{42} & a_{44} & c_{44} \end{vmatrix} - \begin{vmatrix} b_{14} & b_{24} & \cdot & b_{44} \\ c_{21} & c_{22} & a_{22} & c_{24} \\ c_{31} & c_{32} & a_{32} & c_{34} \\ c_{41} & c_{42} & a_{42} & c_{44} \end{vmatrix}$$

$$= -|a_{24}b_{12}|\cdot|c_{32}c_{44}| + \cdots$$

The right-hand member of (1) is equivalent to a direction to substitute for the r^{th} row of R the s^{th} column of Q; similarly, the right-hand member of (2) is a direction, though not so evident, to substitute for the r^{th} and ρ^{th} rows of R the s^{th} and σ^{th} columns of Q; and it is clear that (1) and (2) are but the first two identities of many. On the other hand, (3) is quite diverse in character, being got by the combination of two results analogous to (2). This is best brought out by noting that in the examples the right-hand members of (1) and (2) are got by multiplying

$$\begin{vmatrix} \cdot & 1 & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \cdot & 1 & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \cdot & \cdot & \cdot & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

respectively by $|b_{11}b_{22}b_{33}b_{44}|$; and that similarly the two four-line determinants on the right of (3) are got by multiplying

$$\begin{vmatrix} \cdot & 1 & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \cdot & \cdot & \cdot & 1 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{vmatrix}$$

and

$$\begin{vmatrix} \cdot & \cdot & \cdot & 1 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \cdot & 1 & \cdot & \cdot \\ b_{41} & b_{42} & b_{43} & b_{44} \end{vmatrix}.$$

It should be carefully noted also that, while in (2) the number of terms in the development is $\frac{1}{2}n(n-1)$, in (3) the number is $(n-1)^2$.

Lastly, putting

$$\left.\begin{array}{lcl} b_{11}\dfrac{\partial P}{\partial a_{r1}} + b_{12}\dfrac{\partial P}{\partial a_{r2}} + \cdots\cdots + b_{1n}\dfrac{\partial P}{\partial a_{rn}} & = & H_{r1} \\ b_{21}\dfrac{\partial P}{\partial a_{r1}} + b_{22}\dfrac{\partial P}{\partial a_{r2}} + \cdots\cdots + b_{2n}\dfrac{\partial P}{\partial a_{rn}} & = & H_{r2} \\ \cdots\cdots\cdots\cdots\cdots & & \\ b_{n1}\dfrac{\partial P}{\partial a_{r1}} + b_{n2}\dfrac{\partial P}{\partial a_{r2}} + \cdots\cdots + b_{nn}\dfrac{\partial P}{\partial a_{rn}} & = & H_{rn} \end{array}\right\}$$

so that H_{rs} stands for what P becomes when its r^{th} row is replaced by the s^{th} row of Q, and using the multipliers $\partial Q/\partial b_{11}, \partial Q/\partial b_{21}, \ldots, \partial Q/\partial b_{n1}$ prior to addition, Brioschi obtains

$$Q\frac{\partial P}{\partial a_{r1}} = H_{r1}\frac{\partial Q}{\partial b_{11}} + H_{r2}\frac{\partial Q}{\partial b_{21}} + \cdots\cdots + H_{rn}\frac{\partial Q}{\partial b_{n1}},$$

and similarly

$$Q\frac{\partial P}{\partial a_{r2}} = H_{r1}\frac{\partial Q}{\partial b_{12}} + H_{r2}\frac{\partial Q}{\partial b_{22}} + \cdots\cdots + H_{rn}\frac{\partial Q}{\partial b_{n2}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Q\frac{\partial P}{\partial a_{rn}} = H_{r1}\frac{\partial Q}{\partial b_{1n}} + H_{r2}\frac{\partial Q}{\partial b_{2n}} + \cdots\cdots + H_{rn}\frac{\partial Q}{\partial b_{nn}}.$$

With this derived set of equations the multipliers $a_{r1}, a_{r2}, \ldots, a_{rn}$ are then used, and addition performed, the result being Sylvester's theorem of 1839, namely,

$$QP = H_{r1}K_{r1} + H_{r2}K_{r2} + \cdots\cdots + H_{rn}K_{rn},$$

where K_{rs} stands for what Q becomes when its s^{th} row is replaced by the r^{th} row of P.*

In his treatment of the minors of the adjugate determinant Brioschi (pp. 36–39) closely follows Spottiswoode; that is to say, from a set of linear equations he derives one result, then from the adjugate set another result, and finally draws a deduction from a comparison of the two. His thus obtained extension of Spottiswoode's theorem is open to the same criticism as Spottiswoode's extension of Jacobi's.

* Brioschi does not note the independent importance of his second set of equations, which may be condensed into

$$Q\frac{\partial P}{\partial a_{rs}} = H_{r1}\frac{\partial Q}{\partial b_{1s}} + H_{r2}\frac{\partial Q}{\partial b_{2s}} + \ldots + H_{rn}\frac{\partial Q}{\partial b_{ns}},$$

and which, when $r, s = 1, 1$ and $n = 3$, is

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \beta_2 & \beta_3 \\ \gamma_2 & \gamma_3 \end{vmatrix} - \begin{vmatrix} \beta_1 & \beta_2 & \beta_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \alpha_2 & \alpha_3 \\ \gamma_2 & \gamma_3 \end{vmatrix} + \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{vmatrix}.$$

This, however, may be viewed also as a case of Sylvester's theorem, namely, where the first row of P is 1, 0, 0.

The section (§ 7) on "determinanti di determinanti" is founded on Cauchy, and contains known extensions of two or three theorems above given in the notation of differentiation.

CANTOR [M. B.] (1855, March).

[Théorème sur les déterminants Cramériens. *Nouv. Annales de Math.* (1), xiv. pp. 113–114.]

The theorem in question may be formulated thus—*If the permutations of* 1, 2, 3, . . . , n *be arranged in order of magnitude as if they were integral numbers, the sign of the* k^{th} *permutation is independent of* n. Reference is appropriately made to Reiss' paper of 1825, but the theorem is virtually contained in Hinderburg's rule of the year 1784.

HEGER, I. (1856, July).

[Ueber die Auflösung eines Systemes von mehreren unbestimmten Gleichungen des ersten Grades in ganzen Zahlen. *Denkschr. d. k. Akad. d. Wiss. in Wien: math.-naturw. Cl.*, xiv. (2) pp. 1–122.]

Although in this lengthy paper the vanishing of the determinants of a 2-by-n array is repeatedly under consideration (*e.g.* § 24, p. 87), nothing new on the subject presents itself.

SCHLÖMILCH, O. (1856).

[Brioschi's Theorie der Determinante und ihre hauptsächlichsten Anwendungen. *Zeitschrift f. Math. u. Phys.*, I. *Literaturzeitung*, pp. 80–87.]

After a faithful account of Schellbach's translation of Brioschi's text-book, Schlömilch inveighs against the adoption of "die miserable englische Terminologie," instancing *Unterdeterminante*, *Determinante mit reciproken Elementen*, and *Hessian*, for the last of which he proposes to substitute "Inflexionsdeterminante."

RUBINI, R. (1857, May).

[Applicazione della teorica dei determinanti. *Annali de Sci. Mat. e Fis.*, viii. pp. 179–200.]

This resembles Chio's paper of 1853, having the same fundamental theorem, but different illustrative examples. In the mere enunciation of the theorem Rubini is the more successful. Taking the n-line determinant whose element in the place r,s is $a_{rs}+b_{rs}$, and denoting by A the determinant of the a's, and by A_r a determinant obtainable from A on substituting for r columns of a's the corresponding r columns of b's, he writes the expansion in the form

$$A + \Sigma A_1 + \Sigma A_2 + \ldots\ldots + \Sigma A_{n-1} + A_n .$$

BELLAVITIS, G. (1857, June).

[Sposizione elementare della teorica dei determinante. *Memorie ... Istituto Veneto,* ... vii. pp. 67–144.]

Notwithstanding its place of publication, this writing of Bellavitis' is exactly what its title implies; and as a text-book it could scarcely have failed to be useful, so simple and clear is it in style. It consists of two chapters, one on determinants in general (pp. 3–30), and one on special forms (pp. 30–72): a note of six pages on permutations appears as an appendix.

To Bellavitis we owe the modification of Laplace's notation which is now in common use. The passage introducing it is: "Quando gli elementi sieno indicati in modo che chiaramente apparisca la loro formazione, noi porremo tra le due | | i soli elementi della *diagonale* (intendendo sempre per *diagonale* quella da sinistra verso destra discendendo). Cosi

$$|a_1 b_2 c_3 \ldots\ldots| \text{ equivalerà a } \begin{vmatrix} a_1 & b_1 & c_1 & \ldots\ldots \\ a_2 & b_2 & c_2 & \ldots\ldots \\ a_3 & b_3 & c_3 & \ldots\ldots \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot \end{vmatrix}$$

$$|a_q^{(h)} a_s^{(r)}| \text{ equivalerà a } \begin{vmatrix} a_q^{(h)} & a_q^{(r)} \\ a_s^{(h)} & a_s^{(r)} \end{vmatrix}, \text{ ec."}$$

Throughout the exposition this notation is employed. "Riga" he uses either for a "fila orizzontale" or a "fila verticale," and "colonna" for a "fila perpendicolare a quella che s'intese per riga."

Two well-known developments he specifies thus:—

$$|a_1 b_2 c_3 \ldots| = \left(a_1\frac{\partial}{\partial a_1} + b_1\frac{\partial}{\partial b_1} + c_1\frac{\partial}{\partial c_1} + \ldots\right) |a_1 b_2 c_3 \ldots|,$$

$$\begin{aligned} |a_1 b_2 c_3 \ldots h_n| = \Big(a_1\frac{\partial}{\partial a_1} &+ a_2 b_1\frac{\partial^2}{\partial a_2 \partial b_1} + \ldots\ldots + a_2 h_1\frac{\partial^2}{\partial a_2 \partial h_1} \\ &+ a_3 b_1\frac{\partial^2}{\partial a_3 \partial b_1} + \ldots\ldots + a_3 h_1\frac{\partial^2}{\partial a_3 \partial h_1} \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &+ a_n b_1\frac{\partial^2}{\partial a_n \partial b_1} + \ldots\ldots + a_n h_1\frac{\partial^2}{\partial a_n \partial h_1}\Big) |a_1 b_2 c_3 \ldots h_n| . \end{aligned}$$

In reference to determinants with binomial elements (§ 13) he says: "Compiendo questo sviluppo si ottiene la formula

$$\begin{aligned} | a_1+\alpha_1 \ \ b_2+\beta_2 \ \ c_3+\gamma_3 | = & \ | a_1 b_2 c_3 | + | a_1 b_2 \gamma_3 | + | a_1 \beta_2 c_3 | + | a_1 \beta_2 \gamma_3 | \\ & + | \alpha_1 b_2 c_3 | + | \alpha_1 b_1 \gamma_3 | + | \alpha_1 \beta_2 c_3 | + | \alpha_1 \beta_2 \gamma_3 | \end{aligned}$$

che è facile da tenersi a memoria per la sua perfetta analogia collo sviluppo del prodotto di tre binomii."

After giving a sufficient condition for the vanishing of a determinant, he enunciates (§ 15) the converse, namely, *When a determinant vanishes, one of the rows is equal to a sum of multiples of the other rows,* basing its validity on the fact that the multipliers referred to can actually be found by solving a set of simultaneous linear equations.

The multiplication-theorem for determinants Δ_1, Δ_2 of the third order he seeks to establish (§ 31) by partitioning the product-determinant into twenty-seven determinants, and showing that the sum of the six which do not vanish is $\Delta_1\Delta_2$.

Chio's theorem of 1853 is introduced (§ 38) by noting that the resultant of

$$a_r x + b_r y + c_r = 0 \qquad (r = 1, 2, 3)$$

may be viewed as the resultant of

$$\left. \begin{aligned} | a_1 b_2 | y + | a_1 c_2 | &= 0 \\ | a_1 b_3 | y + | a_1 c_3 | &= 0 \end{aligned} \right\},$$

and that therefore

$$\begin{Vmatrix} | a_1 b_2 | & | a_1 c_2 | \\ | a_1 b_3 | & | a_1 c_3 | \end{Vmatrix} \text{ must be a multiple of } | a_1 b_2 c_3 | .$$

That it is so he proves by diminishing the 2nd and 3rd columns of $|a_1 b_2 c_3|$ by b_1/a_1 times the 1st column and c_1/a_1 times the 1st column respectively. Further, he points out (§§ 39, 40) a practical application, namely, in evaluating a determinant whose elements are given in figures.

The adjugate determinant (unfortunately renamed *associato*) is dealt with (§§ 55–58) in connection with the solution of a set of simultaneous linear equations, the special cases being considered where the determinant of the set is 1 and 0. In the former special case he notes the theorem, *The adjugate of the product of two unit determinants is identical in all its elements with the product of the adjugates of the said determinants*; and in the latter the theorem all but reached by Jacobi in 1835 and 1841, *In a zero determinant the cofactors of the elements of a row are proportional to the cofactors of the elements of any other row.*

Cauchy's "clefs algébriques" (*chiavi algebriche*) are expounded at some length (§§ 81–88).

In the last three paragraphs he draws attention to the existence of expressions which may be viewed as "determinanti simbolici," his first kind being those in which symbols of differentiation take the place of elements; *e.g.* the expression

$$\mathrm{P}\left(\frac{\partial \mathrm{Q}}{\partial z} - \frac{\partial \mathrm{R}}{\partial y}\right) + \mathrm{Q}\left(\frac{\partial \mathrm{R}}{\partial x} - \frac{\partial \mathrm{P}}{\partial z}\right) + \mathrm{R}\left(\frac{\partial \mathrm{P}}{\partial y} - \frac{\partial \mathrm{Q}}{\partial x}\right),$$

whose vanishing is the condition for the derivability of the equation

$$\mathrm{P}\partial x + \mathrm{Q}\partial y + \mathrm{R}\partial z = 0$$

from a single primitive, is denoted by

$$\left| \mathrm{P} \ \frac{\partial}{\partial y} \ \mathrm{R} \right|$$

—a notation which is even less satisfactory than that for which it is a contraction, namely,

$$\begin{vmatrix} \mathrm{P} & \frac{\partial}{\partial x} & \mathrm{P} \\ \mathrm{Q} & \frac{\partial}{\partial y} & \mathrm{Q} \\ \mathrm{R} & \frac{\partial}{\partial z} & \mathrm{R} \end{vmatrix}.$$

The other kind of expressions originated with Binet, who in 1812 gave the identities

$$\begin{aligned} \Sigma ab' &= \Sigma a\Sigma b - \Sigma ab, \\ \Sigma ab'c'' &= \Sigma a\Sigma b\Sigma c + 2\Sigma abc - \Sigma a\Sigma bc - \Sigma b\Sigma ca - \Sigma c\Sigma ab; \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

but in this case, though the close resemblance of the right-hand expressions to the developments of axisymmetric determinants is pointed out, no notation founded on the fact is suggested.

As an appendix there is a note on permutations, explaining circular substitutions, interchanges (*alternazioni*), inversions of order (*rovesciamenti d'ordine*), and their relations to one another. Cauchy's sign-rule depending on the number of circular substitutions is replaced by a simpler rule, which requires the counting of only the *even* circular substitutions. Thus the permutation 3265417 being got from the standard permutation 1234567 by means of the circular substitutions

(316), (2), (54), (7),

and only one of these being even, the sign of 3265417 is $(-)^1$. Bellavitis' enunciation is: "*Il numero delle alternazioni, con cui una disposizione può mutarsi in un' altra è pari o dispari insieme col numero di tutte le* *sostituzioni binomie, quadrinomie, sestinomie, ecc. che occorrono per passare da una disposizione all' altra.*"

BALTZER, R. (1857).

[THEORIE UND ANWENDUNG DER DETERMINANTEN, mit Beziehung auf die Originalquellen, dargestellt von Dr Richard Baltzer . . . ; vi + 129 pp.; Leipzig. French translation by J. Houel, xii + 235 pp.; Paris, 1861.]

The good qualities spoken of above as belonging to Brioschi's text-book are still more conspicuous in the German text-book of three years later, but the historical footnotes in Baltzer's give it special value. The *theory* is dealt with in eight little chapters or sections, and the so-called *applications* in ten; several of the latter, however, might quite well have been classed with the former, as they are merely concerned with determinants of special form.

The first section corresponds closely in subject with Bellavitis' appendix: and in connection therewith may be noted Baltzer's remark (§ 2, 3) that *any term got from the diagonal term by substituting* $k_1, k_2, \ldots, k_n$ *for the second suffixes* 1, 2, ..., n *may also be got by substituting* 1, 2, ..., n *for* $k_1, k_2, \ldots, k_n$ *in the set of first suffixes.*

Brioschi's mode of proving Sylvester's theorem of 1839 is improved upon (§ 3, 11) by taking Q one order lower than P, and using the multipliers $\partial Q/\partial b_{11}, \partial Q/\partial b_{21}, \ldots, \partial Q/\partial b_{n-1,1}$ on the identities

$$\left.\begin{array}{l} a_{11}\dfrac{\partial P}{\partial a_{n1}} + a_{12}\dfrac{\partial P}{\partial a_{n2}} + \ldots\ldots + a_{1n}\dfrac{\partial P}{\partial a_{nn}} = 0 \\ a_{21}\dfrac{\partial P}{\partial a_{n1}} + a_{22}\dfrac{\partial P}{\partial a_{n2}} + \ldots\ldots + a_{2n}\dfrac{\partial P}{\partial a_{nn}} = 0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{n-1,1}\dfrac{\partial P}{\partial a_{n1}} + a_{n-1,2}\dfrac{\partial P}{\partial a_{n2}} + \ldots\ldots + a_{n-1,n}\dfrac{\partial P}{\partial a_{nn}} = 0 \end{array}\right\},$$

the result of addition then being

$$\begin{aligned} &\left(a_{11}\frac{\partial Q}{\partial b_{11}} + a_{21}\frac{\partial Q}{\partial b_{21}} + \ldots\ldots + a_{n-1,1}\frac{\partial Q}{\partial b_{n-1,1}}\right)\frac{\partial P}{\partial a_{n1}} \\ +&\left(a_{12}\frac{\partial Q}{\partial b_{11}} + a_{22}\frac{\partial Q}{\partial b_{21}} + \ldots\ldots + a_{n-1,2}\frac{\partial Q}{\partial b_{n-1,1}}\right)\frac{\partial P}{\partial a_{n2}} \\ +&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ +&\left(a_{1n}\frac{\partial Q}{\partial b_{11}} + a_{2n}\frac{\partial Q}{\partial b_{21}} + \ldots\ldots + a_{n-1,n}\frac{\partial Q}{\partial b_{n-1,1}}\right)\frac{\partial P}{\partial a_{nn}} = 0, \end{aligned}$$

which, if we bear in mind what single determinants the expressions in brackets stand for, is seen to be Sylvester's theorem in its alternative form as the assertion of the vanishing of an aggregate of products of pairs of determinants.

The theorem formulated by Bellavitis regarding a zero determinant is appropriately based (§ 7, 5) on the vanishing of the two-line minors of the adjugate determinant—a course suggested by what Lebesgue did in 1837.

Cayley's development of 1847 is well stated (§ 8, 6) in the form

$$\mathrm{D} + \Sigma a_{ii}\mathrm{D}_i + \Sigma a_{ii}a_{kk}\mathrm{D}_{ik} + \Sigma a_{ii}a_{kk}a_{ll}\mathrm{D}_{ikl} + \ \ldots\ + a_{11}a_{22}\ldots a_{nn}\,,$$

where D is what the given determinant becomes when all its diagonal elements are made 0, and $\mathrm{D}_{ik\ldots}$ is the minor of D got by deleting the i^{th}, k^{th}, . . . rows and the i^{th}, k^{th}, . . . columns; and the proof consists in showing that no term is thus neglected or repeated.

Newman, F. (1857).

[On determinants, better called eliminants. *Proceedings Roy. Soc. London*, viii. pp. 426–431: or *Philos. Magazine* (4), xiv. p. 392.]

The author's object was merely to recommend the introduction of the subject into elementary text-books.

Del Grosso, R. (1857).

[Sulla regola secondo la quale debbono procedere i segni nello sviluppo d'un determinante in prodotti di determinanti minori. *Rendic. . . . Accad. Pontaniana*, Ann. v. pp. 196–198.]

When a determinant is expressed in accordance with Laplace's theorem as an aggregate of products of complementary minors, Del Grosso directs that the sign of any product is to be $(-1)^\sigma$, where σ is the sum of the odd row-numbers and odd column-numbers of one of the factors. The rule is not stated with sufficient care, and the author in reaching it concludes too hastily that the simplest case is all that need be established.

Janni, G. (1858).

[Saggio di una Teorica Elementare de' Determinanti, del Sacerdote Giuseppe Janni. . . . 40 pp. Napoli.]

Janni's professed object was to make determinants more readily accessible, previous text-books having, he says, either totally neglected demonstrations or used those of great difficulty. He speaks of the work as the first of a series, and its contents certainly look like the first five chapters of a text-book planned on a fairly large scale. The theorems, twenty-three in number, are carefully enunciated and are printed in italics; but, although the proofs receive every attention, it is very doubtful whether the object aimed at was to any extent accomplished. There is at any rate nothing sufficiently fresh in the treatment to warrant attention here.

ZEHFUSS, G. (1858).

[Ueber die Auflösung der linearen endlichen Differenzengleichungen mit variabeln Coefficienten. *Zeitschrift f. Math. u. Phys.*, iii. pp. 175–177.]

His solution suggests to Zehfuss the remark (p. 177) that every determinant can be expressed as a multiple integral. It will suffice to give the result in the case of a determinant of the 4th order. Denoting $\cos 2\pi\theta + \sqrt{-1}\sin 2\pi\theta$ by 1^θ, and putting P for

$$1^\alpha 1^\beta 1^\gamma 1^\delta (1^\delta - 1^\gamma)(1^\delta - 1^\beta)(1^\delta - 1^\alpha)(1^\gamma - 1^\beta)(1^\gamma - 1^\alpha)(1^\beta - 1^\alpha)$$

and Q for

$$\begin{aligned} &(a_1 1^{-\alpha} + b_1 1^{-\beta} + c_1 1^{-\gamma} + d_1 1^{-\delta}) \\ \times &(a_2 1^{-2\alpha} + b_2 1^{-2\beta} + c_2 1^{-2\gamma} + d_2 1^{-2\delta}) \\ \times &(a_3 1^{-3\alpha} + b_3 1^{-3\beta} + c_3 1^{-3\gamma} + d_3 1^{-3\delta}) \\ \times &(a_4 1^{-4\alpha} + b_4 1^{-4\beta} + c_4 1^{-4\gamma} + d_4 1^{-4\delta}), \end{aligned}$$

Zehfuss says that

$$\sum \pm a_1 b_2 c_3 d_4 = \underline{\overline{\iiiint}}_0^1 \mathrm{PQ}\,d\alpha d\beta d\gamma d\delta .$$

He does not, however, note in passing that

$$\mathrm{P} = \begin{vmatrix} 1^\alpha & 1^\beta & 1^\gamma & 1^\delta \\ 1^{2\alpha} & 1^{2\beta} & 1^{2\gamma} & 1^{2\delta} \\ 1^{3\alpha} & 1^{3\beta} & 1^{3\gamma} & 1^{3\delta} \\ 1^{4\alpha} & 1^{4\beta} & 1^{4\gamma} & 1^{4\delta} \end{vmatrix} .$$

ZEHFUSS, G.: MAINARDI, G. (1858).

[Ueber die Zeichen der einzelnen Glieder einer Determinante. *Zeitschrift f. Math. u. Phys.*, iii. pp. 249–250.]

[Una regola per attribuire il segno proprio ad ogni parte di un determinante numerico. *Atti . . . Istituto Lombardo* (Milano), i. pp. 105–106.]

Neither of these communications is of importance. Zehfuss, using the recurrent law of formation and giving "dérangement" the very opposite of its original meaning, so that the principal term of an n-line determinant has $\frac{1}{2}n(n-1)$ derangements, seeks to show that the sign of any other term having μ derangements is $(-1)^{\frac{1}{2}n(n-1)+\mu}$.

Mainardi, employing Cauchy's "clefs algébriques," finds himself also face to face with derangements, and seriously advises that in counting them we should say, not 1, 2, 3, 4, 5, . . . , but 1, 2, 1, 2, 1, . . . , the sign being − or + according as we end with 1 or 2.

GALLENKAMP, W. (1858).

[Die einfachsten Eigenschaften und Anwendungen der Determinanten. 12 pp. Sch. Progr. Duisburg.]

A workmanlike twelve-page exposition.

SPERLING, I. (1858).

[Teorija opredělitelej i eja važnějsija priloženija. Č. 1. St Petersburg.]

This dissertation I have failed to see. In English the title is, *The Theory of Determinants and its most important applications.* The letters used here in transliterating the Russian title have German values.

CASORATI, F. (1858, September).

[Intorno ad alcuni punti della teoria dei minimi quadrati. *Annali di Mat.*, i. pp. 329–343.]

The title here refers only to the latter half of the paper, the other half being concerned with an auxiliary series of theorems on the product-determinant. The first of these theorems is avowedly old, being that which concerns the so-called product C of two non-quadrate arrays

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn}, \end{matrix} \qquad \begin{matrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{m1} & b_{m2} & b_{m3} & \dots & b_{mn}, \end{matrix}$$

where $n > m$. The second, though not so spoken of, is only new in form, and concerns any primary minor of C. Unfortunately, Casorati does not observe that any primary minor of C is a determinant formed exactly like

C after omitting a row from the first array and a row from the second, and that therefore his second theorem is unnecessary. Further, his mode of procedure leads him to an expression for a multiple of the minor, namely, for

$$(n-m+1)\frac{\partial C}{\partial c_{rs}},$$

and making an oversight similar to Hesse's of 1853, he does not divide both sides by $n-m+1$.

His third theorem,

$$C\frac{\partial C}{\partial c_{rs}} = \frac{\partial C}{\partial a_{r1}}\frac{\partial C}{\partial b_{s1}} + \frac{\partial C}{\partial a_{r2}}\frac{\partial C}{\partial b_{s2}} + \ldots\ldots + \frac{\partial C}{\partial a_{rm}}\frac{\partial C}{\partial b_{sm}},$$

is more worthy of note. The proof of it depends essentially on substituting for C in the first factor of each term of the right-hand member its equivalent,

$$c_{r1}\frac{\partial C}{\partial c_{r1}} + c_{r2}\frac{\partial C}{\partial c_{r2}} + \ldots\ldots + c_{rm}\frac{\partial C}{\partial c_{rm}},$$

in which, it is important to note, the differential-quotients are necessarily all independent of $a_{r1}, a_{r2}, \ldots\ldots$ The said right-hand member can then be transformed into

$$\begin{aligned} &\frac{\partial C}{\partial b_{s1}}\left(b_{11}\frac{\partial C}{\partial c_{r1}} + b_{21}\frac{\partial C}{\partial c_{r2}} + \ldots\ldots + b_{m1}\frac{\partial C}{\partial c_{rm}}\right) \\ +&\frac{\partial C}{\partial b_{s2}}\left(b_{12}\frac{\partial C}{\partial c_{r1}} + b_{22}\frac{\partial C}{\partial c_{r2}} + \ldots\ldots + b_{m2}\frac{\partial C}{\partial c_{rm}}\right) \\ +&\ldots\ldots\ldots\ldots\ldots\ldots \\ +&\frac{\partial C}{\partial b_{sn}}\left(b_{1n}\frac{\partial C}{\partial c_{r1}} + b_{2n}\frac{\partial C}{\partial c_{r2}} + \ldots\ldots + b_{mn}\frac{\partial C}{\partial c_{rm}}\right), \end{aligned}$$

which, if addition be performed columnwise, becomes

$$0+0+\ldots+C\frac{\partial C}{\partial c_{rs}}+0+\ldots+0,$$

because of the fact that the theorem

$$c_{r1}\frac{\partial C}{\partial c_{s1}} + c_{r2}\frac{\partial C}{\partial c_{s2}} + \ldots\ldots + c_{rn}\frac{\partial C}{\partial c_{sn}} = \left.\begin{matrix} C \\ 0 \end{matrix}\right\} \text{ when } \left\{\begin{matrix} r=s \\ r\neq s \end{matrix}\right.$$

holds in reference to the a's and b's as well as to the c's—a fact which should be noted for other purposes, and which is readily seen to be justifiable if we view C in its composite form AB and bear in mind that the operation

$$\alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial b} + \gamma\frac{\partial}{\partial c} + \ldots,$$

when performed on a homogeneous linear function of $a, b, c, \ldots$ is equivalent to a substitution.

The case where the two given arrays are identical is formulated, due care being taken with the differential-quotients because of C becoming axisymmetric.

We have only to add that the form in which this new theorem of Casorati's is stated obscures to some extent its significance. If we write the case of $AB = C$ where $m=3$, $n=4$ in the form

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{Vmatrix} \cdot \begin{Vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \end{Vmatrix} = \begin{vmatrix} \Sigma al & \Sigma am & \Sigma an \\ \Sigma bl & \Sigma bm & \Sigma bn \\ \Sigma cl & \Sigma cm & \Sigma cn \end{vmatrix},$$

then, freed from all reference to differentiation, the theorem for the case $r=2$, $s=3$ is

$$- \begin{vmatrix} \Sigma al & \Sigma am & \Sigma an \\ \Sigma bl & \Sigma bm & \Sigma bn \\ \Sigma cl & \Sigma cm & \Sigma cn \end{vmatrix} \cdot \begin{vmatrix} \Sigma al & \Sigma am \\ \Sigma cl & \Sigma cm \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma al & \Sigma am & \Sigma an \\ l_1 & m_1 & n_1 \\ \Sigma cl & \Sigma cm & \Sigma cn \end{vmatrix} \cdot \begin{vmatrix} \Sigma al & \Sigma am & a_1 \\ \Sigma bl & \Sigma bm & b_1 \\ \Sigma cl & \Sigma cm & c_1 \end{vmatrix} + \begin{vmatrix} \Sigma al & \Sigma am & \Sigma an \\ l_2 & m_2 & n_2 \\ \Sigma cl & \Sigma cm & \Sigma cn \end{vmatrix} \cdot \begin{vmatrix} \Sigma al & \Sigma am & a_2 \\ \Sigma bl & \Sigma bm & b_2 \\ \Sigma cl & \Sigma cm & c_2 \end{vmatrix}$$

$$+ \begin{vmatrix} \Sigma al & \Sigma am & \Sigma an \\ l_3 & m_3 & n_3 \\ \Sigma cl & \Sigma cm & \Sigma cn \end{vmatrix} \cdot \begin{vmatrix} \Sigma al & \Sigma am & a_3 \\ \Sigma bl & \Sigma bm & b_3 \\ \Sigma cl & \Sigma cm & c_3 \end{vmatrix} + \begin{vmatrix} \Sigma al & \Sigma am & \Sigma an \\ l_4 & m_4 & n_4 \\ \Sigma cl & \Sigma cm & \Sigma cn \end{vmatrix} \cdot \begin{vmatrix} \Sigma al & \Sigma am & a_4 \\ \Sigma bl & \Sigma bm & b_4 \\ \Sigma cl & \Sigma cm & c_4 \end{vmatrix}.$$

Further, no change but substitution is necessary on passing to the case where the two arrays are identical.

SALMON, G. (1859).

[LESSONS INTRODUCTORY TO THE MODERN HIGHER ALGEBRA. By the Rev. George Salmon, A.M. . . . xii+147 pp. Dublin.]

The first three lessons (pp. 1–18) of this historically interesting text-book are devoted to an elementary exposition of determinants. The only fresh matter (§ 20) concerns the determinant formed from

$$\begin{Vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{Vmatrix}, \quad \begin{Vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{Vmatrix}$$

by row-by-row multiplication. This is shown to vanish, not by pointing out that it contains at least one zero determinant of the third order as a factor, but by partitioning it into eight determinants with monomial elements, and showing that all the eight vanish.*

Unfortunately, for *terms* of a determinant the word "elements" is used, and for *adjugate* the word "reciprocal," although the elements of the adjugate are spoken of as the "inverse constituents."

SPERLING, J. F. DE (1860, April).

[Note sur un théorème de M. Sylvester relatif à la transformation du produit de déterminants du même ordre. *Journ.* (*de Liouville*) *de Math.* . . . (2), v. pp. 121–126.]

This is a carefully formulated proof of Sylvester's theorem of 1839 and the extended theorem of 1851, the lines followed being those suggested and illustrated by Cayley in 1843. Unfortunately, however, instead of extending Cayley's method to prove directly and at once the generalisation of 1851, Sperling repeats Cayley's proof of the simpler theorem, and then uses the method of so-called mathematical induction to arrive at the generalisation.

The two determinants whose product is the subject of discussion being $| a_{11} \, a_{22} \, . \, . \, . \, a_{nn} |$ and $| b_{11} \, b_{22} \, . \, . \, . \, b_{nn} |$, or, say, A and B, he forms the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n-m} & \cdot & \cdot & \cdots & a_{1n} & b_{11} & b_{12} & \cdots & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n-m} & \cdot & \cdot & \cdots & a_{2n} & b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n,n-m} & \cdot & \cdot & \cdots & a_{nn} & b_{n1} & b_{n2} & \cdots & b_{nn} \\ \cdot & \cdot & \cdots & \cdot & a_{1,n-m+1} & a_{1,n-m+2} & \cdots & a_{1,n-1} \; a_{1n} & b_{11} & b_{12} & \cdots & b_{1n} \\ \cdot & \cdot & \cdots & \cdot & a_{2,n-m+1} & a_{2,n-m+2} & \cdots & a_{2,n-1} \; a_{2n} & b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & a_{n,n-m+1} & a_{n,n-m+2} & \cdots & a_{n,n-1} \; a_{nn} & b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix},$$

---

* In using the notation $\| \;\; \|$ he is not more explicit than its author, Cayley. If it were explained that

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Vmatrix}$$

stands for

$$| a_1 b_2 |, \quad | a_1 b_3 |, \quad | a_2 b_3 |,$$

it would readily follow that the statement

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Vmatrix} = 0$$

was short for

$$| a_1 b_2 |, \quad | a_1 b_3 |, \quad | a_2 b_3 | = 0 , 0 , 0 ;$$

and that

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Vmatrix} \cdot \begin{Vmatrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \end{Vmatrix}$$

was short for

$$\big(| a_1 b_2 |, \quad | a_1 b_3 |, \quad | a_2 b_3 | \between | a_1 \beta_2 |, \quad | a_1 \beta_3 |, \quad | a_2 \beta_3 | \big),$$

which, he says, is seen to vanish on trying to find Laplace's expansion of it in terms of minors formed from the last $n+1$ columns and the minors that are complementary of those. Then, noting that the like outcome is not met with when the boundary-line necessary for the application of the said expansion-theorem is horizontal and bisects the determinant, he sets about obtaining the terms of the latter development in orderly fashion. Clearly, the first factors of those terms are all alike as regards their first $n-m$ columns, but the remaining m columns may include another column of a's or may not. Making a separation of the terms in accordance with this distinction, and calling the one aggregate Σ_{m-1} and the other Σ_m, where the suffix corresponds with the number of columns of b's appearing in each first factor, and therefore also with the number of columns of a's appearing in each second factor, Sperling gives evidence that Σ_{m-1} is Sylvester's expansion for $|\,a_{11}\,a_{22}\ldots a_{nn}\,|\cdot|\,b_{11}\,b_{22}\ldots b_{nn}\,|$, when in the formation of it there is an interchange of $m-1$ columns, that Σ_m is the corresponding expansion due to an interchange of m columns, and that the two Σ's occur with different signs. The conclusion is thus reached that, if we have previously proved the identity $AB=\Sigma_{m-1}$, the identity $AB=\Sigma_m$ must follow.

It is important to note in passing that if Sperling had put zeros for $a_{1n}, a_{2n}, \ldots, a_{nn}$ in the second half of his $2n$-line determinant, its value then would have been, when obtained in one way, $(-1)^{m-1}AB$, and in another, $(-1)^{m-1}\Sigma_m$. He would thus have made the natural extension of Cayley's simple proof.

LIST OF AUTHORS

whose writings are herein dealt with.

(*Issued separately November* 6, 1908.)

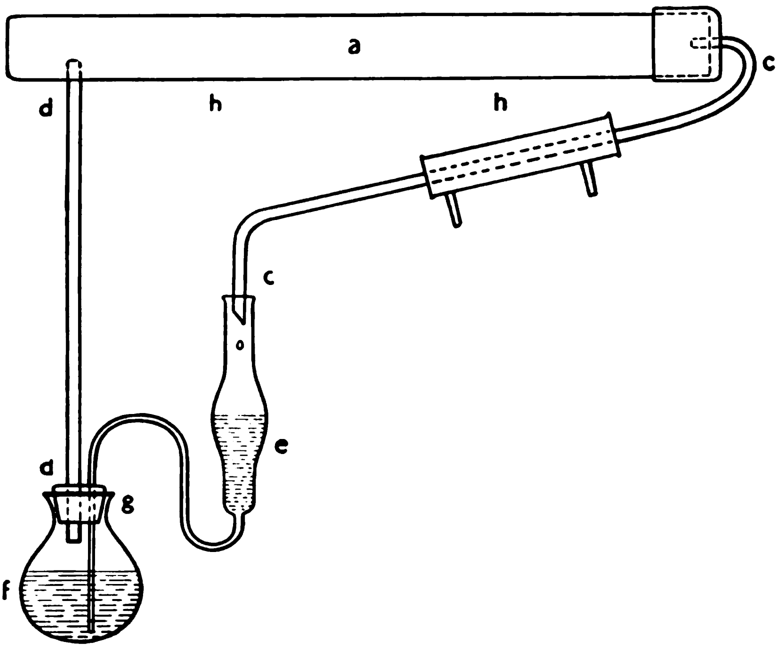
XLII.—An Improved Method of Esterification.

By G. E. Gibson, B.Sc.

(MS. received July 24, 1908. Read July 13, 1908.)

When a mixture of ethyl alcohol, benzene, and water is fractionated, a ternary mixture of all three distils over first at a temperature of 68° C.

Taylor * has made use of this fact in the preparation of the ethyl esters of high boiling-point acids.



His method is a great improvement on the old methods, but it is not continuous, and is tedious when large quantities of ester have to be prepared.

To remedy these defects the apparatus described in this paper was devised.

a is a wrought-iron tube 50 cm. long and of 5 cm. internal diameter. One end is closed and the other is fitted with a screw cap, to which is

* *Proc. Roy. Soc. Edin.*, 1905, p. 831.

attached the metal condenser *c c*. The tube *d d* is screwed in so that about 1 cm. of the end projects into the tube *a*.

The condenser tube fits loosely into a glass siphon tube *e* whose capacity is about 150 c.cm. The lower end of the siphon tube passes through the double-bored cork *g* and is kept well below the level of the liquid in the flask.

The siphon tube is open to the air, but all the other joints are air-tight.

The tube *a* is loosely filled with lumps of quicklime. The end of the entry tube *d* and of the condenser tube *c* are guarded with wads of glass wool to keep back the lime, which falls to fine powder when it is slaked.

The flask *f* is charged with acid to be esterified, mixed with excess of alcohol and about 100 c.cms. of benzene. A few drops of concentrated hydrochloric acid are added, and the apparatus is placed in position.

Two small gas flames at *h h* are then lighted, and when the tube *a* is sufficiently hot to prevent condensation of alcohol or benzene the flask is heated on the steam bath.

As the liquid in the flask boils, the vapour of the ternary mixture of alcohol, benzene, and water passes up the tube *d* and over the quicklime in the tube *a*, where the water is absorbed.

The mixed vapours of benzene and alcohol pass on and are condensed in the condenser *c*. The condensed liquid collects in the siphon tube *e* until the pressure of the liquid in the siphon is greater than the pressure in the flask.

The siphon then overflows, and the recovered benzene and alcohol flow back into the flask.

The steam in the steam bath should be regulated so that the siphon overflows about four times in an hour.

In one experiment the flask was charged with 100 gms. of tartaric acid, 300 gms. of ethyl alcohol, 90 c.cs. of benzene, and a few drops of hydrochloric acid.

After boiling for five hours in the apparatus, the alcohol and benzene were distilled off and the ester was distilled *in vacuo*.

125 gms. of pure tartaric ester were obtained, which is more than 90 per cent. of the theoretical yield. Good yields of tartronic ester have also been obtained, and it is clear that the method is applicable to all cases in which Taylor's method can be used.

(*Issued separately November* 6, 1908.)

XLIII.—Nitric Anhydride as a Nitrating Agent.

By G. E. Gibson, B.Sc.

(MS. received July 24, 1908. Read July 13, 1908.)

In the ordinary methods for the preparation of tartaric acid dinitrate, a mixture of fuming nitric acid with concentrated sulphuric acid is used as a nitrating agent.

The product is treated with iced water to remove the sulphuric and nitric acids.

As the dinitrate is decomposed by water, even at 0° C., it is impossible to avoid some loss in this process.

By using nitric anhydride as a nitrating agent this difficulty is avoided. Nitric acid, which is the only other product of the reaction, can be removed by evaporation *in vacuo*.

The ordinary methods for the preparation of nitric anhydride are somewhat troublesome. By the method described below, several hundred grams of nitric anhydride can be prepared in a day.

In the figure, A is a tubulated retort of 3 litres capacity, the neck of which has been bent to a right angle.

B is a glass dropping funnel of about 700 c.cs. capacity, the delivery tube of which is ground into the tubule of the retort. C is a platinum or glass receiver of about 1 litre capacity. It is surrounded by a freezing mixture of ice and salt. 150 grams of phosphoric anhydride are introduced into the retort, and the dropping funnel containing 500 c.c. of nitric acid (sp. gr. 1·52) is placed in position. From 200–300 c.c. of nitric acid are slowly dropped into the retort, and the mixture is left to stand, with occasional stirring, till nearly all the phosphoric anhydride is hydrated.

A stout glass rod, bent 3 inches from the lower end to an angle of 30°, is used to stir the mixture. To prevent loss of nitric anhydride, the space between the rod and the inside of the tubule of the retort is closed with an asbestos wad, which is made air-tight by means of a little of the syrupy mixture in the retort. The wad is easily made by wrapping asbestos paper round the glass rod.

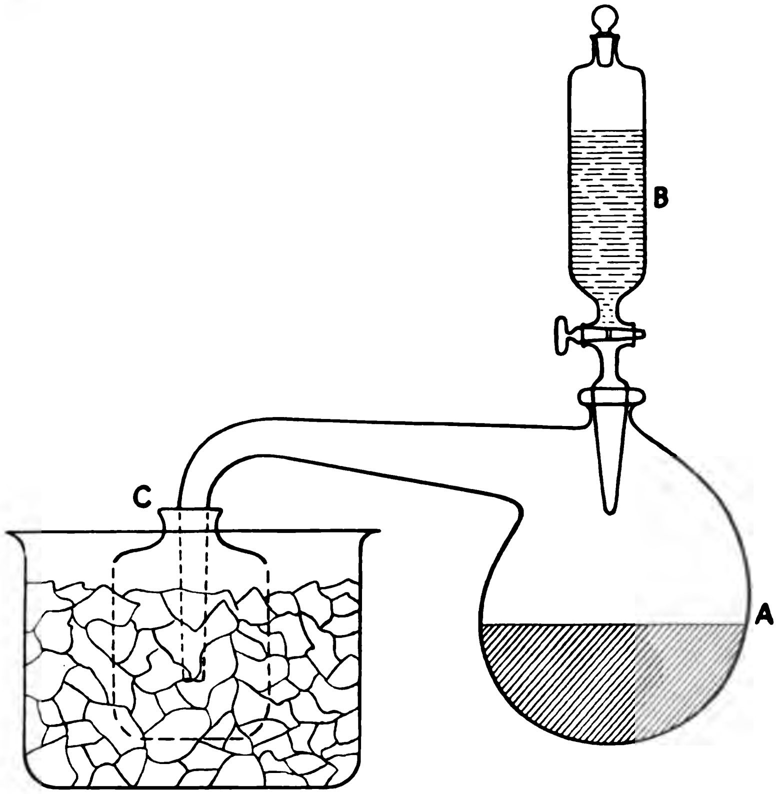
450 grams of phosphoric anhydride are now added and the remainder of the nitric acid is dropped in.

The process is completed by cautiously heating the retort, with occasional stirring.

In one experiment 350 grams of nitric anhydride were obtained. For the preparation of tartaric acid dinitrate a quantity of nitric anhydride slightly in excess of that required by the equation

$$C_4H_4O_4(OH)_2 + 2N_2O_5 = C_4H_4O_4(NO_3)_2 + 2HNO_3$$

is mixed with finely powdered tartaric acid. The mixture is allowed to stand over solid caustic soda in an evacuated desiccator for some hours, till it is practically free from nitric acid.



It is then treated with dry ether, which dissolves the dinitrate, but does not dissolve the unchanged tartaric acid. Most of the ether can be evaporated on the water bath, but the temperature of the solution should not be allowed to rise above 40° C. or decomposition may result. The last portions of ether are removed *in vacuo* over concentrated sulphuric acid.

130 grams of the dinitrate were obtained in this way from 100 grams of tartaric acid. This is 81 per cent. of the theoretical yield.

The following quantitative experiment was carried out in order to decide whether all the tartaric acid which took part in the reaction was converted into the dinitrate.

17·7 grams of very finely powdered tartaric acid were mixed with 30 grams of nitric anhydride. After removal of the nitric acid *in vacuo* over solid caustic soda, the weight was 27·3 grams.

On treatment with ether in a Soxhlet tube, 1·2 grams remained undissolved, and 26·0 grams of pure dinitrate were obtained.

According to the equation, 26·4 grams of dinitrate should be obtained from 16·5 grams of tartaric acid, so that there is a difference of about 1·5 per cent. between the weight as calculated from the formula and the weight of dinitrate actually obtained.

Prepared in this way the dinitrate is a white crystalline powder which decomposes like gun-cotton when touched with a hot glass rod. It is quite stable if kept perfectly dry, but on exposure to the air it is gradually reconverted into tartaric acid, with evolution of fumes of nitric acid.

The solution in water decomposes rapidly into oxalic acid. Very little tartaric acid is formed.

The wet ethereal solution decomposes into diketo-succinic acid, as described by Kekulé.*

On warming the dinitrate with aqueous alcohol, tartronic acid is obtained.† Some tartronic ester was prepared from tartronic acid made in this way, and was purified from oxalic ester by fractionation *in vacuo*.

Analyses and determinations of boiling point proved its identity with tartronic ester prepared in other ways.

There can therefore be no doubt of the identity of tartaric acid dinitrate, prepared by means of nitric anhydride, with the acide nitrotartrique of Dessaigne.

The action of nitric anhydride on other hydroxy acids is at present under investigation.

* *Ann.*, 221, 247. † *Ber.*, x., 1789.

(*Issued separately November* 13, 1908.)

XLIV.—Combustion Analysis. By James Walker, F.R.S., and Thomas Blackadder, B.Sc., University College, Dundee.

(MS. received July 6, 1908. Read same date.)

The process of Dennstedt for the elementary analysis of organic compounds by combustion in oxygen with the help of platinised quartz was tried in this laboratory, and in expert hands was found to be both rapid and accurate. The average student, however, experienced great difficulty in the conduct of the combustion, and it occurred to us that the advantages of the apparatus of Dennstedt, so carefully worked out by him in detail, might be applied to the ordinary method of combustion by means of copper oxide.

If one inspects a tube in which a copper oxide combustion is being conducted, it is found that the oxide actually reduced to metallic copper, after the combustion of the volatile matter is completed, rarely extends for more than an inch or two along the tube, unless the process has been accidentally "rushed." It seemed, therefore, possible to reduce the dimensions of the combustion tube to such an extent as to secure the advantage of the Dennstedt furnace, which from its lightness of construction admits of rapid heating and cooling. This we discovered to be the case, and in reality there is no greater difficulty in performing a copper oxide combustion in a shortened Dennstedt furnace, heated by three Bunsen burners, than in a furnace of the customary type heated by thirty. The saving of initial outlay on the furnace, and on the current consumption of gas is, of course, comparatively great, but a still more considerable advantage is that the combustion may be done on the worker's bench without inconvenience to himself or to his neighbours.

The furnace employed by us, together with the burners, absorption-tubes, and the purifying apparatus for the supply of oxygen, are practically all as described by Dennstedt, and shown to scale in fig. 1. The chief modification is that the furnace is cut down to 60 cm. in length. The combustion tube is of Jena glass, 66 cm. long, and need not be more than 8 mm. internal diameter. The total volume of such a tube is only about 30 c.c., and the charge of copper oxide, including spirals, weighs only 35 g. Two of the burners are supplied with attachments for spreading the flame into a flat sheet: the third is used for local heating, but towards the end of the combustion, when the whole tube is heated, it also may be provided with a flame-spreader.

The copper oxide employed is coarsely powdered and sifted free from fine dust, which during the combustion might clog and stop the tube. The combustion is carried out in a current of oxygen, and the calcium chloride and soda lime absorption tubes are always weighed filled with the same gas.

The method adopted for mixing the substance with copper oxide, and transferring it to the combustion tube, is a slight modification of that used

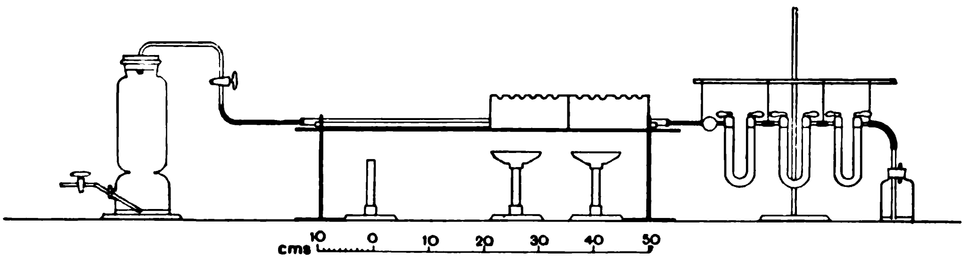


Fig. 1.

by Professor Thiele of Munich. The copper oxide after ignition is transferred while still hot to a tube A, fig. 2, which is then closed with a stopper, through which passes a calcium chloride tube to protect the copper oxide from atmospheric moisture. If the substance to be analysed is a solid, it is weighed off in a small stoppered bottle B, fig. 2, the neck of which fits into

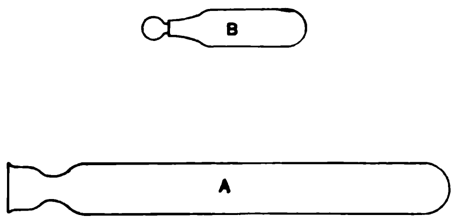


Fig. 2.

the constriction of A, so that the substance may be mixed in B with a quantity of copper oxide from A, with as little exposure to the atmosphere as possible. The end of the combustion tube fits into the wider part of A, so that it also may be conveniently charged with copper oxide from the ignited supply. After the combustion tube has received its charge of copper oxide, the mixture of substance and copper oxide is transferred from B into the combustion tube, the neck of the stoppered bottle being of such a diameter as to fit inside the end of the latter. The bottle is then "washed out" once or twice with copper oxide, received as before from A, and emptied into the combustion tube. The absorption tubes and oxygen supply are next attached and the combustion begun. The two burners at the front (absorption end) of the tube are lit and tiles are placed over them, the heat being as far as possible confined to the portion of the tube covered by the tiles by means of screens of asbestos paper. When the copper oxide has attained dull redness, the third Bunsen is lit at the other extremity of the tube without a tile. The heat from this Bunsen gradually volatilises the substance, with or without decomposition, the volatile products being for the most part burned in the moderately rapid stream of oxygen (2–3 bubbles per second) which is all the time passing through the apparatus. The Bunsen is gradually moved forward as the combustion proceeds, tiles being placed behind it to keep the tube still hot. Under ordinary conditions there is no visible reduction of copper oxide to metallic copper, although towards the end of the combustion the oxide usually glows immediately behind the tiles at the absorption end of the tube. To burn any carbon that may be left on the tube by decomposition, all the tiles are placed in position, and the three burners adjusted so that the tube is heated as uniformly as possible to dull redness. The carbon at this temperature is not graphitised and burns off readily. When the oxygen comes freely through the indicator bottle at the end of the apparatus, the combustion is finished, and very little sweeping out is necessary, owing to the small volume of the apparatus. Immediately after the combustion, the copper oxide is returned to the tube A, and is ready for the next analysis.

The time occupied between attaching and removing the absorption tubes need not exceed half an hour. The amount of oxygen consumed in a combustion averages 800 c.c.

In order that an idea may be formed of the rapidity and accuracy of the method, the following results may be quoted. Four combustions of succinic acid were performed, including weighing, between 9 a.m. and 1 p.m., two sets of absorption tubes being used, in order that no time might be lost in waiting for the tubes to cool before weighing.

Quantity taken.	Percentage Carbon.	Percentage Hydrogen.
0·1866 g.	40·61	5·28
0·1492 g.	40·44	5·20
0·1890 g.	40·48	5·18
0·1733 g.	40·56	5·20

The theoretical percentages for $C_4H_6O_4$ are, carbon = 40·68, hydrogen = 5·09.

If the substance contains nitrogen, a metallic copper spiral 7 cm. long is, as usual, placed at the front of the tube, and the current of oxygen is passed at a slower rate at the beginning of the combustion. For acetanilide the following percentages were obtained:—carbon, 71·03, 71·24; hydrogen, 6·72, 6·89, the numbers for the formula C_8H_9NO being 71·11 and 6·67 respectively.

When the substance is a liquid it is weighed off in a small bulb with a sealed capillary, the tip of the capillary being broken off before its introduction into the tube. The best results are got when the capillary is comparatively wide and about 8 cm. long. The open end of the capillary, surrounded by the copper oxide, faces the stream of oxygen. The liquid, when the copper oxide in the front of the tube has reached dull redness, is slowly distilled out of the bulb into the copper oxide at the cool end of the tube by means of a small Bunsen flame applied directly to the upper side of the tube over the bulb. The combustion is then continued as for a solid.

When a volatile liquid, such as benzene, is burned, it is introduced into the tube in a small sealed bulb terminating in a capillary at either end. The longer capillary (8 cm.) is plugged with fusible metal (compare Hempel, *Gas Analysis*, p. 341), and the shorter (3 cm.) is sealed off in the flame after the bulb has received its charge of benzene. In this case it is advisable not to surround the bulb and capillaries with copper oxide, but to leave the whole clear at the back of the tube. The little plug of fusible metal at the end facing the current of oxygen is melted by the application of the flame above the combustion tube, and the benzene is gradually vaporised by very gentle heating. The following result was obtained by this method:—carbon = 91·94 per cent., hydrogen = 7·83 per cent; calculated for C_6H_6—carbon = 92·31 per cent., hydrogen = 7·69 per cent.

A combustion tube of the same dimensions may be used for estimating nitrogen by the direct method. In this case a current of carbon dioxide is substituted for the current of oxygen, the nitrogen being collected over concentrated potash solution. The reduced copper spiral necessary in this case need not exceed 7 cm. in length. The percentage of nitrogen in aniline found by this method was in two experiments 15·09 and 15·25, the percentage required for the formula C_6H_7N being 15·05. Equally satisfactory results were obtained with acetanilide and hippuric acid.

To show that the method gives satisfactory results in the hands of a beginner, the following consecutive analyses may be quoted from the notebook of a student who had no previous experience of organic combustions:—

		Carbon.	Hydrogen.
Succinic acid.	Found I.	39·91	5·19
	„ II.	40·54	5·20
	Calculated	40·68	5·09
Cane sugar.	Found I.	40·83	6·48
	„ II.	41·81	6·63
	„ III.	42·12	6·41
	Calculated	42·11	6·43

It is apparent, then, that the method as here described is well adapted to ordinary laboratory work, and compares very favourably with the copper oxide method as usually employed in respect of economy in apparatus, gas, and time, and of the greatly increased comfort of the operator.

(*Issued separately November* 13, 1908.)

APPENDIX.

CONTENTS.

PROCEEDINGS OF THE STATUTORY GENERAL MEETING,

The 125th Session.

At the Annual Statutory Meeting, Monday, 28th October 1907,

Dr Robert Munro in the Chair,

the Minutes of last Annual Statutory Meeting of 22nd October 1906, and of the Special General Meeting held on 21st December 1906, were read, approved, and signed.

On the motion of Professor CUNNINGHAM, seconded by Dr R. H. TRAQUAIR, Mr W. L. CALDERWOOD and Dr W. S. BRUCE were appointed Scrutineers, and the Ballot for the New Council commenced.

The SECRETARY laid on the table the TREASURER's Accounts for the past year. These, with the Auditors' Report thereon, were read and approved.

Dr KNOTT made a sympathetic reference to the late Dr BUCHAN, Vice-President of the Society, and moved the following expression of sympathy with his family, seconded by the Chairman:—

"That the Fellows of the Royal Society at this their stated Meeting desire to express the deep regret which they feel at the death of Dr ALEXANDER BUCHAN, Vice-President of the Society, and at the same time to acknowledge their deep indebtedness to Dr BUCHAN for the great service he rendered the Society for many years as a Councillor and as Curator of the Library."

It was further resolved that a copy of this resolution be forwarded to Dr HILL BUCHAN, with an expression of the Society's deep sympathy with him in his loss.

The Scrutineers reported that the following New Council had been duly elected:—

The Right Hon. LORD KELVIN, G.C.V.O., LL.D., D.C.L., F.R.S., President.
ROBERT MUNRO, M.A., M.D., LL.D.,
Professor ANDREW GRAY, M.A., LL.D., F.R.S.,
RAMSAY H. TRAQUAIR, M.D., LL.D., F.R.S.,
Professor CRUM BROWN, M.D., LL.D., F.R.S.,
Professor J. C. EWART, M.D., F.R.S.,
JOHN HORNE, LL.D., F.R.S.,
} Vice-Presidents.
Professor GEORGE CHRYSTAL, LL.D., General Secretary.
Professor D. J. CUNNINGHAM, M.D., LL.D., F.R.S.,
CARGILL G. KNOTT, D.Sc.,
} Secretaries to Ordinary Meetings.
JAMES CURRIE, M.A., Treasurer.
JOHN S. BLACK, M.A., LL.D., Curator of Library and Museum.

COUNCILLORS.

B. N. PEACH, LL.D., F.R.S., F.G.S.
JAMES J. DOBBIE, M.A., D.Sc., F.R.S.
Professor GEORGE A. GIBSON, M.A., LL.D.
Professor J. ARTHUR THOMSON, M.A.
Professor E. A. SCHÄFER, LL.D., F.R.S.
The Hon. LORD M'LAREN, LL.D.
Professor F. O. BOWER, M.A., D.Sc., F.R.S.
Professor T. HUDSON BEARE, M.Inst.C.E.
Professor F. W. DYSON, F.R.S., Astronomer Royal in Scotland.
Professor D'ARCY W. THOMPSON, C.B.
O. CHARNOCK BRADLEY, M.D.
CHARLES TWEEDIE, M.A.

On the motion of Professor T. HUDSON BEARE, seconded by Professor CRUM BROWN, thanks were voted to the Scrutineers.

On the motion of Professor CUNNINGHAM, seconded by LORD M'LAREN, thanks were voted to the Auditors, who were reappointed.

On the motion of the CHAIRMAN thanks were voted to the General Secretary for his services to the Society during the past year; and

On the motion of Dr FERGUSON, seconded by Professor CRUM BROWN, thanks were voted to the Chairman.

RAMSAY H. TRAQUAIR, *V.-P.*,
Chairman.

PROCEEDINGS OF THE ORDINARY MEETINGS,

Session 1907–1908.

FIRST ORDINARY MEETING.

Monday, 4th November 1907.

Dr R. H. Traquair, F.R.S., Vice-President, in the Chair.

The following gentlemen signed the Roll, and were formally admitted Fellows of the Society:—Mr Muhammad Badre and Mr John Thomson Pearce, B.A., B.Sc.

The Chairman read a Statement regarding the gift by the Misses Sang to the Nation of the late Dr Sang's MS. *Logarithmic and Mathematical Tables*, which are now, by deed of gift, in the custody of the Society. (*See* pp. 183–196.)

The following Communications were read:—

1. A Note on the Roman Numerals. By James A. S. Barrett, M.A. Communicated by J. Sutherland Black, M.A., LL.D. pp. 161–182.
2. Brachydactyly: An Account of a number of Individuals, all related to one another by descent, who show a Congenital Abnormality of the Hands and Feet, consisting in an Abortive Middle Phalanx, with other peculiarities, together with references to their normal relatives. By H. Drinkwater, M.D., C.M. (Edin.), M.R.C.S. (Lond.). Communicated by Professor D. J. Cunningham, F.R.S. (*With Lantern Illustrations.*) pp. 35–57.
3. *Notolepis Coatsi*, Poisson pélagique nouveau recueilli par l'Expédition Antarctique Nationale Écossaise. Note préliminaire, par Louis Dollo, Conservateur au Musée royal d'Histoire naturelle à Bruxelles. Présentée par M. R. H. Traquair, M.D., LL.D., F.R.S. pp. 58–65.
4. The Theory of Compound Determinants in the Historical Order of Development up to 1860. By Thomas Muir, LL.D. pp. 197–209.
5. The Product of the Primary Minors of an n-by-$(n+1)$ Array. By the Same. pp. 210–216.

The following gentlemen, nominated for Honorary Fellowships, were balloted for and declared duly elected:—

As British Honorary Fellows.

Sir Alexander B. W. Kennedy, LL.D., F.R.S., Pres. Inst. C.E.
Sir E. Ray Lankester, K.C.B., LL.D., F.R.S., Director of the Natural History Departments, British Museum.
Sir James A. H. Murray, LL.D., D.C.L., Editor of the Oxford English Dictionary.
Charles S. Sherrington, M.A., M.D., LL.D., F.R.S., Holt Professor of Physiology in the University of Liverpool.

As Foreign Honorary Fellows.

Emil Fischer, Professor of Chemistry, University of Berlin.
George William Hill, Ph.D., Sc.D., LL.D. West Nyack, New York.
Friedrich Wilhelm Georg Kohlrausch, Pres. of the Physikalisch-Technische Reichsanstalt, Charlottenburg.
Henry Fairfield Osborn, Professor of Zoology, Columbia University, and Curator of the Department of Vertebrate Palæontology, American Museum of Natural History.
Iwan P. Pawlov, Professor of Physiology, Military Medical Academy, St Petersburg.
Gustaf Retzius, formerly Professor of Anatomy, Stockholm.
Augusto Righi, Professor of Physics in the University of Bologna.
Louis Joseph Troost, Member of the Institute of France, formerly Professor of Chemistry at the Sorbonne, Paris.

SECOND ORDINARY MEETING.

Monday, 18th November 1907.

Professor A. Gray, LL.D., F.R.S., Vice-President, in the Chair.

Mr James Archibald, M.A., signed the Roll, and was duly admitted a Fellow of the Society.

The following Communications were read:—

1. The Effect of Load and Vibrations upon Magnetism in Nickel. By James Russell, Esq. (*With Lantern Illustrations.*)
2. The Shift of the Neutral Points due to Variation of the Intensity of Mechanical Vibrations

or Electric Oscillations superposed upon Cyclic Magnetisation in Iron. By JAMES RUSSELL, Esq. (*With Lantern Illustrations.*)

3. Scottish Rotifers collected by the Lake Survey. By JAMES MURRAY, Esq. *Trans.*, vol. xlvi. pp. 189–201.

THIRD ORDINARY MEETING.

Monday, 2nd December 1907.

Dr Robert Munro, Vice-President, in the Chair.

The following Communications were read:—

1. The Body Temperature of Fishes and other Marine Animals. By SUTHERLAND SIMPSON, M.D., D.Sc. Communicated by Professor SCHÄFER, F.R.S. pp. 66–84.
2. Seismic Radiations through the Earth. By C. G. KNOTT, D.Sc. pp. 217–230.
3. The Theory of Skew Determinants in the Historical Order of Development up to 1865. By THOMAS MUIR, LL.D. pp. 303–310.

FOURTH ORDINARY MEETING.

Monday, 16th December 1907.

Dr R. H. Traquair, F.R.S., Vice-President, in the Chair.

The following Communications were read:—

1. The Medusæ of the Scottish National Antarctic Expedition. By EDWARD T. BROWNE, Zoological Research Laboratory, University College, London. Communicated by Dr W. S. BRUCE. *Trans.*, vol. xlvi. pp. 233–251.
2. The Method of extracting Venom from Poisonous Snakes in India. By Lieut.-Colonel W. B. BANNERMAN, M.D., B.Sc., I.M.S., Director, Bombay Bacteriological Laboratory, Parel, Bombay. (*With Lantern Illustrations.*)

The following Candidates for Fellowship were balloted for, and declared duly elected Fellows of the Society:—Lieut.-Colonel JOHN CAMPBELL and WALTER AUBREY KIDD, M.D., F.Z.S.

FIFTH ORDINARY MEETING.

Monday, 6th January 1908.

Professor Crum Brown, LL.D., F.R.S., Vice-President, in the Chair.

Mr JOHN KEMP, M.A., signed the Roll, and was duly admitted a Fellow of the Society.

The Chairman made the following reference to the death of the President of the Society:—

We meet here to-night for the first time since the death of Lord Kelvin.

This is not the time to enter into an enumeration or a criticism of what he did. Our thoughts now are of the loss which we have sustained. But it is impossible in our mind to separate the man from his work. For the transparent truthfulness, the simplicity and straightforwardness, the absence of the least trace of affectation or trick, which contributed so much to the charm of his manner, felt by everyone who came, even in the slightest and most transient way, into relation with him, are to be seen in all that he did. It was his love of truth and his sympathy with nature that led him in all his investigations directly to the root of the matter, and made him so zealous and successful in his searches for the essential principles underlying the phenomena of nature. And when a truly essential new view was obtained, by himself or by another, of the way in which nature works, he rejoiced greatly and called on his friends to rejoice with him. Nature was to him very real, and no demonstration seemed to him quite satisfactory until it had been "realised." This, and his sympathy with men and with their work, gave everything to him a practical aspect. And so in almost every direction in which he worked he devised working models and instruments of precision. Some of these are known only to specialists, and by them used and valued; but everybody has heard of his compass and of his sounding apparatus, and knows something of the enormous benefits he has conferred on navigation.

It was not only in pure and applied science that he was interested; everything that affects the life of the people, education, politics, religion, occupied his thoughts, and on all subjects which he had seriously considered he had definite opinions. While he would, on occasion, defend with zeal and energy what he believed to be the truth, he was always perfectly fair to his opponents, as he was always courteous to everybody.

We have already had emphatic evidence that the world knows that a great and good man has left us; we who knew him more intimately also mourn a dear, trustworthy, and trusted friend.

The following Communication was then read:—

On the Fossil Osmundaceæ. Part II. By D. T. Gwynne-Vaughan, M.A., Birkbeck College, London, and Robert Kidston, F.R.S., F.G.S. *Trans.*, vol. xlvi. pp. 213–232.

SIXTH ORDINARY MEETING.

Monday, 20th January 1908.

Professor J. C. Ewart, M.D., F.R.S., Vice-President, in the Chair.

The following Communications were read:—

1. The Arterial Pressure in Man. I.—Methods. By G. A. Gibson, M.D., D.Sc., F.R.C.P.E. (*With Lantern Illustrations.*) pp. 343–355.
2. Seismic Radiations. Part II. By C. G. Knott, D.Sc. pp. 217–230.

SEVENTH ORDINARY MEETING.

Monday, 3rd February 1908.

Professor Andrew Gray, LL.D., F.R.S., Vice-President, in the Chair.

The following Communications were read:—

1. Sensitive State induced in Magnetic Materials by Thermal Treatment. By James G. Gray, B.Sc., Lecturer on Physics in the University of Glasgow, and Alexander D. Ross, M.A., B.Sc., Assistant to the Professor of Natural Philosophy in the University of Glasgow. Communicated by Professor A. Gray, F.R.S. pp. 239–248.
2. The Growth and Development of the Limbs of the Penguin. By David Waterston, M.A., M.D., and A. C. Geddes, M.B. (*With Lantern Illustrations.*)

The following Candidates for Fellowship were balloted for, and duly elected Fellows of the Society:—Archibald Hewat, President of the Faculty of Actuaries in Scotland, F.I.A., Theophilus Bulkeley Hyslop, M.D., C.M., M.R.C.P.E., and Thomas Francis Cavanagh, M.D.

EIGHTH ORDINARY MEETING.

Monday, 17th February 1908.

Dr John Horne, F.R.S., Vice-President, in the Chair.

Mr Archibald Hewat signed the Roll, and was duly admitted a Fellow of the Society.

The following Communications were read:—

1. The Systematic Motions of the Stars. By Professor F. W. Dyson, F.R.S. pp. 231–233.
2. Preliminary Note on *Lepidophloios Scottii*, a New Species from the Calciferous Sandstone Series at Pettycur. By W. T. Gordon, M.A., B.Sc. Communicated by Professor Geikie. *Trans.*, vol. xlvi. pp. 443–453.
3. The Middle Cells of the Grey Matter of the Spinal Cord. By Dr J. H. Harvey Pirie. Communicated by Dr Alex. Bruce. pp. 595–614.
4. On *q*-Functions and a certain Difference Operator. By the Rev. F. H. Jackson. Communicated by Professor George Chrystal. *Trans.*, vol. xlvi. pp. 253–281.

The following Candidates for Fellowship were balloted for, and duly elected Fellows of the Society:—James Ireland Craig, M.A., B.A., William Dawson Henderson, M.A., B.Sc., Ph.D., Andrew William Kerr, F.S.A.Scot., William Davidge Page, F.C.S., F.G.S., M.Inst.M.E., Linsdall Richardson, F.G.S., F.L.S., Henry Charles Williamson, M.A., D.Sc., and George Cossar Pringle, M.A.

NINTH ORDINARY MEETING.

Monday, 2nd March 1908.

Dr John Horne, F.R.S., Vice-President, in the Chair.

The following Communications were read:—

1. A Preliminary Notice of New Iron-Bacteria. By Dr David Ellis. pp. 338–342.
2. The Effect of Load and Vibrations upon Magnetism in Nickel. Supplementary Communication. (*With Lantern Illustrations.*) By James Russell, Esq.
3. A Simplified Calendar. By Alex. Philip, LL.B. Communicated by Dr C. G. Knott.

TENTH ORDINARY MEETING.

Monday, 16*th March* 1908.

Dr R. H. Traquair, F.R.S., Vice-President, in the Chair.

Mr J. P. F. Bell, signed the Roll, and was admitted a Fellow of the Society.

The following Communications were read :—

1. On the Lamellibranch Fauna found in the Millstone Grit of Scotland. By Dr Wheelton Hind, F.R.C.S., F.G.S. Communicated by Dr J. Horne, F.R.S. *Trans.*, vol. xlvi. pp. 331–359.
2. The Lamellibranchs from the Silurian Rocks of Girvan. By Dr Wheelton Hind, F.R.C.S., F.G.S. Communicated by Dr J. Horne, F.R.S.
3. On a Test for Continuity. By Dr W. H. Young, F.R.S. Communicated by J. H. Maclagan Wedderburn, M.A. pp. 249–258.
4. The Theory of Hessians in the Historical Order of Development up to 1860. By Dr Thomas Muir. pp. 413–432.

ELEVENTH ORDINARY MEETING.

Monday, 4*th May* 1908.

Professor Crum Brown, F.R.S., Vice-President, in the Chair.

Mr Henry Charles Williamson signed the Roll, and was formally admitted a Fellow of the Society.

The following Communications were read :—

1. On Iodine Concentration Cells. By Principal A. P. Laurie. pp. 382–393.
2. Sunset and Twilight Curves, and related Phenomena. By D. M. Y. Sommerville. Communicated by Professor Dyson. pp. 311–337.
3. Preliminary Statement on the Morphology of the Cone of *Lycopodium Cernuum*, and its bearing on the Affinities of *Spencerites*. By William H. Lang, M.B., D.Sc., Lecturer in Botany, Glasgow University. Communicated by Professor F. O. Bower, Sc.D., F.R.S.L. & E. (*With Lantern Illustrations.*) pp. 356–368.
4. On the Origin of the Adaxially Curved Leaf-trace in the Filicales. By D. T. Gwynne-Vaughan, M.A., and R. Kidston, LL.D., F.R.S.L. & E. (*With Lantern Illustrations.*) pp. 433–436.
5. On a New Species of *Dineuron* and of *Botryopteris*, from Pettycur, Fife. By R. Kidston, LL.D., F.R.S.L. & E. (*With Lantern Illustrations.*) *Trans.*, vol. xlvi. pp. 361–364.
6. The Inca Bone: Its Homology and Nomenclature. By Dr W. Ramsay Smith. pp. 586–594.

FIRST SPECIAL MEETING.

Monday, 11*th May* 1908.

Dr R. H. Traquair, F.R.S., Vice-President, in the Chair.

At the request of the Council, Professor F. O. Bower, F.R.S., gave an Address on :—

The Origin of a Land Flora. (*With Lantern Illustrations.*)

TWELFTH ORDINARY MEETING.

Monday, 18*th May* 1908.

Professor Andrew Gray, LL.D., F.R.S., Vice-President, in the Chair.

The following Communications were read :—

1. Obituary Notice of the Right Hon. Lord Kelvin. By the Chairman.
2. On the Cohesion of Steel, and on the Relation between the Yield Points in Tension and in Compression. By G. H. Gulliver, B.Sc. pp. 374–381.
3. The Preparation of a Glass to Conduct Electricity. By Charles E. S. Phillips, Esq. (*With Lantern Illustrations.*) pp. 627–642.

The following Candidates for Fellowship were balloted for, and declared duly elected Fellows of the Society:—George Freeland Barbour Simpson, M.D., F.R.C.P.E., F.R.C.S.E., and Devendra Nath Mallik, B.A. (Cantab.), B.Sc. (Lond.).

THIRTEENTH ORDINARY MEETING.

Monday, 1st June 1908.

Dr John Horne, F.R.S., Vice-President, in the Chair.

Dr W. D. HENDERSON signed the Roll, and was duly admitted a Fellow of the Society.

The following Communications were read:—

1. Note on some points in the Anatomy of a Trilobite, *Calymene blumenbachii*. By MALCOLM LAURIE, B.A., D.Sc. (*With Lantern Illustrations and a Model.*)
2. Experiments with Heusler's Magnetic Alloy. By JAS. G. GRAY, B.Sc., Lecturer on Physics in the University of Glasgow. Communicated by Professor A. GRAY, F.R.S. pp. 403–412.
3. Note on the Electrical Resistance of Spark Gaps. By ROBERT A. HOUSTON, Ph.D., D.Sc. Communicated by Professor A. GRAY, F.R.S. pp. 369–378.
4. Treatment of Aneurism by Electrolysis. By Dr DAWSON TURNER. (*With Lantern Illustrations.*) pp. 400–402.
5. Exhibition of Professor Leduc's Photographs of Growth due to Osmosis, and the Microscopic Structure of such Growth. By Dr DAWSON TURNER.
6. On the "Negative" Viscosity of Aqueous Solutions. By Dr W. W. TAYLOR and T. W. MOORE, M.A., B.Sc. Communicated by Professor A. CRUM BROWN. pp. 461–471.

FOURTEENTH ORDINARY MEETING.

Monday, 15th June 1908.

Dr R. H. Traquair, F.R.S., Vice-President, in the Chair.

Mr J. I. CRAIG signed the Roll, and was admitted a Fellow of the Society.

The following Communications were read:—

1. On the Reducing Action of Electrolytic Hydrogen on Arsenious and Arsenic Acids when liberated from the surface of different Metals. By WM. THOMSON, Esq. (*With Lantern Illustrations.*)
2. The Theory of the Micro-barograph, and on some Observations with the Dines-Shaw Instrument. By Professor CHRYSTAL. pp. 437–460.
3. On the Effect on the Metabolism of Chloroform administered by Different Channels. By Professor D. NOËL PATON. pp. 472–496.
4. On the Rate of Elimination of Chloroform when administered by Different Channels. By DOROTHY E. LINDSAY and Professor D. NOËL PATON. pp. 497–502.
5. Astéries, Ophiures et Échinides de l'Expédition Antarctique Nationale Écossaise. Par R. KOEHLER, Professeur de Zoologie à l'Université de Lyon. Présentée par M. le Dr W. S. BRUCE. *Trans.*, vol. xlvi.
6. Les Holothuries de l'Expédition Antarctique Nationale Écossaise. Par CLÉMENT VANEY, Maître de Conférences de Zoologie à la Faculté des Sciences de Lyon. Présentée par M. le Dr W. S. BRUCE. *Trans.*, vol. xlvi. pp. 405–441.

The following Candidates for Fellowship were balloted for, and declared duly elected Fellows of the Society:—JOHN RENNIE, D.Sc., and THOMAS WOOD, M.D.

SECOND SPECIAL MEETING.

Monday, 22nd June 1908.

Professor J. C. Ewart, F.R.S., Vice-President, in the Chair.

The following Communications were read:—

1. Equilibrium in the System Water, and a pair of Enantiomorph Solids. By W. W. TAYLOR, D.Sc., and THEODORE RETTIE, D.Sc. Communicated by Professor CRUM BROWN.
2. On the Electrolytic Conductivity of Aqueous Solutions of Lactic Acid. By JOHN GIBSON, Ph.D., and ANDREW KING, F.I.C.
3. On Changes in Electrolytic Conductivity accompanying the Alcoholic Fermentation. By JOHN GIBSON, Ph.D., and ANDREW KING, F.I.C.
4. On an Improved Thermostat, electrically controlled, and other Apparatus for the accurate determination of Electrolytic Conductivity. By JOHN GIBSON, Ph.D., and G. E. GIBSON, B.Sc.
5. Determinations of the Electrolytic Conductivity of Concentrated Aqueous Solutions of good Electrolytes:—

    (1) Hydriodic and Hydrobromic Acids. By JOHN GIBSON, Ph.D., and ANDREW KING, F.I.C.

(2) Hydrochloric Acid. By JOHN GIBSON, Ph.D., and W. H. PATERSON.
(3) Ammonium Bromide, Lithium Bromide, and Sodium Bromide. By JOHN GIBSON, Ph.D., and E. B. R. PRIDEAUX, D.Sc.

6. On the Precipitation of certain Chlorides by Hydrochloric Acid. By JOHN GIBSON, Ph.D., and R. B. DENISON, D.Sc., Ph.D. (*With Lantern Illustrations.*)
7. Andrews' Measurements of the Compression of Carbon Dioxide and of Mixtures of Carbon Dioxide and Nitrogen. By Dr C. G. KNOTT.

FIFTEENTH ORDINARY MEETING.

Monday, 6th July 1908.

Professor A. Crum Brown, F.R.S., Vice-President, in the Chair.

The following Communications were read:—

1. The Craniology of the Aborigines of Tasmania. By Principal Sir WM. TURNER, K.C.B. *Trans.*, vol. xlvi. pp. 365-403.
2. Inversion Temperatures, and the Form of the Equation of State. By Professor W. PEDDIE. pp. 394–399.
3. Magnetic Quality in the most open Cubic Arrangement of Molecular Magnets. By Professor W. PEDDIE. pp. 643–651.
4. On Energy Accelerations and Partition of Energy. By C. W. FOLLETT, Esq. Communicated by Professor W. PEDDIE.
5. Combustion Analysis. By Professor JAMES WALKER, F.R.S., and THOMAS BLACKADDER, B.Sc. pp. 708–712.

THIRD SPECIAL MEETING.

Monday, 13th July 1908.

Professor A. Crum Brown, F.R.S., Vice-President, in the Chair.

The following Communications were read:—

1. An Improved Method of Esterification. By G. E. GIBSON, B.Sc. Communicated by Professor A. CRUM BROWN. pp. 703–704.
2. Nitric Anhydride as a Nitrating Agent. By G. E. GIBSON, B.Sc. Communicated by Professor A. CRUM BROWN. pp. 705–707.
3. On the Significance of Maximum Electrolytic Conductivity. By Professor JOHN GIBSON. (*With Lantern Illustrations.*)
4. The Variation of Young's Modulus under an Electric Current. Part II. By HENRY WALKER, M.A., B.Sc. Communicated by Professor J. G. MACGREGOR, F.R.S. pp. 652–675.
5. The Theory of General Determinants in the Historical Order of Development up to 1860. By THOMAS MUIR, LL.D. pp. 676–702.

SIXTEENTH AND LAST ORDINARY MEETING.

Monday, 20th July 1908.

Dr R. H. Traquair, F.R.S., Vice-President, in the Chair.

Mr ALEXANDER GALBRAITH signed the Roll, and was duly admitted a Fellow of the Society.

PRIZES.

In presenting the KEITH PRIZE the Chairman read the following statement:—

The Council have awarded the Keith Prize for the biennial period 1905–1907 to Dr ALEXANDER BRUCE, for his paper entitled "Distribution of the Cells in the Intermedio-Lateral Tract of the Spinal Cord," published in the *Transactions* of the Society within the period.

Dr Bruce early turned his attention to the elucidation of the finer anatomy of the brain and spinal cord. Few fields of investigation require a greater amount of patience and more laborious treatment, and yet, although Dr Bruce has only been able to devote the few leisure hours he has been able to snatch from the responsible and arduous duties of a consulting physician, he has attained a high reputation both at home and abroad in this line of work.

Dr Bruce has published many important papers and memoirs on Neurology, both in the *Proceedings* of this Society and elsewhere; but the particular essay which the Council have deemed worthy of the award of the Keith Medal and Prize is one which appeared in the *Transactions* in 1906, and entitled "The Distribution of the Cells in the Intermedio-Lateral Tract of the Spinal Cord." It is a memoir of very high merit. It provides for the first time a complete anatomical picture of an exceedingly important column of cells, which since 1851, when first described by Lockhart Clarke, has been much in the mind of the anatomist and the physiologist.

Dr Bruce shows us that this column is neither continuous nor yet uniform. Its continuity is broken at certain well-defined points, whilst its contour is distinctly moniliform. He further lays stress upon the suggestive correlation which exists between the regions of outflow of sympathetic fibres from the cord and the distribution of the cells in the cord of the intermedio-lateral tract. This correlation offers strong presumptive evidence that these sympathetic fibres arise as the axons of these cells.

In presenting the Neill Prize for the triennial period 1904–1907 to Mr Frank J. Cole, B.Sc., for his paper entitled "A Monograph on the General Morphology of the Myxinoid Fishes, based on a study of Myxine," published in the *Transactions* of the Society, regard being also paid to Mr Cole's other valuable contributions to the Anatomy and Morphology of Fishes, the Chairman said:—

The Neill Prize for the triennial period 1904–1907 has been awarded to Mr Frank J. Cole for his beautiful and valuable papers on the anatomy of the skeleton and of the muscles of the Hagfish (*Myxine*) published in the *Transactions* of the Society, and which form the first instalments of a comprehensive monograph "On the General Morphology of the Myxinoid Fishes." Regard has also been paid by the Council to Mr Cole's other contributions to the anatomy and morphology of Fishes. In these works his attention has been principally devoted to the nervous system and sense organs, and the principal result obtained has been the formulation of the component theory of the nervous system, now almost universally accepted. This theory was initiated by Strong in America and by Mr Cole in Great Britain, and although it has been considerably elaborated and somewhat modified since by many workers, they were the pioneers. Numerous other subsidiary theoretical questions have also been dealt with by Mr Cole, such as the origin of the lateral line system of sense organs in Fishes, the facial nerve of man, the morphology of the skull, and the asymmetry of the Flat-fishes.

The following Communications were read:—

1. On a Sensitive State induced in Magnetic Materials by Thermal Treatment. Part II. By James G. Gray, B.Sc., Lecturer on Physics in the University of Glasgow, and Alexander D. Ross, M.A., B.Sc., Assistant to the Professor of Natural Philosophy in the University of Glasgow. Communicated by Professor A. Gray, F.R.S. pp. 615–626.
2. The Structure of *Turrilepas peachi* and its Allies. By F. R. Cowper Reed, M.A., F.G.S. Communicated by Dr Horne, F.R.S. *Trans.*, vol. xlvi. pp. 519–528.
3. On the Recalescence of Nickel. By T. A. Lindsay, M.A., B.Sc., Carnegie Scholar, Physical Laboratory, Edinburgh University. Communicated by Professor J. G. MacGregor, F.R.S.
4. Note on the Study of Polarisation by means of the Dolezalek Electrometer. By A. F. Ewan, Esq., Physical Laboratory, Edinburgh University. Communicated by Professor J. G. MacGregor, F.R.S.
5. Preliminary Note on the Action of Nitric Anhydride on Mucic Acid. By Professor A. Crum Brown, F.R.S., and G. E. Gibson, B.Sc.
6. The Meteorology of the Weddell Quadrant and Adjacent Areas. By R. C. Mossman, Esq.

The following Candidates for Fellowship were balloted for, and declared duly elected Fellows of the Society:—James Hunter Harvey Pirie, B.Sc., M.D., M.R.C.P.E., Alexander Cameron Miller, M.D., F.S.A.Scot., and Harry Drinkwater, M.D., M.R.C.S. (Eng.).

FOURTH SPECIAL MEETING.

Monday, 19th October 1908.

Dr Robert Munro, Vice-President, in the Chair.

The following Address was read:—

Prehistoric Japan. By Dr Neil Gordon Munro. (*With Lantern Illustrations.*)

LAWS OF THE SOCIETY,

As revised 26th October 1908.

[By the Charter of the Society (printed in the *Transactions*, vol. vi. p. 5), the Laws cannot be altered, except at a Meeting held one month after that at which the Motion for alteration shall have been proposed.]

I.

THE ROYAL SOCIETY OF EDINBURGH shall consist of Ordinary and Honorary Fellows. Title.

II.

Every Ordinary Fellow, within three months after his election, shall pay Two Guineas as the fee of admission, and Three Guineas as his contribution for the Session in which he has been elected; and annually at the commencement of every Session, Three Guineas into the hands of the Treasurer. This annual contribution shall continue for ten years after his admission, and it shall be limited to Two Guineas for fifteen years thereafter.* Fellows may compound for these contributions on such terms as the Council may from time to time fix. The fees of Ordinary Fellows residing in Scotland.

III.

All Fellows who shall have paid Twenty-five years' annual contribution shall be exempted from further payment. Payment to cease after 25 years.

IV.

The fees of admission of an Ordinary Non-Resident Fellow shall be £26, 5s., payable on his admission; and in case of any Non-Resident Fellow coming to reside at any time in Scotland, he shall, during each year of his residence, pay the usual annual contribution of £3, 3s., payable by each Resident Fellow; but after payment of such annual contribution for eight years, he shall be exempt from any further payment. In the case of any Resident Fellow ceasing to reside in Scotland, and wishing to continue a Fellow of the Society, it shall be in the power of the Council to determine on what terms, in the circumstances of each case, the privilege of remaining a Fellow of the Society shall be continued to such Fellow while out of Scotland. Fees of Non-Resident Ordinary Fellows. Case of Fellows becoming Non-Resident.

* A modification of this rule, in certain cases, was agreed to at a Meeting of the Society held on the 3rd January 1831.

At the Meeting of the Society, on the 5th January 1857, when the reduction of the Contributions from £3, 3s. to £2, 2s., from the 11th to the 25th year of membership, was adopted, it was resolved that the existing Members shall share in this reduction, so far as regards their future annual Contributions.

V.

Defaulters.

Members failing to pay their contributions for three successive years (due application having been made to them by the Treasurer) shall be reported to the Council, and, if they see fit, shall be declared from that period to be no longer Fellows, and the legal means for recovering such arrears shall be employed.

VI.

Privileges of Ordinary Fellows.

None but Ordinary Fellows shall bear any office in the Society, or vote in the choice of Fellows or Office-Bearers, or interfere in the patrimonial interests of the Society.

VII.

Numbers unlimited.

The number of Ordinary Fellows shall be unlimited.

VIII.

Fellows entitled to Transactions and Proceedings.

All Ordinary Fellows of the Society who are not in arrear of their Annual Contributions shall be entitled to receive, gratis, copies of the parts of the Transactions of the Society which shall be published subsequent to their admission, upon application, either personally or by an authorised agent, to the Librarian, provided they apply for them within five years of the date of publication of such parts.

Copies of the parts of the Proceedings shall be distributed to all Fellows of the Society, by post or otherwise, as soon as may be convenient after publication.

IX.

Mode of Recommending Ordinary Fellows.

Candidates for admission as Ordinary Fellows shall make an application in writing, and shall produce along with it a certificate of recommendation to the purport below,* signed by at least *four* Ordinary Fellows, two of whom shall certify their recommendation from personal knowledge. This recommendation shall be delivered to the Secretary, and by him laid before the Council, and shall be exhibited publicly in the Society's rooms for one month, after which it shall be considered by the Council. If the Candidate be approved by the Council, notice of the day fixed for the election shall be given in the circulars of at least two Ordinary Meetings of the Society.

X.

Honorary Fellows, British and Foreign.

Honorary Fellows shall not be subject to any contribution. This class shall consist of persons eminently distinguished for science or literature. Its number shall not exceed Fifty-six, of whom Twenty may be British subjects, and Thirty-six may be subjects of foreign states.

* "A. B., a gentleman well versed in science (*or Polite Literature, as the case may be*), being "to our knowledge desirous of becoming a Fellow of the Royal Society of Edinburgh, we hereby "recommend him as deserving of that honour, and as likely to prove a useful and valuable "Member."

XI.

Personages of Royal Blood may be elected Honorary Fellows, without regard to the limitation of numbers specified in Law X. **Royal Personages.**

XII.

Honorary Fellows may be proposed by the Council, or by a recommendation (in the form given below*) subscribed by three Ordinary Fellows; and in case the Council shall decline to bring this recommendation before the Society, it shall be competent for the proposers to bring the same before a General Meeting. The election shall be by ballot, after the proposal has been communicated *viva voce* from the Chair at one Meeting, and printed in the circulars for Two Ordinary Meetings of the Society, previous to the day of election. **Recommendation of Honorary Fellows.** **Mode of election.**

XIII.

The election of Ordinary Fellows shall take place only at one Afternoon Ordinary Meeting of each month during the Session. The election shall be by ballot, and shall be determined by a majority of at least two-thirds of the votes, provided Twenty-four Fellows be present and vote. **Election of Ordinary Fellows.**

XIV.

The Ordinary Meetings shall be held on the first and third Mondays of each month from November to March, and from May to July, inclusive; with the exception that when there are five Mondays in January, the Meetings for that month shall be held on its second and fourth Mondays. Regular Minutes shall be kept of the proceedings, and the Secretaries shall do the duty alternately, or according to such agreement as they may find it convenient to make. **Ordinary Meetings.**

XV.

The Society shall from time to time publish its Transactions and Proceedings. For this purpose the Council shall select and arrange the papers which they shall deem it expedient to publish in the Transactions of the Society, and shall superintend the printing of the same. **The Transactions.**

XVI.

The Transactions shall be published in parts or *Fasciculi* at the close of each Session, and the expense shall be defrayed by the Society. **How Published.**

* We hereby recommend ____________________________________
for the distinction of being made an Honorary Fellow of this Society, declaring that each of us from our own knowledge of his services to (*Literature or Science, as the case may be*) believe him to be worthy of that honour.

(To be signed by three Ordinary Fellows.)

____

____

____

To the President and Council of the Royal Society of Edinburgh.

XVII.

The Council.

That there shall be formed a Council, consisting—First, of such gentlemen as may have filled the office of President; and Secondly, of the following to be annually elected, viz.:—a President, Six Vice-Presidents (two at least of whom shall be Resident), Twelve Ordinary Fellows as Councillors, a General Secretary, Two Secretaries to the Ordinary Meetings, a Treasurer, and a Curator of the Museum and Library.

The Council shall have power to regulate the private business of the Society. At any Meeting of the Council the Chairman shall have a casting as well as a deliberative vote.

XVIII.

Retiring Councillors.

Four Councillors shall go out annually, to be taken according to the order in which they stand on the list of the Council.

XIX.

Election of Office-Bearers.

An Extraordinary Meeting for the election of Office-Bearers shall be held annually on the fourth Monday of October, or on such other lawful day in October as the Council may fix, and each Session of the Society shall be held to begin at the date of the said Extraordinary Meeting.

XX.

Special Meetings; how called.

Special Meetings of the Society may be called by the Secretary, by direction of the Council; or on a requisition signed by six or more Ordinary Fellows. Notice of not less than two days must be given of such Meetings.

XXI.

Treasurer's Duties.

The Treasurer shall receive and disburse the money belonging to the Society, granting the necessary receipts, and collecting the money when due.

He shall keep regular accounts of all the cash received and expended, which shall be made up and balanced annually; and at the Extraordinary Meeting in October, he shall present the accounts for the preceding year, duly audited. At this Meeting, the Treasurer shall also lay before the Council a list of all arrears due above two years, and the Council shall thereupon give such directions as they may deem necessary for recovery thereof.

XXII.

Auditor.

At the Extraordinary Meeting in October, a professional accountant shall be chosen to audit the Treasurer's accounts for that year, and to give the necessary discharge of his intromissions.

XXIII.

General Secretary's Duties.

The General Secretary shall keep Minutes of the Extraordinary Meetings of the Society, and of the Meetings of the Council, in two distinct books. He shall, under the direction of the Council, conduct the correspondence of the Society, and superintend its publications. For these purposes he shall, when necessary, employ a clerk, to be paid by the Society.

XXIV.

The Secretaries to the Ordinary Meetings shall keep a regular Minute-book, in which a full account of the proceedings of these Meetings shall be entered; they shall specify all the Donations received, and furnish a list of them, and of the Donors' names, to the Curator of the Library and Museum; they shall likewise furnish the Treasurer with notes of all admissions of Ordinary Fellows. They shall assist the General Secretary in superintending the publications, and in his absence shall take his duty. **Secretaries to Ordinary Meetings.**

XXV.

The Curator of the Museum and Library shall have the custody and charge of all the Books, Manuscripts, objects of Natural History, Scientific Productions, and other articles of a similar description belonging to the Society; he shall take an account of these when received, and keep a regular catalogue of the whole, which shall lie in the hall, for the inspection of the Fellows. **Curator of Museum and Library.**

XXVI.

All articles of the above description shall be open to the inspection of the Fellows at the Hall of the Society, at such times and under such regulations as the Council from time to time shall appoint. **Use of Museum and Library.**

XXVII.

A Register shall be kept, in which the names of the Fellows shall be enrolled at their admission, with the date. **Register Book.**

XXVIII.

If, in the opinion of the Council of the Society, the conduct of any Fellow is unbecoming the position of a Member of a learned Society, or is injurious to the character and interests of this Society, the Council may request such Fellow to resign; and, if he fail to do so within one month of such request being addressed to him, the Council shall call a General Meeting of the Fellows of the Society to consider the matter; and, if a majority of the Fellows present at such Meeting agree to the expulsion of such Member, he shall be then and there expelled by the declaration of the Chairman of the said Meeting to that effect; and he shall thereafter cease to be a Fellow of the Society, and his name shall be erased from the Roll of Fellows, and he shall forfeit all right or claim in or to the property of the Society. **Power of Expulsion.**

THE KEITH, MAKDOUGALL-BRISBANE, NEILL, AND GUNNING VICTORIA JUBILEE PRIZES.

The above Prizes will be awarded by the Council in the following manner:—

I. KEITH PRIZE.

The Keith Prize, consisting of a Gold Medal and from £40 to £50 in Money, will be awarded in the Session 1909–1910 for the "best communication on a scientific subject, communicated,* in the first instance, to the Royal Society during the Sessions 1907–1908 and 1908–1909." Preference will be given to a paper containing a discovery.

II. MAKDOUGALL-BRISBANE PRIZE.

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some paper written on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded at the commencement of the Session 1908–1909, for an Essay or Paper having reference to any branch of scientific inquiry, whether Material or Mental.

2. Competing Essays to be addressed to the Secretary of the Society, and transmitted not later than 8th July 1908.

3. The Competition is open to all men of science.

4. The Essays may be either anonymous or otherwise. In the former case, they must be distinguished by mottoes, with corresponding sealed billets, superscribed with the same motto, and containing the name of the Author.

5. The Council impose no restriction as to the length of the Essays, which may be, at the discretion of the Council, read at the Ordinary Meetings of the Society.

* For the purposes of this award the word "communicated" shall be understood to mean the date on which the manuscript of a paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

They wish also to leave the property and free disposal of the manuscripts to the Authors; a copy, however, being deposited in the Archives of the Society, unless the paper shall be published in the Transactions.

6. In awarding the Prize, the Council will also take into consideration any scientific papers presented to the Society during the Sessions 1906–07, 1907–08, whether they may have been given in with a view to the prize or not.

III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr Patrick Neill of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate:

1. The Neill Prize, consisting of a Gold Medal and a sum of Money, will be awarded during the Session 1909–1910.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented * to the Society during the two years preceding the 23rd October 1909,—or failing presentation of a paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award.

IV. GUNNING VICTORIA JUBILEE PRIZE.

This Prize, founded in the year 1887 by Dr R. H. Gunning, is to be awarded quadrennially by the Council of the Royal Society of Edinburgh, in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics.

Evidence of such work may be afforded either by a Paper presented to the Society, or by a Paper on one of the above subjects, or some discovery in them elsewhere communicated or made, which the Council may consider to be deserving of the Prize.

The Prize consists of a sum of money, and is open to men of science resident in or connected with Scotland. The first award was made in the year 1887.

In accordance with the wish of the Donor, the Council of the Society may on fit occasions award the Prize for work of a definite kind to be undertaken during the three succeeding years by a scientific man of recognised ability.

* For the purposes of this award the word "presented" shall be understood to mean the date on which the manuscript of a paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

AWARDS OF THE KEITH, MAKDOUGALL-BRISBANE, NEILL, AND GUNNING VICTORIA JUBILEE PRIZES, FROM 1827 TO 1907.

I. KEITH PRIZE.

1st Biennial Period, 1827-29.—Dr Brewster, for his papers "on his Discovery of Two New Immiscible Fluids in the Cavities of certain Minerals," published in the Transactions of the Society.

2nd Biennial Period, 1829-31.—Dr Brewster, for his paper "on a New Analysis of Solar Light," published in the Transactions of the Society.

3rd Biennial Period, 1831-33.—Thomas Graham, Esq., for his paper "on the Law of the Diffusion of Gases," published in the Transactions of the Society.

4th Biennial Period, 1833-35.—Professor J. D. Forbes. for his paper "on the Refraction and Polarization of Heat," published in the Transactions of the Society.

5th Biennial Period, 1835-37.—John Scott Russell, Esq., for his researches "on Hydrodynamics," published in the Transactions of the Society.

6th Biennial Period, 1837-39.—Mr John Shaw, for his experiments "on the Development and Growth of the Salmon," published in the Transactions of the Society.

7th Biennial Period, 1839-41.—Not awarded.

8th Biennial Period, 1841-1843.—Professor James David Forbes, for his papers "on Glaciers," published in the Proceedings of the Society.

9th Biennial Period, 1843-45.—Not awarded.

10th Biennial Period, 1845-47.—General Sir Thomas Brisbane, Bart., for the Makerstoun Observations on Magnetic Phenomena, made at his expense, and published in the Transactions of the Society.

11th Biennial Period, 1847-49.—Not awarded.

12th Biennial Period, 1849-51.—Professor Kelland, for his papers "on General Differentiation, including his more recent Communication on a process of the Differential Calculus, and its application to the solution of certain Differential Equations," published in the Transactions of the Society.

13th Biennial Period, 1851-53.—W. J. Macquorn Rankine, Esq., for his series of papers "on the Mechanical Action of Heat," published in the Transactions of the Society.

14th Biennial Period, 1853-55.—Dr Thomas Anderson, for his papers "on the Crystalline Constituents of Opium, and on the Products of the Destructive Distillation of Animal Substances," published in the Transactions of the Society.

15th Biennial Period, 1855-57.—Professor Boole, for his Memoir "on the Application of the Theory of Probabilities to Questions of the Combination of Testimonies and Judgments," published in the Transactions of the Society.

16th Biennial Period, 1857-59.—Not awarded.

17th Biennial Period, 1859-61.—John Allan Broun, Esq., F.R.S., Director of the Trevandrum Observatory, for his papers "on the Horizontal Force of the Earth's Magnetism, on the Correction of the Bifilar Magnetometer, and on Terrestrial Magnetism generally," published in the Transactions of the Society.

18th Biennial Period, 1861-63.—Professor William Thomson, of the University of Glasgow, for his Communication "on some Kinematical and Dynamical Theorems."

19th Biennial Period, 1863-65.—Principal Forbes, St Andrews, for his "Experimental Inquiry into the Laws of Conduction of Heat in Iron Bars," published in the Transactions of the Society.

20TH BIENNIAL PERIOD, 1865–67.—Professor C. PIAZZI SMYTH, for his paper "on Recent Measures at the Great Pyramid," published in the Transactions of the Society.

21ST BIENNIAL PERIOD, 1867–69.—Professor P. G. TAIT, for his paper "on the Rotation of a Rigid Body about a Fixed Point," published in the Transactions of the Society.

22ND BIENNIAL PERIOD, 1869–71.—Professor CLERK MAXWELL, for his paper "on Figures, Frames, and Diagrams of Forces," published in the Transactions of the Society.

23RD BIENNIAL PERIOD, 1871–73.—Professor P. G. TAIT, for his paper entitled "First Approximation to a Thermo-electric Diagram," published in the Transactions of the Society.

24TH BIENNIAL PERIOD, 1873–75.—Professor CRUM BROWN, for his Researches "on the Sense of Rotation, and on the Anatomical Relations of the Semicircular Canals of the Internal Ear."

25TH BIENNIAL PERIOD, 1875–77.—Professor M. FORSTER HEDDLE, for his papers "on the Rhombohedral Carbonates," and "on the Felspars of Scotland," published in the Transactions of the Society.

26TH BIENNIAL PERIOD, 1877–79.—Professor H. C. FLEEMING JENKIN, for his paper "on the Application of Graphic Methods to the Determination of the Efficiency of Machinery," published in the Transactions of the Society; Part II. having appeared in the volume for 1877–78.

27TH BIENNIAL PERIOD, 1879–81.—Professor GEORGE CHRYSTAL, for his paper "on the Differential Telephone," published in the Transactions of the Society.

28TH BIENNIAL PERIOD, 1881–83.—THOMAS MUIR, Esq., LL.D., for his "Researches into the Theory of Determinants and Continued Fractions," published in the Proceedings of the Society.

29TH BIENNIAL PERIOD, 1883–85.—JOHN AITKEN, Esq., for his paper "on the Formation of Small Clear Spaces in Dusty Air," and for previous papers on Atmospheric Phenomena, published in the Transactions of the Society.

30TH BIENNIAL PERIOD, 1885–87.—JOHN YOUNG BUCHANAN, Esq., for a series of communications, extending over several years, on subjects connected with Ocean Circulation, Compressibility of Glass, etc.; two of which, viz., "On Ice and Brines," and "On the Distribution of Temperature in the Antarctic Ocean," have been published in the Proceedings of the Society.

31ST BIENNIAL PERIOD, 1887–89.—Professor E. A. LETTS, for his papers on the Organic Compounds of Phosphorus, published in the Transactions of the Society.

32ND BIENNIAL PERIOD, 1889–91.—R. T. OMOND, Esq., for his contributions to Meteorological Science, many of which are contained in vol. xxxiv. of the Society's Transactions.

33RD BIENNIAL PERIOD, 1891–93.—Professor THOMAS R. FRASER, F.R.S., for his papers on *Strophanthus hispidus*, Strophanthin, and Strophanthidin, read to the Society in February and June 1889 and in December 1891, and printed in vols. xxxv., xxxvi., and xxxvii. of the Society's Transactions.

34TH BIENNIAL PERIOD, 1893–95.—Dr CARGILL G. KNOTT, for his papers on the Strains produced by Magnetism in Iron and in Nickel, which have appeared in the Transactions and Proceedings of the Society.

35TH BIENNIAL PERIOD, 1895–97.—Dr THOMAS MUIR, for his continued communications on Determinants and Allied Questions.

36TH BIENNIAL PERIOD, 1897–99.—Dr JAMES BURGESS, for his paper "on the Definite Integral $\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt$, with extended Tables of Values," printed in vol. xxxix. of the Transactions of the Society.

37TH BIENNIAL PERIOD, 1899–1901.—Dr HUGH MARSHALL, for his discovery of the Persulphates, and for his Communications on the Properties and Reactions of these Salts, published in the Proceedings of the Society.

38TH BIENNIAL PERIOD, 1901–03.—Sir WILLIAM TURNER, K.C.B., LL.D., F.R.S., &c., for his memoirs entitled "A Contribution to the Craniology of the People of Scotland," published in the Transactions of the Society, and for his "Contributions to the Craniology of the People of the Empire of India," Parts I., II., likewise published in the Transactions of the Society.

39TH BIENNIAL PERIOD, 1903–05.—THOMAS H. BRYCE, M.A., M.D., for his two papers on "The Histology of the Blood of the Larva of *Lepidosiren paradoxa*," published in the Transactions of the Society within the period.

40TH BIENNIAL PERIOD, 1905–07.—ALEXANDER BRUCE, M.A., M.D., F.R.C.P.E., for his paper entitled "Distribution of the Cells in the Intermedio-Lateral Tract of the Spinal Cord," published in the Transactions of the Society within the period.

II. MAKDOUGALL-BRISBANE PRIZE.

1st Biennial Period, 1859.—Sir Roderick Impey Murchison, on account of his Contributions to the Geology of Scotland.

2nd Biennial Period, 1860-62.—William Seller, M.D., F.R.C.P.E., for his "Memoir of the Life and Writings of Dr Robert Whytt," published in the Transactions of the Society.

3rd Biennial Period, 1862-64.—John Denis Macdonald, Esq., R.N., F.R.S., Surgeon of H.M.S. "Icarus," for his paper "on the Representative Relationships of the Fixed and Free Tunicata, regarded as Two Sub-classes of equivalent value; with some General Remarks on their Morphology," published in the Transactions of the Society.

4th Biennial Period, 1864-66.—Not awarded.

5th Biennial Period, 1866-68.—Dr Alexander Crum Brown and Dr Thomas Richard Fraser, for their conjoint paper "on the Connection between Chemical Constitution and Physiological Action," published in the Transactions of the Society.

6th Biennial Period, 1868-70.—Not awarded.

7th Biennial Period, 1870-72.—George James Allman, M.D., F.R.S., Emeritus Professor of Natural History, for his paper "on the Homological Relations of the Cœlenterata," published in the Transactions, which forms a leading chapter of his Monograph of Gymnoblastic or Tubularian Hydroids—since published.

8th Biennial Period, 1872-74.—Professor Lister, for his paper "on the Germ Theory of Putrefaction and the Fermentive Changes," communicated to the Society, 7th April 1873.

9th Biennial Period, 1874-76.—Alexander Buchan, A.M., for his paper "on the Diurnal Oscillation of the Barometer," published in the Transactions of the Society.

10th Biennial Period, 1876-78.—Professor Archibald Geikie, for his paper "on the Old Red Sandstone of Western Europe," published in the Transactions of the Society.

11th Biennial Period, 1878-80.—Professor Piazzi Smyth, Astronomer-Royal for Scotland, for his paper "on the Solar Spectrum in 1877-78, with some Practical Idea of its probable Temperature of Origination," published in the Transactions of the Society.

12th Biennial Period, 1880-82.—Professor James Geikie, for his "Contributions to the Geology of the North-West of Europe," including his paper "on the Geology of the Faroes," published in the Transactions of the Society.

13th Biennial Period, 1882-84.—Edward Sang, Esq., LL.D., for his paper "on the Need of Decimal Subdivisions in Astronomy and Navigation, and on Tables requisite therefor," and generally for his Recalculation of Logarithms both of Numbers and Trigonometrical Ratios,—the former communication being published in the Proceedings of the Society.

14th Biennial Period, 1884-86.—John Murray, Esq., LL.D., for his papers "On the Drainage Areas of Continents, and Ocean Deposits," "The Rainfall of the Globe, and Discharge of Rivers," "The Height of the Land and Depth of the Ocean," and "The Distribution of Temperature in the Scottish Lochs as affected by the Wind."

15th Biennial Period, 1886-88.—Archibald Geikie, Esq., LL.D., for numerous Communications, especially that entitled "History of Volcanic Action during the Tertiary Period in the British Isles," published in the Transactions of the Society.

16th Biennial Period, 1889-90.—Dr Ludwig Becker, for his paper on "The Solar Spectrum at Medium and Low Altitudes," printed in vol. xxxvi. Part I. of the Society's Transactions.

17th Biennial Period, 1890-92.—Hugh Robert Mill, Esq., D.Sc., for his papers on "The Physical Conditions of the Clyde Sea Area," Part I. being already published in vol. xxxvi. of the Society's Transactions.

18th Biennial Period, 1892-94.—Professor James Walker, D.Sc., Ph.D., for his work on Physical Chemistry, part of which has been published in the Proceedings of the Society, vol. xx. pp. 255-263. In making this award, the Council took into consideration the work done by Professor Walker along with Professor Crum Brown on the Electrolytic Synthesis of Dibasic Acids, published in the Transactions of the Society.

19th Biennial Period, 1894-96.—Professor John G. M'Kendrick, for numerous Physiological papers, especially in connection with Sound, many of which have appeared in the Society's publications.

20th Biennial Period, 1896-98.—Dr William Peddie, for his papers on the Torsional Rigidity of Wires.

21st Biennial Period, 1898-1900.—Dr Ramsay H. Traquair, for his paper entitled "Report on Fossil Fishes collected by the Geological Survey in the Upper Silurian Rocks of Scotland," printed in vol. xxxix. of the Transactions of the Society.

22ND BIENNIAL PERIOD, 1900-02.—Dr ARTHUR T. MASTERMAN, for his paper entitled "The Early Development of *Cribrella oculata* (Forbes), with remarks on Echinoderm Development," printed in vol. xl. of the Transactions of the Society.

23RD BIENNIAL PERIOD, 1902-04.—Mr JOHN DOUGALL, M.A., for his paper on "An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate," published in vol. xli. of the Transactions of the Society.

24TH BIENNIAL PERIOD, 1904-06.—JACOB E. HALM, Ph.D., for his two papers entitled "Spectroscopic Observations of the Rotation of the Sun," and "Some Further Results obtained with the Spectroheliometer," and for other astronomical and mathematical papers published in the Transactions and Proceedings of the Society within the period.

III. THE NEILL PRIZE.

1ST TRIENNIAL PERIOD, 1856-59.—Dr W. LAUDER LINDSAY, for his paper "on the Spermogones and Pycnides of Filamentous, Fruticulose, and Foliaceous Lichens," published in the Transactions of the Society.

2ND TRIENNIAL PERIOD, 1859-61.—ROBERT KAYE GREVILLE, LL.D., for his Contributions to Scottish Natural History, more especially in the department of Cryptogamic Botany, including his recent papers on Diatomaceæ.

3RD TRIENNIAL PERIOD, 1862-65.—ANDREW CROMBIE RAMSAY, F.R.S., Professor of Geology in the Government School of Mines, and Local Director of the Geological Survey of Great Britain, for his various works and memoirs published during the last five years, in which he has applied the large experience acquired by him in the Direction of the arduous work of the Geological Survey of Great Britain to the elucidation of important questions bearing on Geological Science.

4TH TRIENNIAL PERIOD, 1865-68.—Dr WILLIAM CARMICHAEL M'INTOSH, for his paper "on the Structure of the British Nemerteans, and on some New British Annelids," published in the Transactions of the Society.

5TH TRIENNIAL PERIOD, 1868-71.—Professor WILLIAM TURNER, for his papers "on the Great Finner Whale; and on the Gravid Uterus, and the Arrangement of the Fœtal Membranes in the Cetacea," published in the Transactions of the Society.

6TH TRIENNIAL PERIOD, 1871-74.—CHARLES WILLIAM PEACH, Esq., for his Contributions to Scottish Zoology and Geology, and for his recent contributions to Fossil Botany.

7TH TRIENNIAL PERIOD, 1874-77.—Dr RAMSAY H. TRAQUAIR, for his paper "on the Structure and Affinities of *Tristichopterus alatus* (Egerton)," published in the Transactions of the Society, and also for his contributions to the Knowledge of the Structure of Recent and Fossil Fishes.

8TH TRIENNIAL PERIOD, 1877-80.—JOHN MURRAY, Esq., for his paper "on the Structure and Origin of Coral Reefs and Islands," published (in abstract) in the Proceedings of the Society.

9TH TRIENNIAL PERIOD, 1880-83.—Professor HERDMAN, for his papers "on the Tunicata," published in the Proceedings and Transactions of the Society.

10TH TRIENNIAL PERIOD, 1883-86.—B. N. PEACH, Esq., for his Contributions to the Geology and Palæontology of Scotland, published in the Transactions of the Society.

11TH TRIENNIAL PERIOD, 1886-89.—ROBERT KIDSTON, Esq., for his Researches in Fossil Botany, published in the Transactions of the Society.

12TH TRIENNIAL PERIOD, 1889-92.—JOHN HORNE, Esq., F.G.S., for his Investigations into the Geological Structure and Petrology of the North-West Highlands.

13TH TRIENNIAL PERIOD, 1892-95.—ROBERT IRVINE, Esq., for his papers on the Action of Organisms in the Secretion of Carbonate of Lime and Silica, and on the solution of these substances in Organic Juices. These are printed in the Society's Transactions and Proceedings.

14TH TRIENNIAL PERIOD, 1895-98.—Professor COSSAR EWART, for his recent Investigations connected with Telegony.

15TH TRIENNIAL PERIOD, 1898-1901.—Dr JOHN S. FLETT, for his papers entitled "The Old Red Sandstone of the Orkneys" and "The Trap Dykes of the Orkneys," printed in vol. xxxix. of the Transactions of the Society.

16TH TRIENNIAL PERIOD, 1901-04.—Professor J. GRAHAM KERR, M.A., for his Researches on *Lepidosiren paradoxa*, published in the Philosophical Transactions of the Royal Society, London.

17TH TRIENNIAL PERIOD, 1904-07.—FRANK J. COLE, B.Sc., for his paper entitled "A Monograph on the General Morphology of the Myxinoid Fishes, based on a study of Myxine," published in the Transactions of the Society, regard being also paid to Mr Cole's other valuable contributions to the Anatomy and Morphology of Fishes.

IV. GUNNING VICTORIA JUBILEE PRIZE.

1st Triennial Period, 1884-87.—Sir William Thomson, Pres. R.S.E., F.R.S., for a remarkable series of papers "on Hydrokinetics," especially on Waves and Vortices, which have been communicated to the Society.

2nd Triennial Period, 1887-90.—Professor P. G. Tait, Sec. R.S.E., for his work in connection with the "Challenger" Expedition, and his other Researches in Physical Science.

3rd Triennial Period, 1890-93.—Alexander Buchan, Esq., LL.D., for his varied, extensive, and extremely important Contributions to Meteorology, many of which have appeared in the Society's Publications.

4th Triennial Period, 1893-96.—John Aitken, Esq., for his brilliant Investigations in Physics, especially in connection with the Formation and Condensation of Aqueous Vapour.

1st Quadrennial Period, 1896-1900.—Dr T. D. Anderson, for his discoveries of New and Variable Stars.

2nd Quadrennial Period, 1900-04.—Sir James Dewar, LL.D., D.C.L., F.R.S., etc., for his researches on the Liquefaction of Gases, extending over the last quarter of a century, and on the Chemical and Physical Properties of Substances at Low Temperatures: his earliest papers being published in the Transactions and Proceedings of the Society.

THE COUNCIL OF THE SOCIETY,

October 1908.

PRESIDENT.

SIR WILLIAM TURNER, K.C.B., M.B., F.R.C.S.E., LL.D., D.C.L., D.Sc.Dub., F.R.S., Principal of the University of Edinburgh.

VICE-PRESIDENTS.

ANDREW GRAY, M.A., LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow.
RAMSAY H. TRAQUAIR, M.D., LL.D., F.R.S., F.G.S., late Keeper of the Natural History Collections in the Royal Scottish Museum, Edinburgh.
ALEXANDER CRUM BROWN, M.D., D.Sc., F.R.C.P.E., LL.D., F.R.S., Emeritus Professor of Chemistry in the University of Edinburgh.
JAMES COSSAR EWART, M.D., F.R.C.S.E., F.R.S., F.L.S., Regius Professor of Natural History in the University of Edinburgh.
JOHN HORNE, LL.D., F.R.S., F.G.S., Director of the Geological Survey of Scotland.
JAMES BURGESS, C.I.E., LL.D., M.R.A.S.

GENERAL SECRETARY.

GEORGE CHRYSTAL, M.A., LL.D., Professor of Mathematics in the University of Edinburgh.

SECRETARIES TO ORDINARY MEETINGS.

DANIEL JOHN CUNNINGHAM, M.D., D.Sc., LL.D., D.C.L., F.R.S., Professor of Anatomy in the University of Edinburgh.
CARGILL G. KNOTT, D.Sc., Lecturer on Applied Mathematics in the University of Edinburgh.

TREASURER.

JAMES CURRIE, M.A.

CURATOR OF LIBRARY AND MUSEUM.

JOHN SUTHERLAND BLACK, M.A., LL.D.

COUNCILLORS.

EDWARD ALBERT SCHÄFER, M.R.C.S., LL.D., F.R.S., Professor of Physiology in the University of Edinburgh.
THE HON. LORD M'LAREN, LL.D. Edin. and Glas., F.R.A.S., one of the Senators of the College of Justice.
FREDERICK O. BOWER, M.A., D.Sc., F.R.S., F.L.S., Regius Professor of Botany in the University of Glasgow.
THOMAS HUDSON BEARE, B.Sc., Memb. Inst. C.E., Professor of Engineering in the University of Edinburgh.
FRANK WATSON DYSON, M.A., F.R.S., Astronomer-Royal for Scotland, and Professor of Astronomy in the University of Edinburgh.
D'ARCY W. THOMPSON, C.B., B.A., F.L.S., Professor of Natural History in University College, Dundee.
O. CHARNOCK BRADLEY, M.D., D.Sc.
CHARLES TWEEDIE, M.A., B.Sc., Lecturer on Mathematics in the University of Edinburgh.
JOHN WALTER GREGORY, D.Sc., F.R.S., Professor of Geology in the University of Glasgow.
A. P. LAURIE, M.A., D.Sc., Principal of the Heriot-Watt College, Edinburgh.
WM. PEDDIE, D.Sc., Professor of Natural Philosophy in University College, Dundee.
HECTOR MUNRO MACDONALD, M.A., F.R.S., Professor of Mathematics in the University of Aberdeen.

ALPHABETICAL LIST OF THE ORDINARY FELLOWS OF THE SOCIETY,

Corrected to October 1908.

N. B.—*Those marked * are Annual Contributors.*

B. prefixed to a name indicates that the Fellow has received a Makdougall-Brisbane Medal.
K. ,, ,, ,, Keith Medal.
N. ,, ,, ,, Neill Medal.
V. J. ,, ,, ,, the Gunning Victoria Jubilee Prize.
C. ,, ,, ,, contributed one or more Communications to the Society's TRANSACTIONS or PROCEEDINGS.

Date of Election.		
1898	C.	* Abercromby, The Hon. John, 62 Palmerston Place
1898		Adami, Prof. J. G., M.A., M.D. Cantab., F.R.S., Professor of Pathology in M'Gill University, Montreal
1896		* Affleck, Jas. Ormiston, M.D., F.R.C.P.E., 38 Heriot Row
1871		Agnew, Sir Stair, K.C.B., M.A., Registrar-General for Scotland, 22 Buckingham Terrace
1875	C. K. V. J.	Aitken, John, LL.D., F.R.S., Ardenlea, Falkirk 5
1895		* Alford, Robert Gervase, Memb. Inst. C.E., 1 Windmill Hill, Hampstead, London, N.W.
1889		* Alison, John, M.A., Headmaster, George Watson's College, Edinburgh
1894		Allan, Francis John, M.D., C.M. Edin., M.O.H., City of Westminster, Westminster City Hall, Charing Cross Road, London
1888	C.	* Allardice, R. E., M.A., Professor of Mathematics in Stanford University, Palo Alto, Santa Clara Co., California
1878		Allchin, Sir William H., M.D., F.R.C.P.L., Senior Physician to the Westminster Hospital, 5 Chandos Street, Cavendish Square, London 10
1906		Anderson, Daniel E., M.D., B.A., B.Sc., 121 Avenue des Champs Elysées, Paris, France
1893		Anderson, J. Macvicar, Architect, 6 Stratton Street, London
1883		Anderson, Sir Robert Rowand, LL.D., 16 Rutland Square
1905		Anderson, William, F.G.S., 52 Lancis Buildings, Loveday Street, Johannesburg, Transvaal, South Africa
1905		* Anderson, William, M.A., Head Science Master, George Watson's College, Edinburgh, 29 Lutton Place 15
1903		Anderson-Berry, David, M.D., C.M. Edin., F.S.A. Scot., West Brow, St Leonards-on-Sea
1905		* Andrew, George, M.A., B.A., H.M.I.S., Glenhuntly, Hyndford Road, Lanark
1881	C.	Anglin, A. H., M.A., LL.D., M.R.I.A., Professor of Mathematics, Queen's College, Cork
1906		Appleton, Arthur Frederick, F.R.C.V.S., Lieut.-Colonel, Army Veterinary Department, Heworth Croft, York
1899		Appleyard, James R., Royal Technical Institute, Salford, Manchester 20
1893		* Archer, Walter E., 17 Sloan Court, London
1907		* Archibald, James, M.A., Headmaster, St Bernard's School, 52 Polwarth Gardens
1907		* Badre, Muhammad, 30 Minto Street
1894		* Bailey, Frederick, Lieut.-Col. (*late*) R.E., 7 Drummond Place
1896		* Baily, Francis Gibson, M.A., Professor of Applied Physics, Heriot-Watt College 25
1877	C.	Balfour, I. Bayley, M.A., Sc.D., M.D., LL.D., F.R.S., F.L.S., King's Botanist in Scotland, Professor of Botany in the University of Edinburgh and Keeper of the Royal Botanic Gardens, Inverleith House

Date of Election.			
1905		Balfour-Browne, William Alexander Francis, M.A., Barrister-at-Law, Claremont, Holywood, Co. Down, Ireland	
1892		* Ballantyne, J. W., M.D., F.R.C.P.E., 24 Melville Street	
1902	C.	Bannerman, W. B., M.D., B.Sc., Lt.-Colonel, Indian Medical Service, Director, Bacteriological Laboratory, Parel, Bombay, India	
1889		* Barbour, A. H. F., M.A., M.D., F.R.C.P.E., 4 Charlotte Square	30
1886		* Barclay, A. J. Gunion, M.A., 729 Great Western Road, Glasgow	
1872		Barclay, George, M.A., 17 Coates Crescent	
1883	C.	Barclay, G. W. W., M.A., 91 Union Street, Aberdeen	
1903		Bardswell, Noël Dean, M.D., M.R.C.P. Ed. and Lond., King Edward VII. Sanatorium, Midhurst	
1882	C.	Barnes, Henry, M.D., LL.D., 6 Portland Square, Carlisle	35
1904		Barr, Sir James, M.D., F.R.C.P. Lond., 72 Rodney Street, Liverpool	
1874		Barrett, William F., F.R.S., M.R.I.A., Prof. of Physics, Royal College of Science, Dublin	
1889		Barry, T. D. Collis, M.R.C.S., F.I.C., Lt.-Colonel I.M.S., Chemical Analyser to the Government of Bombay, and Prof. of Chemistry and Medical Jurisprudence, Grant Medical College, Bombay	
1887		* Bartholomew, J. G., F.R.G.S., The Geographical Institute, Dalkeith Road	
1895	C.	Barton, Edwin H., D.Sc., A.M.I.E.E., Memb. Phys. Soc. of London, Professor of Experimental Physics, University College, Nottingham	40
1904		* Baxter, William Muirhead, 2A Merchiston Place	
1888		* Beare, Thomas Hudson, B.Sc., Memb. Inst. C.E., Professor of Engineering in the University of Edinburgh	
1897	C.	* Beattie, John Carruthers, D.Sc., Professor of Physics, South African College, Cape Town	
1892		Beck, J. H. Meiring, M.D., M.R.C.P.E., Rondebosch, Cape Town	
1893	B. C.	* Becker, Ludwig, Ph.D., Regius Professor of Astronomy in the University of Glasgow, The Observatory, Glasgow	45
1882	C.	Beddard, Frank E., M.A. Oxon., F.R.S., Prosector to the Zoological Society of London, Zoological Society's Gardens, Regent's Park, London	
1887		* Begg, Ferdinand Faithfull, Bartholomew House, London	
1886		* Bell, A. Beatson, 17 Lansdowne Crescent	
1906		Bell, John Patrick Fair, F.Z.S., Fulforth, Witton Gilbert, Durham	
1874		Bell, Joseph, M.D., F.R.C.S.E., 2 Melville Crescent	50
1900		* Bennett, James Bower, Memb. Inst. C.E., 42 Frederick Street	
1887		* Bernard, J. Mackay, of Dunsinnan, B.Sc., Dunsinnan, Perth	
1893	C.	* Berry, George A., M.D., C.M., F.R.C.S., 31 Drumsheugh Gardens	
1897	C.	Berry, Richard J., M.D., F.R.C.S.E., Professor of Anatomy in the University of Melbourne, Victoria	
1904		* Beveridge, Erskine, LL.D., St Leonards Hill, Dunfermline	55
1880	C.	Birch, De Burgh, M.D., Professor of Physiology in the University of Leeds, 16 De Grey Terrace, Leeds	
1900		* Bisset, James, M.A., F.L.S., F.G.S., 9 Greenhill Park	
1907		* Black, Frederick Alexander, Solicitor, 59 Academy Street, Inverness	
1884		Black, John S., M.A., LL.D. (CURATOR OF LIBRARY AND MUSEUM), 6 Oxford Terrace	
1850		Blackburn, Hugh, M.A., LL.D., Emeritus Professor of Mathematics in the University of Glasgow, Roshven, Lochailort	60
1897		* Blaikie, Walter Biggar, The Loan, Colinton	
1904	C.	* Bles, Edward J., M.A., D.Sc., The Mill House, Iffley, Oxford	
1898	C.	* Blyth, Benjamin Hall, M.A., Memb. Inst. C.E., 17 Palmerston Place	
1894		* Bolton, Herbert, F.G.S., F.L.S., Curator of the Bristol Museum, Queen's Road, Bristol	
1884		Bond, Francis T., B.A., M.D., M.R.C.S., Gloucester	65
1872	C.	Bottomley, J. Thomson, M.A., D.Sc., LL.D., F.R.S., F.C.S., 13 University Gardens, Glasgow	
1869	C.	Bow, Robert Henry, C.E., 7 South Gray Street	
1886		* Bower, Frederick O., M.A., D.Sc., F.R.S., F.L.S., Regius Professor of Botany in the University of Glasgow, 1 St John's Terrace, Hillhead, Glasgow	
1884	C.	Bowman, Frederick Hungerford, D.Sc., F.C.S. (Lond. and Berl.), F.I.C., Assoc. Inst. C.E., Assoc. Inst. M.E., M.I.E.E., &c., 4 Albert Square, Manchester	
1901		Bradbury, J. B., M.D., Downing Professor of Medicine, University of Cambridge	70
1903	C.	* Bradley, O. Charnock, M.D., D.Sc., Royal Veterinary College, Edinburgh	
1886		* Bramwell, Byrom, M.D., F.R.C.P.E., 23 Drumsheugh Gardens	
1907		* Bramwell, Edwin, M.B., F.R.C.P.E., M.R.C.P. Lond., 23 Drumsheugh Gardens	

Date of Election.			
1895		* Bright, Charles, Assoc. Memb. Inst. C.E., Memb. Inst. E.E., F.R.A.S., F.G.S., Parliament Chambers, London, S.W.	
1877		Broadrick, George, Memb. Inst. C.E., Broughton House, Broughton Road, Ipswich	75
1893		Brock, G. Sandison, M.D., 6 Corso d'Italia, Rome, Italy	
1901	C.	* Brodie, W. Brodie, M.B., Thaxted, Essex	
1907		Brown, Alexander, M.A., B.Sc., Professor of Applied Mathematics, South African College, Cape Town	
1864	C. K. B.	Brown, Alex. Crum, M.D., D.Sc., F.R.C.P.E., LL.D., F.R.S. (Vice-President), Emeritus Professor of Chemistry in the University of Edinburgh, 8 Belgrave Crescent	
1898		* Brown, David, F.C.S., F.I.C., Willowbrae House, Midlothian	80
1883	C.	Brown, J. J. Graham, M.D., F.R.C.P.E., 3 Chester Street	
1885	C.	Brown, J. Macdonald, M.D., F.R.C.S., 2 Frognal, London, N.W.	
1888	K. C.	Bruce, Alexander, M.A., M.D., F.R.C.P.E., 8 Ainslie Place	
1906		* Bruce, William Speirs, LL.D., Antarctica, Joppa, Midlothian	
1898	K. C.	* Bryce, T. H., M.A., M.D. (Edin.), 2 Granby Terrace, Glasgow	85
1870	C. K.	Buchanan, John Young, M.A., F.R.S., Christ's College, Cambridge	
1902		* Buchanan, Robert M., M.B., F.F.P.S.G., 2 Northbank Terrace, Glasgow	
1882		Buchanan, T. R., M.A., M.P., 12 South Street, Park Lane, London, W.	
1887	C.	* Buist, J. B., M.D., F.R.C.P.E., 1 Clifton Terrace	
1905		Bunting, Thomas Lowe, M.D., Scotswood, Newcastle-on-Tyne	90
1902		* Burgess, A. G., M.A., Mathematical Master, Edinburgh Ladies' College, 64 Strathearn Road	
1894	C. K.	* Burgess, James, C.I.E., LL.D., M.R.A.S., M. Soc. Asiatique de Paris, *H.*A.R.I.B.A. (Vice-President), 22 Seton Place	
1902		* Burn, The Rev. John Henry, B.D., The Parsonage, Ballater	
1887		* Burnet, John James, Architect, 18 University Avenue, Hillhead, Glasgow	
1888		* Burns, Rev. T., D.D., F.S.A. Scot., Minister of Lady Glenorchy's Parish Church, Croston Lodge, Chalmers Crescent	95
1896		* Butters, J. W., M.A., B.Sc., Rector of Ardrossan Academy	
1887	C.	* Cadell, Henry Moubray, of Grange, B.Sc., Bo'ness	
1897		* Caird, Robert, LL.D., Shipbuilder, Greenock	
1893	C.	Calderwood, W. L., Inspector of Salmon Fisheries of Scotland, 7 East Castle Road, Merchiston	
1894		* Cameron, James Angus, M.D., Medical Officer of Health, Firhall, Nairn	100
1905	C.	Cameron, John, M.D., D.Sc., M.R.C.S. Eng., Anatomy Department, Middlesex Hospital Medical School, London	
1904		* Campbell, Charles Duff, 21 Montague Terrace, Inverleith Row	
1908		* Campbell, Lt.-Colonel John, Westwood, Cupar, Fife	
1899	C.	* Carlier, Edmund W. W., M.D., B.Sc., Prof. of Physiology in Mason College, Birmingham	
1906		Carruthers, John Bennett, F.L.S., Assoc. R.B.S., Director of Agriculture and Government Botanist, F.M.S.. Kuala Lumpur, Federated Malay States	105
1905	C.	* Carse, George Alexander, M.A., D.Sc., Lecturer on Natural Philosophy, University of Edinburgh, 120 Lauriston Place	
1901		Carslaw, H. S., M.A., D.Sc., Professor of Mathematics in the University of Sydney, New South Wales	
1905		Carter, Joseph Henry, F.R.C.V.S., Rowley Hall, Burnley, Lancashire	
1898		* Carter, Wm. Allan, Memb. Inst. C.E., 32 Great King Street	
1898		Carus-Wilson, Cecil, F.R.G.S., F.G.S., 16 Waldegrave Park, Strawberry Hill, Middlesex	110
1908		Cavanagh, Thomas Francis, M.D., 396 Eccleshall Road, Sheffield	
1882		Cay, W. Dyce, Memb. Inst. C.E., 39 Victoria Street, Westminster, London	
1890		Charles, John J., M.A., M.D., C.M., late Prof. of Anatomy and Physiology, Queen's College, Cork, Karlsruhe, Port Stewart, Co. Derry	
1899		Chatham, James, Actuary, 7 Belgrave Crescent	
1874		Chiene, John, C.B., M.D., LL.D., F.R.C.S.E., Professor of Surgery in the University of Edinburgh, 21 Alva Street	115
1880	C. K.	Chrystal, George, M.A., LL.D., Professor of Mathematics in the University of Edinburgh (General Secretary), 5 Belgrave Crescent	
1891		* Clark, John B., M.A., Head Master of Heriot's Hospital School, Lauriston, Garleffin, Craiglea Drive	
1903		* Clarke, William Eagle, F.L.S., Keeper of the Natural History Collections in the Royal Scottish Museum, Edinburgh, 35 Braid Road	
1875		Clouston, T. S., M.D., LL.D., Vice-President of the Royal College of Physicians, 26 Heriot Row	

Date of Election.			
1892		* Coates, Henry, Pitcullen House, Perth	120
1887		* Cockburn, John, F.R.A.S., The Abbey, North Berwick	
1904	C.	Coker, Ernest George, M.A., D.Sc., Professor of Mechanical Engineering and Applied Mechanics, City and Guilds Technical College, Finsbury, Leonard Street, City Road, London, E.C.	
1904		Coles, Alfred Charles, M.D., D.Sc., York House, Poole Road, Bournemouth, W.	
1888	C.	Collie, John Norman, Ph.D., F.R.S., F.C.S., Professor of Organic Chemistry in the University College, Gower Street, London	
1904	C.	* Colquhoun, Walter, M.A., M.B., 7 Stanley Street, Glasgow, W.	125
1886		Connan, Daniel M., M.A.	
1872		Constable, Archibald, LL.D., 11 Thistle Street	
1894		Cook, John, M.A., late Principal of the Government Central College, Bangalore, India, The Bungalow, 14 Elm Grove, Wimbledon, Surrey	
1891		* Cooper, Charles A., LL.D., 41 Drumsheugh Gardens	
1905		* Corrie, David, F.C.S., Nobel's Explosives Company, Polmont Station	130
1908		Craig, James Ireland, M.A., B.A., Director of the Computation Office, Survey Department, Egypt, Mataria, Egypt	
1875		Craig, William, M.D., F.R.C.S.E., Lecturer on Materia Medica to the College of Surgeons, 71 Bruntsfield Place	
1907		* Cramer, William, Ph.D., Lecturer in Physiological Chemistry in the University of Edinburgh, Physiological Department, The University	
1903		Crawford, Lawrence, M.A., D.Sc., Professor of Mathematics in the South African College, Cape Town	
1887		* Crawford, William Caldwell, 1 Lockharton Gardens, Colinton Road	135
1870		Crichton-Browne, Sir Jas., M.D., LL.D., F.R.S., Lord Chancellor's Visitor and Vice-President of the Royal Institution of Great Britain, 72 Queen's Gate, and Royal Courts of Justice, Strand, London	
1886		* Croom, Sir John Halliday, M.D., F.R.C.P.E., Professor of Midwifery in the University of Edinburgh, Vice-President, Royal College of Surgeons, Edinburgh, 25 Charlotte Square	
1898		* Cullen, Alexander, F.S.A. Scot., Millburn House, by Hamilton	
1878	C.	Cunningham, Daniel John, M.D., LL.D., D.C.L., F.R.S., F.Z.S., Professor of Anatomy in the University of Edin. (SECRETARY), 18 Grosvenor Crescent	
1898		* Currie, James, M.A. Cantab. (TREASURER) Larkfield, Goldenacre	140
1904		* Cuthbertson, John, Secretary, West of Scotland Agricultural College, 4 Charles Street, Kilmarnock	
1885		* Daniell, Alfred, M.A., LL.B., D.Sc., Advocate, The Athenæum Club, Pall Mall, London	
1884		Davy, R., F.R.C.S. Eng., Consulting Surgeon to Westminster Hospital, Burstone House, Bow, North Devon	
1894		* Denny, Archibald, Braehead, Dumbarton	
1869	C. V. J.	Dewar, Sir James, M.A., LL.D., D.C.L., D.Sc. Dub., F.R.S., F.C.S., Jacksonian Professor of Natural and Experimental Philosophy in the University of Cambridge, and Fullerian Professor of Chemistry at the Royal Institution of Great Britain, London	145
1905		* Dewar, James Campbell, C.A., 27 Douglas Crescent	
1906		* Dewar, Thomas Wm., M.D., F.R.C.P., Kincairn, Dunblane	
1904		Dickinson, Walter George Burnett, F.R.C.V.S., Boston, Lincolnshire	
1884		Dickson, The Right Hon. Charles Scott, K.C., LL.D., 22 Moray Place	
1888	C.	* Dickson, Henry Newton, M.A., D.Sc., The Lawn, Upper Redlands Road, Reading	150
1876	C.	Dickson, J. D. Hamilton, M.A., Fellow and Tutor, St Peter's College, Cambridge	
1885	C.	Dixon, James Main, M.A., Litt. Hum. Doctor, Professor of English, University of Southern California, Wesley Avenue, Los Angelos, California, United States	
1897		* Dobbie, James Bell, F.Z.S., 12 South Inverleith Avenue	
1904	C.	* Dobbie, James Johnston, M.A., D.Sc., LL.D., F.R.S., Director of the Royal Scottish Museum, Edinburgh, 27 Polwarth Terrace	
1881	C.	Dobbin, Leonard, Ph.D., Lecturer on Chemistry in the University of Edinburgh, 6 Wilton Road	155
1902		Dollar, John A. W., M.R.C.V.S., 56 New Bond Street, London	
1867	C.	Donaldson, Sir James, M.A., LL.D., Principal of the University of St Andrews, St Andrews	
1896		* Donaldson, William, M.A., Viewpark House, Spylaw Road	
1905		* Donaldson, Rev. Wm. Galloway, Minister of St Paul's Parish, 11 Claremont Crescent	
1882		Dott, David B., F.I.C., Memb. Pharm. Soc., Ravenslea, Musselburgh	160
1892		Doyle, Patrick, C.E., M.R.I.A., F.G.S., Editor of *Indian Engineering*, Calcutta	

Date of Election.			
1901		* Douglas, Carstairs Cumming, M.D., D.Sc., Professor of Medical Jurisprudence and Hygiene, Anderson's College, Glasgow, 2 Royal Crescent, Glasgow	
1866		Douglas, David, 22 Drummond Place	
1908	C.	Drinkwater, Harry, M.D., M.R.C.S. (Eng.), Grosvenor Lodge, Wrexham, North Wales	
1901		* Drinkwater, Thomas W., L.R.C.P.E., L.R.C.S.E., Chemical Laboratory, Surgeons' Hall	165
1878		Duncanson, J. J. Kirk, M.D., F.R.C.P.E., 22 Drumsheugh Gardens	
1904		* Dunlop, William Brown, M.A., 7 Carlton Street	
1859	C.	Duns, Rev. Professor, D.D., 5 Greenhill Place	
1903		* Dunstan, John, M.R.C.V.S., 1 Dean Terrace, Liskeard, Cornwall	
1892	C.	Dunstan, M. J. R., M.A., F.I.C., F.C.S., Principal, South-Eastern Agricultural College, Wye, Kent	170
1899		* Duthie, George, M.A., Inspector General of Education, Salisbury, Rhodesia	
1906	C.	* Dyson, Frank Watson, M.A., F.R.S., Astronomer Royal for Scotland, and Professor of Astronomy in the University of Edinburgh, Royal Observatory, Blackford Hill, Edinburgh	
1893		Edington, Alexander, M.D., 20 Kilmaurs Road	
1904		* Edwards, John, 4 Great Western Terrace, Kelvinside, Glasgow	
1904		* Elder, William, M.D., F.R.C.P.E., 4 John's Place, Leith	175
1885		Elgar, Francis, Memb. Inst. C.E., LL.D., F.R.S., 18 Cornwall Terrace, Regent's Park, London	
1875		Elliot, Daniel G., Curator of Department of Zoology, Field Columbian Museum, Chicago, U.S.	
1906	C.	* Ellis, David, D.Sc., Ph.D., Lecturer in Botany and Bacteriology, Glasgow and West of Scotland Technical College, Glasgow	
1897	C.	* Erskine-Murray, James Robert, D.Sc., 77 Kingsfield Road, Watford, Herts	
1884		Evans, William, F.F.A., 38 Morningside Park	180
1879	C. N.	Ewart, James Cossar, M.D., F.R.C.S.E., F.R.S., F.L.S., Regius Professor of Natural History, University of Edinburgh (VICE-PRESIDENT), Duddingston House, Duddingston, Midlothian	
1902		* Ewen, J. T., B.Sc., Memb. Inst. Mech. E., H.M.I.S., 104 King's Gate, Aberdeen	
1878	C.	Ewing, James Alfred, M.A., B.Sc., LL.D., Memb. Inst. C.E., F.R.S., Director of Naval Education, Royal Naval College, Greenwich	
1900		Eyre, John W. H., M.D., M.S. (Dunelm), D.P.H. (Camb.), Guy's Hospital (Bacteriological Department), London	
1875		Fairley, Thomas, Lecturer on Chemistry, 8 Newton Grove, Leeds	185
1907	C.	Falconer, John Downie, M.A., D.Sc., F.G.S., Director, Mineral Survey of Northern Nigeria, The Limes, Little Berkhampstead, Hertford, and Imperial Institute, London	
1888	C.	* Fawsitt, Charles A., 9 Foremount Terrace, Dowanhill, Glasgow	
1883	C.	Felkin, Robert W., M.D., F.R.G.S., Fellow of the Anthropological Society of Berlin, 47 Bassett Road, North Kensington, London, W.	
1899		* Fergus, Andrew Freeland, M.D., 22 Blythswood Square, Glasgow	
1907		* Fergus, Edward Oswald, 12 Clairmont Gardens, Glasgow	190
1904		* Ferguson, James Haig, M.D., F.R.C.P.E., F.R.C.S.E., 7 Coates Crescent	
1888		* Ferguson, John, M.A., LL.D., Professor of Chemistry in the University of Glasgow	
1868	C.	Ferguson, Robert M., Ph.D., LL.D. (SOCIETY'S REPRESENTATIVE ON GEORGE HERIOT'S TRUST), 5 Douglas Gardens	
1898		* Findlay, John R., M.A. Oxon, 27 Drumsheugh Gardens	
1899		* Finlay, David W., B.A., M.D., LL.D., F.R.C.P., D.P.H., Professor of Medicine in the University of Aberdeen, Honorary Physician to His Majesty in Scotland, 2 Queen's Terrace, Aberdeen	195
1906		* Fleming, Robert Alexander, M.D., F.R.C.P.E., Assistant Physician, Royal Infirmary, 10 Chester Street	
1900	C. N.	* Flett, John S., M.A., D.Sc., Geological Survey Office, 28 Jermyn Street, London	
1880		Flint, Robert, D.D., Corresponding Member of the Institute of France, Corresponding Member of the Royal Academy of Sciences of Palermo, Emeritus Professor of Divinity in the University of Edinburgh, 5 Royal Terrace	
1872	C.	Forbes, Professor George, M.A., Memb. Inst. C.E., Memb. Inst. E.E., F.R.S., F.R.A.S., 34 Great George Street, Westminster	
1904		Forbes, Norman Hay, F.R.C.S.E., L.R.C.P. Lond., Corres. Memb. Soc. d'Hydrologie de Paris, Druminnor, Church Stretton, Salop	200
1892		* Ford, John Simpson, F.C.S., 4 Nile Grove	
1858		Fraser, A. Campbell, Fellow of the British Academy, Hon. D.C.L. Oxford, LL.D, Litt. D., Emeritus Professor of Logic and Metaphysics in the University of Edinburgh, Gorton House, Hawthornden	

Date of Election.		
1896		* Fraser, John, M.B., F.R.C.P.E., one of H.M. Commissioners in Lunacy for Scotland, 13 Heriot Row
1867	C. K. B.	Fraser, Sir Thomas R., M.D., LL.D., F.R.C.P.E., F.R.S., Professor of Materia Medica in the University of Edinburgh, Honorary Physician to the King in Scotland, 13 Drumsheugh Gardens
1891		* Fullarton, J. H., M.A., D.Sc., Brodick, Arran 205
1891		* Fulton, T. Wemyss, M.D., Scientific Superintendent, Scottish Fishery Board, 417 Great Western Road, Aberdeen
1907		* Galbraith, Alexander, Organiser of Continuation Classes in Science, Glasgow and West of Scotland Technical College, 4 Maxwell Square, Pollokshields, Glasgow
1888	C.	* Galt, Alexander, D.Sc., Keeper of the Technological Department, Royal Scottish Museum, Edinburgh
1901		Ganguli, Sanjiban, M.A., Principal, Maharaja's College, and Director of Public Instruction, Jaipur States, Jaipur, India
1899		Gatehouse, T. E., Assoc. Memb. Inst. C.E., Memb. Inst. M.E., Memb. Inst. E.E., Tulse Hill Lodge, 100 Tulse Hill, London 210
1867		Gayner, Charles, M.D., F.L.S.
1900		Gayton, William, M.D., M.R.C.P.E., Ravensworth, Regent's Park Road, Finchley, London, N.
1880	C.	Geddes, Patrick, Professor of Botany in University College, Dundee, and Lecturer on Zoology, Ramsay Garden, University Hall, Edinburgh
1861	C. B.	Geikie, Sir Archibald, K.C.B., D.C.L. Oxf., D.Sc. Camb. Dub., LL.D. St And., Glasg., Aberdeen, Edin., Ph.D. Upsala, Sec. R.S., Pres. G.S., Foreign Member of the Reale Accad. Lincei, Rome, of the National Acad. of the United States, of the Academies of Stockholm, Christiania, Göttingen, Corresponding Member of the Institute of France and of the Academies of Berlin, Vienna, Munich, Turin, Belgium, Philadelphia, New York, &c., Shepherd's Down, Haslemere, Surrey
1871	C. B.	Geikie, James, LL.D., D.C.L., F.R.S., F.G.S., Professor of Geology in the University of Edinburgh, Kilmorie, Colinton Road 215
1881	C.	Gibson, George Alexander, D.Sc., M.D., LL.D., F.R.C.P.E., 3 Drumsheugh Gardens
1890		* Gibson, George A., M.A., LL.D., Professor of Mathematics in the Glasgow and West of Scotland Technical College, 8 Sandyford Place, Glasgow
1877	C.	Gibson, John, Ph.D., Professor of Chemistry in the Heriot-Watt College, 16 Woodhall Terrace, Juniper Green
1900		Gilchrist, Douglas A., B.Sc., Professor of Agriculture and Rural Economy, Armstrong College, Newcastle-upon-Tyne
1880		Gilruth, George Ritchie, Surgeon, 53 Northumberland Street 220
1907		Gilruth, John Anderson, M.R.C.V.S., Chief Veterinarian, N.Z. Government, and Pathologist to Public Health Department N.Z., Wellington, New Zealand
1898		* Glaister, John, M.D., F.F.P.S. Glasgow, D.P.H. Camb., Professor of Forensic Medicine in the University of Glasgow, 3 Newton Place, Glasgow
1901		Goodwillie, James, M.A., B.Sc., Liberton, Edinburgh
1899		* Goodwin, Thomas S., M.B., C.M., F.C.S., 1 Heron Terrace, St Margaret's, Middlesex
1897		Gordon-Munn, John Gordon, M.D., 34 Dover Street, London, W. 225
1891		* Graham, Richard D., 11 Strathearn Road
1898	C.	* Gray, Albert A., M.D., 14 Newton Terrace, Glasgow
1883		Gray, Andrew, M.A., LL.D., F.R.S. (Vice-President), Professor of Natural Philosophy in the University of Glasgow
1880	C.	Gray, Thomas, B.Sc., Professor of Physics, Rose Polytechnic Institute, Terre Haute, Indiana, U.S.
1886		* Greenfield, W. S., M.D., F.R.C.P.E., Professor of General Pathology in the University of Edinburgh, 7 Heriot Row 230
1897		Greenlees, Thomas Duncan, M.D. Edin., Amana, Tulse Hill, London
1905		* Gregory, John Walter, D.Sc., F.R.S., Professor of Geology in the University of Glasgow, 4 Park Quadrant, Glasgow
1906		Greig, Edward David Wilson, M.D., B.Sc., Captain, H.M.'s Indian Medical Service, Byculla Club, Bombay, India
1905		Greig, Robert Blyth, F.Z.S., Fordyce Lecturer in Agriculture, University of Aberdeen, Torloisk, Cults, Aberdeenshire
1899		* Guest, Edward Graham, M.A., B.Sc., 5 Church Hill 235
1907	C.	* Gulliver, Gilbert Henry, B.Sc., A.M.I. Mech. E., Lecturer in Experimental Engineering in the University of Edinburgh, 10 Stanley Street, Portobello
1888	C.	Guppy, Henry Brougham, M.B., Rosario, Salcombe, Devon
1905	B. C.	* Halm, Jacob E., Ph.D., Chief Assistant Astronomer, Royal Observatory, Cape Town, Cape of Good Hope

Date of Election.			
1899		Hamilton, Allan M'Lane, M.D., 44 East Twenty-ninth Street, New York	
1881	C.	Hamilton, D. J., M.B., F.R.C.S.E., LL.D., F.R.S., late Professor of Pathological Anatomy in the University of Aberdeen, 35 Queen's Road, Aberdeen	240
1876	C.	Hannay, J. Ballantyne, Cove Castle, Loch Long	
1902		* Hargreaves, Andrew Fuller, F.C.S., Eskhill House, Roslin	
1896		* Harris, David, Fellow of the Statistical Society, Lyncombe Rise, Prior Park Road, Bath	
1896	C.	* Harris, David Fraser, B.Sc. (Lond.), M.D., F.S.A. Scot., Abbey Cottage, St Andrews	
1888		* Hart, D. Berry, M.D., F.R.C.P.E., 5 Randolph Cliff	245
1869		Hartley, Sir Charles A., K.C.M.G., Memb. Inst. C.E., 26 Pall Mall, London	
1877	C.	Hartley, W. N., D.Sc., F.R.S., F.I.C., Prof. of Chemistry, Royal College of Science for Ireland, Dublin	
1881		Harvie-Brown, J. A., of Quarter, F.Z.S., Dunipace House, Larbert, Stirlingshire	
1880	C.	Haycraft, J. Berry, M.D., D.Sc., Professor of Physiology in the University College of South Wales and Monmouthshire, Cardiff	
1892	C.	* Heath, Thomas, B.A., Assistant Astronomer, Royal Observatory, Edinburgh	250
1893		Hehir, Patrick, M.D., F.R.C.S.E., M.R.C.S.L., L.R.C.P.E., Surgeon-Captain, Indian Medical Service, Principal Medical Officer, H.H. the Nizam's Army, Hyderabad, Deccan, India	
1890	C.	Helme, T. Arthur, M.D., M.R.C.P.L., M.R.C.S., 3 St Peter's Square, Manchester	
1900		Henderson, John, D.Sc., Assoc. Inst. E.E., Kinnoul, Warwick's Bench Rd., Guildford, Surrey	
1908		* Henderson, William Dawson, M.A., B.Sc., Ph.D., Assistant Professor, Zoological Department, University College, Dundee	
1890	C.	* Hepburn, David, M.D., Professor of Anatomy in the University College of South Wales and Monmouthshire, Cardiff	255
1881	C. N.	Herdman, W. A., D.Sc., F.R.S., Pres.L.S., Prof. of Natural History in the University of Liverpool, Croxteth Lodge, Ullet Road, Liverpool	
1908		* Hewat, Archibald, F.F.A., F.I.A., 13 Eton Terrace	
1894		Hill, Alfred, M.D., M.R.C.S., F.I.C., Valentine Mount, Freshwater Bay, Isle of Wight	
1902		* Hinxman, Lionel W., B.A., Geological Survey Office, 33 George Square	
1904		Hobday, Frederick T. G., F.R.C.V.S., 6 Berkeley Gardens, Kensington, London	260
1885		Hodgkinson, W. R., Ph.D., F.I.C., F.C.S., Prof. of Chem. and Physics at the Royal Military Acad. and Royal Artillery Coll., Woolwich, 18 Glenluce Road, Blackheath, Kent	
1881	C. N.	Horne, John, LL.D., F.R.S., F.G.S., Director of the Geological Survey of Scotland (VICE-PRESIDENT), 33 George Square, Edinburgh	
1896		Horne, J. Fletcher, M.D., F.R.C.S.E., The Poplars, Barnsley	
1904		* Horsburgh, Ellice Martin, M.A., B.Sc., Lecturer in Technical Mathematics, University of Edinburgh, 11 Granville Terrace	
1897		Houston, Alex. Cruikshanks, M.B., C.M., D.Sc., 19 Fairhazel Gardens, South Hampstead, London, N.W.	265
1893		Howden, Robert, M.A., M.B., C.M., Professor of Anatomy in the University of Durham, 14 Burdon Terrace, Newcastle-on-Tyne	
1899		Howie, W. Lamond, F.C.S., 26 Neville Court, Abbey Road, Regent's Park, London, N.W.	
1883	C.	Hoyle, William Evans, M.A., D.Sc., M.R.C.S., 25 Brunswick Road, Withington, Manchester	
1886		Hunt, Rev. H. G. Bonavia, Mus.D. Dub., Mus.B. Oxon., The Vicarage, Burgess Hill, Sussex	
1887	C.	* Hunter, James, F.R.C.S.E., F.R.A.S., Rosetta, Liberton, Midlothian	270
1887	C.	* Hunter, William, M.D., M.R.C.P.L. and E., M.R.C.S., 54 Harley Street, London	
1908		Hyslop, Theophilus Bulkeley, M.D., M.R.C.P.E., Senior Physician, Bethlem Royal Hospital, London, S.E.	
1882	C.	Inglis, J. W., Memb. Inst. C.E., 26 Pitt Street	
1906		* Innes, Alexander Taylor, M.A., Advocate, 48 Morningside Park	
1904	C.	Innes, R. T. A., Director, Government Observatory, Johannesburg, Transvaal	275
1904		* Ireland, Alexander Scott, S.S.C., 2 Buckingham Terrace	
1875		Jack, William, M.A., LL.D., Professor of Mathematics in the University of Glasgow	
1894		Jackson, Sir John, LL.D., 48 Belgrave Square, London	
1889		* James, Alexander, M.D., F.R.C.P.E., 14 Randolph Crescent	
1882		Jamieson, Prof. A., Memb. Inst. C.E., 16 Rosslyn Terrace, Kelvinside, Glasgow	280

Date of Election.			
1901		* Jardine, Robert, M.D., M.R.C.S. Eng., F.F.P. and S. Glas., 20 Royal Crescent, Glasgow	
1900		Jee, Sir Bhagvat Sinh, G.C.I.E., M.D., LL.D. Edin., H.H. The Thakore Sahib of Gondal, Gondal, Kathiawar, Bombay	
1906	C.	* Jehu, Thomas James, M.A., M.D., F.G.S., Lecturer in Geology, University of St Andrews, Strathmartine, Hepburn Gardens, St Andrews	
1900		* Jerdan, David Smiles, M.A., D.Sc., Ph.D., Temora, Colinton, Midlothian	
1895		Johnston, Lieutenant-Colonel Henry Halcro, C.B., R.A.M.S., D.Sc., M.D., F.L.S., Orphir House, Kirkwall, Orkney	285
1903	C.	* Johnston, Thomas Nicol, M.B., C.M., Corstorphine House, Corstorphine	
1902		Johnstone, George, Lieut. R.N.R., Marine Superintendent, British India Steam Navigation Co., 16 Strand Road, Calcutta, India	
1874		Jones, Francis, M.Sc., Lecturer on Chemistry, Beaufort House, Alexandra Park, Manchester	
1888		Jones, John Alfred, Memb. Inst. C.E., Fellow of the Univ. of Madras, Sanitary Engineer to the Government of Madras, c/o Messrs Parry & Co., 70 Gracechurch St., London	
1905		Jones, George William, M.A., B.Sc., Coraldene, Kirk Brae, Liberton	290
1907		* Kemp, John, M.A., Headmaster, High School, Kelso	
1908		* Kerr, Andrew William, F.S.A. Scot., Royal Bank House, St Andrew Square	
1892		* Kerr, Rev. John, M.A., Manse, Dirleton	
1903	C. N.	* Kerr, John Graham, M.A., Professor of Zoology in the University of Glasgow	
1891		Kerr, Joshua Law, M.D., Biddenden Hall, Cranbrook, Kent	295
1908		Kidd, Walter Aubrey, M.D., F.Z.S., 12 Montpelier Row, Blackheath, London	
1886	C. N.	* Kidston, Robert, LL.D., F.R.S., F.G.S., 12 Clarendon Place, Stirling	
1907		* King, Archibald, M.A., B.Sc., Rector of the Academy, Castle-Douglas, Hazeldene, Castle-Douglas, Kirkcudbrightshire	
1877		King, Sir James, of Campsie, Bart., LL.D., 115 Wellington Street, Glasgow	
1880		King, W. F., Lonend, Russell Place, Trinity	300
1883		Kinnear, The Rt. Hon. Lord, one of the Senators of the College of Justice, 2 Moray Place	
1878		Kintore, The Right Hon. the Earl of, M.A. Cantab., LL.D., Cambridge, Aberdeen and Adelaide, Keith Hall, Inverurie, Aberdeenshire	
1901		* Knight, The Rev. G. A. Frank, M.A., St Leonard's United Free Church, Perth	
1907		* Knight, James, M.A., D.Sc., F.C.S., F.G.S., Headmaster, St James School, Glasgow, The Shieling, Uddingston, by Glasgow	
1880	C. K.	Knott, C. G., D.Sc., Lecturer on Applied Mathematics in the University of Edinburgh (late Prof. of Physics, Imperial University, Japan) (SECRETARY), 42 Upper Gray Street, Edinburgh	305
1886		* Laing, Rev. George P., 17 Buckingham Terrace	
1907		* Lanchester, William Forster, M.A., Den of Gryffe, Kilmalcolm	
1878	C.	Lang, P. R. Scott, M.A., B.Sc., Professor of Mathematics, University of St Andrews	
1885	C.	* Laurie, A. P., M.A., D.Sc., Principal of the Heriot-Watt College, Edinburgh	
1894	C.	* Laurie, Malcolm, B.A., D.Sc., F.L.S., Royal College of Surgeons, Edinburgh	310
1870		Laurie, Simon S., M.A., LL.D., Emeritus Professor of Education in the University of Edinburgh, 22 George Square	
1905		* Lawson, David, M.A., M.D., L.R.C.P. and S.E., Druimdarroch, Banchory, Kincardineshire	
1903		* Leighton, Gerald Rowley, M.D., Sunnyside, Russell Place	
1874	C. K.	Letts, E. A., Ph.D., F.I.C., F.C.S., Professor of Chemistry, Queen's College, Belfast	
1905		* Lightbody, Forrest Hay, 56 Queen Street	315
1889		* Lindsay, Rev. James, D.D., B.Sc., F.G.S., M.R.A.S., Corresponding Member of the Royal Academy of Sciences, Letters and Arts, of Padua, Associate of the Philosophical Society of Louvain, Minister of St Andrew's Parish, Springhill Terrace, Kilmarnock	
1870	C. B.	Lister, The Right Hon. Lord, O.M., P.C., M.D., F.R.C.S.L., F.R.C.S.E., LL.D., D.C.L., F.R.S., Foreign Associate of the Institute of France, Emeritus Professor of Clinical Surgery, King's College, Surgeon Extraordinary to the King, 12 Park Crescent, Portland Place, London	
1903		Liston, William Glen, M.D., Captain, Indian Medical Service, c/o Grindlay Groom & Co., Bombay, India	
1903		* Littlejohn, Henry Harvey, M.A., M.B., B.Sc., F.R.C.S.E., Professor of Forensic Medicine in the University of Edinburgh, 11 Rutland Street	
1898		* Lothian, Alexander Veitch, M.A., B.Sc., Glendoune, Manse Road, Bearsden, Glasgow	320

Date of Election.		Name	
1884		Low, George M., Actuary, 11 Moray Place	
1888		* Lowe, D. F., M.A., LL.D., late Head Master of Heriot's Hospital School, Lauriston, 19 George Square	
1904		* Lowson, Charles Stewart, M.B., C.M., Captain, Indian Medical Service, c/o Messrs Thomas Cook & Son, Bombay, India.	
1900		Lusk, Graham, Ph.D., M.A., Prof. of Physiology, Univ. and Bellevue Medical College, N.Y.	
1894		* Mabbott, Walter John, M.A., Rector of County High School, Duns, Berwickshire	325
1887		M'Aldowie, Alexander M., M.D., Glengarriff, Leckhampton, Cheltenham	
1907		MacAlister, Donald Alexander, A.R.S.M., F.G.S., 20 Hanover Square, London, W.	
1891		Macallan, John, F.I.C., 3 Rutland Terrace, Clontarf, Dublin	
1888	C.	M'Arthur, John, F.C.S., 196 Trinity Road, Wandsworth Common, London	
1883		M'Bride, P., M.D., F.R.C.P.E., 16 Chester Street	330
1903		* M'Cormick, W. S., M.A., LL.D., 13 Douglas Crescent	
1899		* M'Cubbin, James, B.A., Rector of the Burgh Academy, Kilsyth	
1905		* Macdonald, Hector Munro, M.A., F.R.S., Professor of Mathematics, University of Aberdeen, 52 College Bounds, Aberdeen	
1894		* Macdonald, James, Secretary of the Highland and Agricultural Society of Scotland, 2 Garscube Terrace	
1897	C.	* Macdonald, James A., M.A., B.Sc., H.M. Inspector of Schools, Glengarry, Dingwall	335
1904		* Macdonald, John A., M.A., B.Sc., High School, Stellenbosch, Cape Colony	
1886		* Macdonald, The Rt. Hon. Sir J. H. A., K.C.B., K.C., LL.D., F.R.S., M.I.E.E., Lord Justice-Clerk, and Lord President of the Second Division of the Court of Session, 15 Abercromby Place	
1904		Macdonald, William, B.Sc., M.Sc., Agriculturist, Editor *Transvaal Agricultural Journal*, Department of Agriculture, Pretoria Club, Pretoria, Transvaal	
1886		* Macdonald, William J., M.A., 15 Comiston Drive	
1901	C.	* MacDougal, R. Stewart, M.A., D.Sc., 13 Archibald Place	340
1888	C.	* M'Fadyean, Sir John, M.B., B.Sc., LL.D., Principal, and Professor of Comparative Pathology in the Royal Veterinary College, Camden Town, London	
1878	C.	Macfarlane, Alexander, M.A., D.Sc., LL.D., Lecturer in Physics in Lehigh University, Pennsylvania, Gowrie Grove, Chatham, Ontario, Canada	
1885	C.	* Macfarlane, J. M., D.Sc., Professor of Botany and Director of the Botanic Garden, University of Pennsylvania, Philadelphia, Pennsylvania, U.S.A.	
1897		* M'Gillivray, Angus, C.M., M.D., South Tay Street, Dundee	
1878		M'Gowan, George, F.I.C., Ph.D., 21 Montpelier Road, Ealing, Middlesex	345
1886		* MacGregor, Rev. James, D.D., 3 Eton Terrace	
1880	C.	MacGregor, James Gordon, M.A., D.Sc., LL.D., F.R.S., Prof. of Natural Philosophy in the University of Edinburgh, 24 Dalrymple Crescent	
1903		* M'Intosh, D. C., M.A., B.Sc., 37 Warrender Park Terrace	
1869	C. N.	M'Intosh, William Carmichael, M.D., LL.D., F.R.S., F.L.S., Professor of Natural History in the University of St Andrews, 2 Abbotsford Crescent, St Andrews	
1895	C.	* Macintyre, John, M.D., 179 Bath Street, Glasgow	350
1882		Mackay, John Sturgeon, M.A., LL.D., late Mathematical Master in the Edinburgh Academy, 69 Northumberland Street	
1873	C. B.	M'Kendrick, John G., M.D., F.R.C.P.E., LL.D., F.R.S., Emeritus Professor of Physiology in the University of Glasgow, Maxieburn, Stonehaven	
1900	C.	* M'Kendrick, John Souttar, M.D., F.F.P.S.G., 2 Buckingham Terrace, Glasgow	
1894		* Mackenzie, Robert, M.D., Napier, Nairn	
1898		Mackenzie, W. Cossar, D.Sc., Alderston, Haddington	355
1904		* Mackenzie, W. Leslie, M.A., M.D., D.P.H., Medical Member of the Local Government Board for Scotland, 1 Stirling Road, Trinity	
1905		Mackenzie, William Colin, M.D., F.R.C.S., Demonstrator of Anatomy in the University of Melbourne, Elizabeth Street North, Melbourne, Victoria	
1904		* Mackintosh, Donald James, M.V.O., M.B., Supt. of the Western Infirmary, Glasgow	
1869	C.	Maclagan, R. C., M.D., F.R.C.P.E., 5 Coates Crescent	
1869	C.	M'Laren, The Hon. Lord, LL.D., Edin. & Glasg., F.R.A.S., one of the Senators of the College of Justice, 46 Moray Place	360
1899		Maclean, Ewan John, M.D., M.R.C.P. London, 12 Park Place, Cardiff	
1888	C.	* Maclean, Magnus, M.A., D.Sc., Memb. Inst. E.E., Prof. of Electrical Engineering in the Glasgow and West of Scotland Technical College, 51 Kerrsland Terrace, Hillhead, Glasgow	

Date of Election.		
1876		Macleod, Very Rev. Norman, D.D., 74 Murrayfield Gardens
1876		Macmillan, John, M.A., D.Sc., M.B., C.M., F.R.C.P.E., 48 George Square
1893		* M'Murtrie, The Rev. John, M.A., D.D., 13 Inverleith Place 365
1906		* Macnair, Duncan Scott, Ph.D., B.Sc., H.M. Inspector of Schools, 67 Braid Avenue
1907		* Macnair, Peter, Curator of the Natural History Collections in the Glasgow Museums, Kelvingrove Museum, Glasgow
1884		Macpherson, Rev. J. Gordon, M.A., D.Sc., Ruthven Manse, Meigle
1890		* M'Vail, John C., M.D., LL.D., 20 Eton Place, Hillhead, Glasgow
1898	C.	Mahalanobis, S. C., B.Sc., Professor of Physiology, Presidency College, Calcutta, India 370
1908		Mallik, Devendra Nath, B.A., B.Sc., Professor of Physics and Mathematics, Patna College, Bankipur, Bengal, India
1880	C.	Marsden, R. Sydney, M.D., C.M., D.Sc., M.R.I.A., F.I.C, F.C.S., Rowallan House, Cearns Road, and Town Hall, Birkenhead
1882	C.	Marshall, D. H., M.A., Professor of Physics in Queen's University and College, Kingston, Ontario, Canada
1901	C.	* Marshall, F. H. A., M.A., D.Sc., Lecturer on Agricultural Physiology in the University of Cambridge, Christ's College, Cambridge
1888	C. K.	* Marshall, Hugh, D.Sc., F.R.S., Professor of Chemistry in the University College, Dundee 375
1892		* Martin, Francis John, W.S., 17 Rothesay Place
1903		Martin, Nicholas Henry, F.L.S., F.C.S., Ravenswood, Low Fell, Gateshead
1885	C.	* Masson, Orme, D.Sc., F.R.S., Professor of Chemistry in the University of Melbourne
1898	C. B.	* Masterman, Arthur Thomas, M.A., D.Sc., Inspector of Fisheries, Board of Agriculture, Whitehall, London
1906		* Mathieson, Robert, F.C.S., Rillbank, Innerleithen 380
1902		Matthews, Ernest Romney, Assoc. Memb. Inst. C.E., F.G.S., Bessemer Prizeman, Soc. Engineers, Bridlington, Yorkshire
1901		* Menzies, Alan W. C., M.A., B.Sc., F.C.S., Kent Chemical Laboratory, University, Chicago, U.S.A.
1888		* Methven, Cathcart W., Memb. Inst. C.E., F.R.I.B.A., Durban, Natal, S. Africa
1902	C.	Metzler, William H., A.B., Ph.D., Corresponding Fellow of the Royal Society of Canada, Professor of Mathematics, Syracuse University, Syracuse, N.Y.
1885	C. B.	* Mill, Hugh Robert, D.Sc., LL.D., 62 Camden Square, London 385
1908		* Miller, Alexander Cameron, M.D., F.S.A. Scot., Craig Linnhe, Fort-William, Inverness-shire
1905		* Miller-Milne, C. H., M.A., Rector, The High School, Arbroath, 8 Dalhousie Place, Arbroath
1905		* Milne, Archibald, M.A., B.Sc., Lecturer on Mathematics and Science, Edinburgh Provincial Training College, 108 Comiston Drive
1904	C.	* Milne, James Robert, D.Sc., 56 Manor Place
1886		* Milne, William, M.A., B.Sc., 70 Beechgrove Terrace, Aberdeen 390
1899		* Milroy, T. H., M.D., B.Sc., Professor of Physiology in Queen's College, Belfast, Thomlea, Malone Park, Belfast
1866		Mitchell, Sir Arthur, K.C.B., M.A., M.D., LL.D., 34 Drummond Place
1889	C.	Mitchell, A. Crichton, D.Sc., Professor of Pure and Applied Mathematics, and Principal of the Maharajah's College, Trivandrum, Travancore, India
1897		* Mitchell, George Arthur, M.A., 9 Lowther Terrace, Kelvinside, Glasgow
1900		* Mitchell, James, M.A., B.Sc., 4 Manse Street, Kilmarnock 395
1899		* Mitchell-Thomson, Sir Mitchell, Bart., 6 Charlotte Square
1906	C.	Moffat, The Rev. Alexander, M.A., B.Sc., Professor of Physical Science, Christian College, Madras, India
1890	C.	Mond, R. L., M.A., Cantab., F.C.S., The Poplars, 20 Avenue Road, Regent's Park, London
1887	C.	Moos, N. A. F., L.C.E., B.Sc., Professor of Physics, Elphinstone College, and Director of the Government Observatory, Colaba, Bombay
1896		* Morgan, Alexander, M.A., D.Sc., Principal, Edinburgh Provincial Training College, 1 Midmar Gardens 400
1892		Morrison, J. T., M.A., B.Sc., Professor of Physics and Chemistry, Victoria College, Stellenbosch, Cape Colony
1901		Moses, O. St John, M.D., B.Sc., F.R.C.S.E., Captain, Indian Medical Service, 8 Lansdowne Road, Calcutta, India
1892	C.	Mossman, Robert C., Superintendent of Publications, Argentine Meteorological Office, Viamonte, 640, Buenos Ayres

Date of Election.			
1874	C. K.		Muir, Thomas, C.M.G., M.A., LL.D., F.R.S., Superintendent-General of Education for Cape Colony, Education Office, Cape Town, and Mowbray Hall, Rosebank, Cape Colony
1888	C.	*	Muirhead, George, Commissioner to His Grace the Duke of Richmond and Gordon, K.G., Speybank, Fochabers 405
1907			Muirhead, James M. P., Bredisholm, Claremont, near Cape Town, Cape Colony
1887			Mukhopâdhyay, Âsûtosh, M.A., LL.D., F.R.A.S., M.R.I.A., Professor of Mathematics at the Indian Association for the Cultivation of Science, 77 Russa Road North, Bhowanipore, Calcutta
1894		*	Munro, J. M. M., Memb. Inst. E.E., 136 Bothwell Street, Glasgow
1891	C.	*	Munro, Robert, M.A., M.D., LL.D., Hon. Memb. R.I.A., Hon. Memb. Royal Soc. of Antiquaries of Ireland, Elmbank, Largs, Ayrshire
1896		*	Murray, Alfred A., M.A., LL.B., Westfield House, Cramond Bridge 410
1892	C.	*	Murray, George Robert Milne, F.R.S., F.L.S., 32 Market Square, Stonehaven
1907	C.	*	Murray, James, Park Road, Maxwelltown, Dumfries
1877	C. B. N.		Murray, Sir John, K.C.B., LL.D., D.C.L., Ph D., D.Sc., F.R.S., Member of the Prussian Order *Pour le Mérite*, Director of the Challenger Expedition Publications. Office, Villa Medusa, Boswell Road. House, Challenger Lodge, Wardie, and United Service Club
1907		*	Musgrove, James, M.D., F.R.C.S. Edin. and Eng., Bute Professor of Anatomy, University of St Andrews, 56 South Street, St Andrews
1887			Muter, John, M.A., F.C.S., South London Central Public Laboratory, 325 Kennington Road, London 415
1902			Mylne, The Rev. R. S., M.A., B.C.L. Oxford, F.S.A. Lond., Great Amwell, Herts
1888			Napier, A. D. Leith, M.D., C.M., M.R.C.P.L., 28 Angas Street, Adelaide, S. Australia
1897			Nash, Alfred George, C.E., B.Sc., Engineer, Department of Public Works, Jamaica, Belretiro, Mandeville, Jamaica, W.I.
1887		*	Nasmyth, T. Goodall, M.D., C.M., D.Sc., 27 Palmerston Place
1906		*	Newington, Frank A., Memb. Inst. C.E., Memb. Inst. E.E., 4 Osborne Terrace 420
1898			Newman, George, M.D., D.P.H. Cambridge, Lecturer on Preventive Medicine, St Bartholomew's Hospital, University of London; Dene, Hatch End, Middlesex
1884			Nicholson, J. Shield, M.A., D.Sc., Professor of Political Economy in the University of Edinburgh, 3 Belford Park
1880	C.		Nicol, W. W. J., M.A., D.Sc., 15 Blacket Place
1878			Norris, Richard, M.D., M.R.C.S. Eng., 3 Walsall Road, Birchfield, Birmingham
1906		*	O'Connor, Henry, C.E., Assoc. Memb. Inst. C.E., 1 Drummond Place 425
1888		*	Ogilvie, F. Grant, C.B., M.A., B.Sc., Principal Assistant Secretary for Science, Art, and Technology, Board of Education, South Kensington, London
1888		*	Oliphant, James, M.A., 11 Heathfield Park. Willesden, London
1886			Oliver, James, M.D., F.L.S., Physician to the London Hospital for Women, 18 Gordon Square, London
1895	C.		Oliver, Thomas, M.D., LL.D., F.R.C.P., Professor of Physiology in the University of Durham, 7 Ellison Place, Newcastle-upon-Tyne
1884	C. K.		Omond, R. Traill, 3 Church Hill 430
1908			Page, William Davidge, F.C.S., F.G.S., M. Inst. M.E., Clun House, Surrey Street, Strand, London
1905			Pallin, William Alfred, F.R.C.V.S., Captain in the Army Veterinary Department, c/o Messrs Holt & Co., 3 Whitehall Place, London
1892			Parker, Thomas, Memb. Inst. C.E., Severn House, Iron Bridge, Salop
1901		*	Paterson, David, F.C.S., Lea Bank, Rosslyn, Midlothian
1886	C.	*	Paton, D. Noël, M.D., B.Sc., F.R.C.P.E., Professor of Physiology in the University of Glasgow, University, Glasgow 435
1889		*	Patrick, David, M.A., LL.D., c/o W. & R. Chambers, 339 High Street
1892		*	Paulin, David, Actuary, 6 Forres Street
1881	C. N.		Peach, Benjamin N., LL D., F.R.S., F.G.S., late District Superintendent and Acting Palæontologist of the Geological Survey of Scotland, 72 Grange Loan
1907		*	Pearce, John Thomson, B.A., B.Sc., School House, Tranent
1904		*	Peck, James Wallace, M.A., Principal Assistant to Executive Officer (Education) of the London County Council, Heath House, New End Square, Hampstead, London 440

Date of Election.			
1889		* Peck, William, F.R.A.S., Town's Astronomer, City Observatory, Calton Hill, Edinburgh	
1887	C. B.	* Peddie, Wm., D.Sc., Professor of Natural Philosophy in University College, Dundee, Rosemount, Forthill Road, Broughty Ferry	
1900		Penny, John, M.B., C.M., D.Sc., Great Broughton, near Cockermouth, Cumberland	
1893		Perkin, Arthur George, F.R.S., 8 Montpellier Terrace, Hyde Park, Leeds	
1889		* Philip, R. W., M.A., M.D., F.R.C.P.E., 45 Charlotte Square	445
1907	C.	Phillips, Charles E. S., Castle House, Shooter's Hill, Kent	
1905		* Pinkerton, Peter, M.A., Head Mathematical Master, George Watson's College, Edinburgh, 36 Morningside Grove	
1908	C.	* Pirie, James Hunter Harvey, B.Sc., M.D., M.R.C.P.E., 13 Alva Street	
1906		Pitchford, Herbert Watkins, F.R.C.V.S., Bacteriologist and Analyst, Natal Government, The Laboratory, Pietermaritzburg, Natal	
1886		Pollock, Charles Frederick, M.D., F.R.C.S.E., 1 Buckingham Terrace, Hillhead, Glasgow	450
1888		Prain, David, Lt.-Col., Indian Medical Service, M.A., M.B., LL.D., F.L.S., F.R.S., Hon. Memb. Soc. Lett. ed Arti d. Zelanti, Acireale; Corr. Memb. Pharm. Soc. Gt. Britain, etc.; Director, Royal Botanic Gardens, Kew (late Director, Botanical Survey of India, Calcutta), Botanic Gardens, Kew	
1902		* Preller, Charles Du Riche, M.A., Ph.D., Assoc. Memb. Inst. C.E., 61 Melville Street	
1892		* Pressland, Arthur, J., M.A. Camb., Edinburgh Academy	
1875	C.	Prevost, E. W., Ph.D., Weston, Ross, Herefordshire	
1908		* Pringle, George Cossar, M.A , Rector of Peebles Burgh and County High School, Bloomfield, Peebles	455
1885		* Pullar, J. F., Rosebank, Perth	
1903		* Pullar, Laurence, The Lea, Bridge of Allan	
1880		Pullar, Sir Robert, LL.D., M.P. for the City of Perth, Tayside, Perth	
1898		* Purves, John Archibald, D.Sc., 13 Albany Street	
1897		* Rainy, Harry, M.B., C.M., F.R.C.P. Ed., 16 Gt. Stuart Street	460
1899		* Ramage, Alexander G., 8 Western Terrace, Murrayfield	
1884		Ramsay, E. Peirson, M.R.I.A., F.L.S., C.M.Z.S., F.R.G.S., F.G.S., Fellow of the Imperial and Royal Zoological and Botanical Society of Vienna, Curator of Australian Museum, Sydney, N.S.W.	
1891		* Rankine, John, M.A., LL.D., Advocate, Professor of the Law of Scotland in the University of Edinburgh, 23 Ainslie Place	
1904		Ratcliffe, Joseph Riley, M.B., C.M., Elmdon, Wake Green Road, Morley, Birmingham	
1900		Raw, Nathan, M.D., Mill Road Infirmary, Liverpool	465
1883	C.	Readman, J. B., D.Sc., F.C.S., Staffield Hall, Kirkoswald, R.S.O., Cumberland	
1889		Redwood, Sir Boverton, D.Sc. (Hon.), F.I.C., F.C.S., Assoc. Inst. C.E., Wadham Lodge, Wadham Gardens, London	
1902		Rees-Roberts, John Vernon, M.D., D.Sc., D.P.H., Barrister-at-Law, National Liberal Club, Whitehall Place, London	
1902		Reid, George Archdall O'Brien, M.B., C.M., 9 Victoria Road South, Southsea, Hants	
1908	C.	* Rennie, John, D.Sc., Lecturer on Parasitology, and Assistant to the Professor of Natural History, University of Aberdeen, 60 Desswood Place, Aberdeen	470
1908		Richardson, Linsdall, F.G.S., F.L.S., Director, Cheltenham School of Science and Technology, 10 Oxford Parade, Cheltenham	
1875		Richardson, Ralph, W.S., 10 Magdala Place	
1906	C.	* Ritchie, William Thomas, M.D., F.R.C.P.E., 9 Atholl Place	
1898	C.	Roberts, Alexander William, D.Sc., F.R.A.S., Lovedale, South Africa	
1880		Roberts, D. Lloyd, M.D., F.R.C.P.L., 23 St John Street, Manchester	475
1872		Robertson, D. M. C. L. Argyll, M.D., F.R.C.S.E., LL.D., Surgeon Oculist to the King in Scotland, Mon Plaisir, St Aubins, Jersey	
1900		* Robertson, Joseph M'Gregor, M.B., C.M., 26 Buckingham Terrace, Glasgow	
1896		* Robertson, Robert, M.A., 25 Mansionhouse Road	
1902	C.	* Robertson, Robert A., M.A., B.Sc., Lecturer on Botany in the University of St Andrews	
1896	C.	* Robertson, W. G. Aitchison, D.Sc., M.D., F.R.C.P.E., 2 Mayfield Gardens	480
1881		Rosebery, The Right Hon. the Earl of, K.G., K.T., LL.D., D.C.L., F.R.S., Dalmeny Park, Edinburgh	
1880		Rowland, L. L., M.A., M.D., President of the Oregon State Medical Society, and Professor of Physiology and Microscopy in Williamette University, Salem, Oregon	

Date of Election.		
1906		* Russell, Alexander Durie, B.Sc., Mathematical Master, Falkirk High School, Dunaura, Heugh Street, Falkirk
1902	C.	* Russell, James, 11 Argyll Place
1880		Russell, Sir James A., M.A., B.Sc., M.B., F.R.C.P.E., LL.D., Woodville, Canaan Lane 485
1904		Sachs, Edwin O., Architect, 7 Waterloo Place, Pall Mall, London, S.W.
1906		Saleeby, Caleb William, M.D., 13 Greville Place, London
1903		* Samuel, John S., 8 Park Avenue, Glasgow
1903		* Sarolea, Charles, Ph.D., D.Litt., Lecturer on French Language, Literature, and Romance Philology, University of Edinburgh, Eden Grove, Falcon Avenue
1891		Sawyer, Sir James, Knt., M.D., F.R.C.P., F.S.A., J.P., Consulting Physician to the Queen's Hospital, 31 Temple Row, Birmingham 490
1900	C.	* Schäfer, Edward Albert, M.R.C.S., LL.D., F.R.S., Professor of Physiology in the University of Edinburgh
1885	C.	Scott, Alexander, M.A., D.Sc., F.R.S., The Davy-Faraday Research Laboratory of the Royal Institution, London
1880		Scott, J. H., M.B., C.M., M.R.C.S., Prof. of Anatomy in the University of Otago, New Zealand
1905		Scougal, A. E., M.A., H.M.C.I.S., 1 Wester Coates Avenue
1902		Senn, Nicholas, M.D., LL.D., Professor of Surgery, Rush Medical College, Chicago, U.S.A. 495
1872	C.	Seton, George, M.A., Advocate, Ayton House, Abernethy, Perthshire
1897		* Shepherd, John William, Carrickarden, Bearsden, Glasgow
1894		* Shield, Wm., Memb. Inst. C.E., 33 Old Queen Street, Westminster, London
1871		Simpson, Sir A. R., M.D., Emeritus Professor of Midwifery in the University of Edinburgh, 52 Queen Street
1908		* Simpson, George Freeland Barbour, M.D., F.R.C.P.E., F.R.C.S.E., 50 Melville Street 500
1900	C.	* Simpson, James Young, M.A., D.Sc., Professor of Natural Science in the New College, Edinburgh, 25 Chester Street
1903		* Skinner, Robert Taylor, M.A., Governor and Headmaster, Donaldson's Hospital, Edinburgh
1901		* Smart, Edward, B.A., B.Sc., Benview, Craigie, Perth
1891	C.	* Smith, Alexander, B.Sc., Ph.D., Professor of General Chemistry, University of Chicago, Ills., U.S.
1882	C.	Smith, C. Michie, B.Sc., F.R.A.S., Director of the Kodaikánal and Madras Observatories, The Observatory, Kodaikánal, South India 505
1885		* Smith, George, F.C.S., Polmont Station
1871	C.	Smith, John, M.D., F.R.C.S.E., LL.D., 11 Wemyss Place
1904		* Smith, William Charles, K.C., M.A., LL.B., Advocate, 10 Doune Terrace
1907	C.	Smith, William Ramsay, D.Sc., M.B., C.M., Permanent Head of the Health Department, South Australia, Winchester Street, East Adelaide, South Australia
1880		Smith, William Robert, M.D., D.Sc., Barrister-at-Law, Professor of Forensic Medicine in King's College, 74 Great Russell Street, Bloomsbury Square, London 510
1899		Snell, Ernest Hugh, M.D., B.Sc., D.P.H. Camb., Coventry
1880		Sollas, W. J., M.A., D.Sc., LL.D., F.R.S., late Fellow of St John's College, Cambridge, and Professor of Geology and Palæontology in the University of Oxford
1889	C.	Somerville, Wm., M.A., D.Sc., D.Oec., Sibthorpian Professor of Rural Economy in the University of Oxford, 121 Banbury Road, Oxford
1882		Sorley, James, F.I.A., C.A., 82 Onslow Gardens, London
1896		* Spence, Frank, M.A., B.Sc., 25 Craiglea Drive 515
1874	C.	Sprague, T. B., M.A., LL.D., Actuary, 29 Buckingham Terrace
1906		Squance, Thomas Coke, M.D., Physician and Pathologist in the Sunderland Infirmary, 15 Grange Crescent, Sunderland
1891		* Stanfield, Richard, Professor of Mechanics and Engineering in the Heriot-Watt College
1886	C.	* Stevenson, Charles A., B.Sc., Memb. Inst. C.E., 28 Douglas Crescent
1884		Stevenson, David Alan, B.Sc., Memb. Inst. C.E., 45 Melville Street 520
1888	C.	* Stewart, Charles Hunter, D.Sc., M.B., C.M., Professor of Public Health in the University of Edinburgh, 9 Learmonth Gardens
1904		* Stewart, Thomas W., M.A., B.Sc., Science Master, Edinburgh Ladies' College, 29 Bruntsfield Gardens
1873		Stewart, Walter, 3 Queensferry Gardens

Date of Election.			
1877		Stirling, William, D.Sc., M.D., LL.D., Brackenbury Professor of Physiology and Histology in Owens College and Victoria University, Manchester	
1902		* Stockdale, Herbert Fitton, Clairinch, Upper Helensburgh, Dumbartonshire	525
1889		* Stockman, Ralph, M.D., F.R.C.P.E., Professor of Materia Medica and Therapeutics in the University of Glasgow	
1906		Story, Fraser, Lecturer in Forestry, University College, Bangor, North Wales	
1907		* Strong, John, B.A., Rector of Montrose Academy, 11 Union Place, Montrose	
1903		Sutherland, David W., M.D., M.R.C.P. Lond., Captain, Indian Medical Service, Professor of Pathology and Materia Medica, Medical College, Lahore, India	
1896		* Sutherland, John Francis, M.D., Dep. Com. in Lunacy for Scotland, Scotsburn Road, Tain, Ross-shire	530
1905		Swithinbank, Harold William, Denham Court, Denham, Bucks	
1885	C.	* Symington, Johnson, M.D., F.R.C.S.E., F.R.S., Prof. of Anatomy in Queen's College, Belfast	
1904		* Tait, John W., B.Sc., Rector of Leith Academy, 18 Netherby Road, Leith	
1898	C.	Tait, William Archer, B.Sc., Memb. Inst. C.E., 38 George Square	
1895		Talmage, James Edward, D.Sc., Ph.D., F.R.M.S., F.G.S., Professor of Geology, Univ. of Utah, Salt Lake City, Utah	535
1890	C.	Tanakadate, Aikitu, Prof. of Nat. Phil. in the Imperial University of Japan, Tokyo, Japan	
1870		Tatlock, Robert R., F.C.S., City Analyst's Office, 156 Bath Street, Glasgow	
1899		* Taylor, James, M.A., Mathematical Master in the Edinburgh Academy, Edinburgh Academy	
1892		Thackwell, J. B., M.B., C.M.	
1885	C.	* Thompson, D'Arcy W., C.B., B.A., F.L.S., Professor of Natural History in University College, Dundee	540
1907		* Thompson, John Hannay, M. Inst. C.E., M. Inst. Mech. E., Engineer to the Dundee Harbour Trust, Earlville, Broughty Ferry	
1905		* Thoms, Alexander, 7 Playfair Terrace, St Andrews	
1887		* Thomson, Andrew, M.A., D.Sc., F.I.C., Rector, Perth Academy, Ardenlea, Pitcullen, Perth	
1896		* Thomson, George Ritchie, M.B., C.M., Cumberland House, Von Brandis Square, Johannesburg, Transvaal	
1903		Thomson, George S., F.C.S., Dairy Commissioner for Queensland, Department of Agriculture, Brisbane, Queensland	545
1906		* Thomson, Gilbert, C.E., 164 Bath Street, Glasgow	
1887	C.	* Thomson, J. Arthur, M.A., Regius Prof. of Natural History in the Univ. of Aberdeen	
1906		Thomson, James Stuart, F.L.S. (Assistant Professor of Zoology, South African College, Cape Town), 24 Brückfeldstrasse, Bern, Switzerland	
1880		Thomson, John Millar, LL.D., F.R.S., Prof. of Chem. in King's College, Lond., 9 Campden Hill Gardens, London	
1899		* Thomson, R. Tatlock, F.C.S., 156 Bath Street, Glasgow	550
1870		Thomson, Spencer C., Actuary, 10 Eglinton Crescent	
1882		Thomson, Wm., M.A., B.Sc., LL.D., Registrar, University of the Cape of Good Hope, University Buildings, Cape Town	
1876	C.	Thomson, William, Royal Institution, Manchester	
1874	C. B. N.	Traquair, R. H., M.D., LL.D., F.R.S., F.G.S., late Keeper of the Natural History Collections in the Royal Scottish Museum, Edinburgh (VICE-PRESIDENT), The Bush, Colinton	
1874		Tuke, Sir J. Batty, M.D., D.Sc., LL.D., F.R.C.P.E., M.P. for the Universities of Edinburgh and St Andrews, 20 Charlotte Square	555
1888		* Turnbull, Andrew H., Actuary, The Elms, Whitehouse Loan	
1905		* Turner, Arthur Logan, M.D., F.R.C.S.E., 27 Walker Street	
1906	C.	* Turner, Dawson F. D., B.A., M.D., F.R.C.P.E., M.R.C.P. Lond., Lecturer on Physics, Surgeon's Hall, and Physician in charge of Electrical Department, Royal Infirmary, Edinburgh, 37 George Square	
1861	K. N. C.	Turner, Sir William, K.C.B., M.B., F.R.C.S.E., LL.D., D.C.L., D.Sc. Dub., F.R.S., Principal of the University of Edinburgh (PRESIDENT), 6 Eton Terrace	
1895		Turton, Albert H., M.I.M.M., 18 Harrow Road, Bowenbrook, Birmingham	560
1898	C.	* Tweedie, Charles, M.A., B.Sc., Lecturer on Mathematics in the University of Edinburgh, 40 Gillespie Crescent	
1889		Underhill, T. Edgar, M.D., F.R.C.S.E., Dunedin, Barnt Green, Worcestershire	

Date of Election.			
1906		Vandenbergh, William J., Barrister-at-Law, S.S.C., F.R.S.L., F.R.M.S., 29-32 Exchange Buildings, Pirie Street, Adelaide, S. Australia	
1888		Walker, James, Memb. Inst. C.E., Engineer's Office, Tyne Improvement Commission, Newcastle-on-Tyne	
1891	C. B.	* Walker, James, D.Sc., Ph.D., F.R.S., Professor of Chemistry in the University of Edinburgh	565
1873	C.	Walker, Robert, M.A., LL.D., University, Aberdeen	
1902		* Wallace, Alexander G., M.A., 154 Forrest Avenue, Aberdeen	
1886	C.	* Wallace, R., F.L.S., Professor of Agriculture and Rural Economy in the University of Edinburgh	
1898		Wallace, Wm., M.A., Belvedere, Alta, Canada	
1891		* Walmsley, R. Mullineux, D.Sc., Prin. of the Northampton Inst., Clerkenwell, London	570
1907		Waters, E. Wynston, Medical Officer, H.B.M. Administration, E. Africa, Lamu, British East Africa Protectorate, *via* Mombasa	
1901	C.	* Waterston, David, M.A., M.D., F.R.C.S.E., Lecturer on Regional Anatomy in the University of Edinburgh, 1 Coates Place	
1904		* Watson, Charles B. Boog, Huntly Lodge, 1 Napier Road	
1862	C.	Watson, Rev. Robert Boog, B.A., LL.D., F.L.S., Past President of the Conchological Society, 11 Strathearn Place	
1900		* Watson, Thomas P., M.A., B.Sc., Principal, Keighley Institute, Keighley	575
1907		* Watt, Andrew, M.A., Secretary to the Scottish Meteorological Society, 6 Woodburn Terrace	
1896		Webster, John Clarence, B.A., M.D., F.R.C.P.E., Professor of Obstetrics and Gynæcology, Rush Medical College, Chicago, 706 Reliance Buildings, 100 State Street, Chicago	
1907	C.	* Wedderburn, Ernest Maclagan, M.A., LL.B., 6 Succoth Gardens	
1903	C.	* Wedderburn, J. H. Maclagan, M.A., Lecturer on Mathematics in the University of Edinburgh, 6 Succoth Gardens	
1904		Wedderspoon, William Gibson, M.A., LL.D., Indian Educational Service, Senior Inspector of Schools, Burma, The Education Office, Rangoon, Burma	580
1896		Wenley, R. M., M.A., D.Sc., D.Phil., LL.D., Professor of Philosophy in the University of Michigan, U.S.	
1896	C.	White, Philip J., M.B., Prof. of Zoology in University College, Bangor, North Wales	
1890		White, Sir William Henry, K.C.B., Memb. Inst. C.E., LL.D., F.R.S., late Assistant Controller of the Navy, and Director of Naval Construction, Cedarscroft, Putney Heath, London	
1881		Whitehead, Walter, F.R.C.S.E., late Professor of Clinical Surgery, Owens College and Victoria University, Birchfield, Rusholme, Manchester	
1894		Whymper, Edward, F.R.G.S., Holmwood, Waldegrave Road, Teddington, Middlesex	585
1879		Will, John Charles Ogilvie, of Newton of Pitfodels, M.D., 17 Bon-Accord Square, Aberdeen	
1897		* Williams, W. Owen, F.R.C.V.S., Professor of Veterinary Medicine and Surgery, University of Liverpool, The Veterinary School, The University, Liverpool	
1908		* Williamson, Henry Charles, M.A., D.Sc., Naturalist to the Fishery Board for Scotland, 28 Polmuir Road, Aberdeen	
1900		Wilson, Alfred C., F.C.S., Voewood Croft, Stockton-on-Tees	
1879		Wilson, Andrew, Ph.D., F.L.S., Lecturer on Zoology and Comparative Anatomy, 110 Gilmore Place	590
1902		* Wilson, Charles T. R., M.A., F.R.S., Glencorse House, Peebles, and Sidney Sussex College, Cambridge	
1895		Wilson-Barker, David, F.R.G.S., Captain-Superintendent Thames Nautical Training College, H.M.S. "Worcester," Greenhithe, Kent	
1882		Wilson, George, M.A., M.D., LL.D., 7 Avon Place, Warwick	
1891		* Wilson, John Hardie, D.Sc., University of St Andrews, 39 South Street, St Andrews	
1902		Wilson, William Wright, F.R.C.S.E., M.R.C.S. Eng., Cottesbrook House, Acock's Green, Birmingham	595
1908		* Wood, Thomas, M.D., Eastwood, 182 Ferry Road, Bonnington, Leith	
1886	C.	* Woodhead, German Sims, M.D., F.R.C.P.E., Professor of Pathology in the University of Cambridge	
1884		Woods, G. A., M.R.C.S., Eversleigh, 1 Newstead Road, Lee, Kent	
1890		* Wright, Johnstone Christie, Northfield, Colinton	
1896		* Wright, Robert Patrick, Professor of Agriculture, West of Scotland Agricultural College, 6 Blythswood Square, Glasgow	600

Date of Election.		
1882		Young, Frank W., F.C.S., H.M. Inspector of Science and Art Schools, 32 Buckingham Terrace, Botanic Gardens, Glasgow
1892		Young, George, Ph.D., 79 Harvard Court Mansions, Honeybourne Road, West Hampstead, London, N.W.
1896	C.	* Young, James Buchanan, M.B., D.Sc., Dalveen, Braeside, Liberton
1900		* Young, J. M'Lauchlan, F.R.C.V.S., Lecturer on Veterinary Hygiene, University of Aberdeen
1904		Young, R. B., M.A., B.Sc., Transvaal Technical Institute, Johannesburg, Transvaal 605

LIST OF HONORARY FELLOWS OF THE SOCIETY

At October 1908.

HIS MOST GRACIOUS MAJESTY THE KING.

FOREIGNERS (LIMITED TO THIRTY-SIX BY LAW X.).

Elected		
1897	Alexander Agassiz,	*Cambridge (Mass.).*
1897	E.-H. Amagat,	*Paris.*
1900	Arthur Auwers,	*Berlin.*
1900	Adolf Ritter von Baeyer,	*Munich.*
1905	Waldemar Chr. Brögger,	*Christiania.*
1897	Stanislao Cannizzaro,	*Rome.*
1905	Moritz Cantor,	*Heidelberg.*
1902	Jean Gaston Darboux,	*Paris.*
1902	Anton Dohrn,	*Naples.*
1905	Paul Ehrlich,	*Frankfurt-a.-M.*
1908	Emil Fischer,	*Berlin.*
1902	Albert Gaudry,	*Paris.*
1905	Paul Heinrich Groth,	*Munich.*
1888	Ernst Haeckel,	*Jena.*
1883	Julius Hann,	*Graz.*
1908	George William Hill,	*New York.*
1879	Jules Janssen,	*Paris.*
1908	Friedrich Wilhelm Georg Kohlrausch,	*Charlottenburg.*
1897	Gabriel Lippmann,	*Paris.*
1895	Éleuthère-Élie-Nicolas Mascart,	*Paris.*
1895	Carl Menger,	*Vienna.*
1897	Fridtjof Nansen,	*Christiania.*
1881	Simon Newcomb,	*Washington.*
1908	Henry Fairfield Osborn,	*New York.*
1908	Iwan P. Pawlov,	*St Petersburg.*
1905	Eduard Pflüger,	*Bonn.*
1895	Jules Henri Poincaré,	*Paris.*
1889	Georg Hermann Quincke,	*Heidelberg.*
1908	Gustaf Retzius,	*Stockholm.*
1908	Augusto Righi,	*Bologna.*
1897	Giovanni V. Schiaparelli,	*Milan.*
1905	Eduard Suess,	*Vienna.*
1908	Louis Joseph Troost,	*Paris.*
1905	Wilhelm Waldeyer,	*Berlin.*
1905	Wilhelm Wundt,	*Leipzig.*
1897	Ferdinand Zirkel,	*Leipzig.*

Total, 36.

BRITISH SUBJECTS (LIMITED TO TWENTY BY LAW X.).

Elected		
1889	Sir Robert Stawell Ball, Kt., LL.D., F.R.S., M.R.I.A., Lowndean Professor of Astronomy in the University of Cambridge,	*Cambridge.*
1900	Edward Caird, LL.D., Master of Balliol College, Oxford,	*Oxford.*
1892	Colonel Alexander Ross Clarke, C.B., R.E., F.R.S.,	*Redhill, Surrey.*
1897	Sir George Howard Darwin, K.C.B., M.A., LL.D., F.R.S., Plumian Professor of Astronomy in the University of Cambridge,	*Cambridge.*
1900	David Ferrier, M.D., LL.D., F.R.S., Professor of Neuro-Pathology, King's College, London,	*London.*
1900	Andrew Russell Forsyth, D.Sc., F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge,	*Cambridge.*
1892	Sir David Gill, K.C.B., LL.D., F.R.S., formerly His Majesty's Astronomer at the Cape of Good Hope,	*London.*
1895	Albert C. L. G. Günther, Ph.D., F.R.S.,	*London.*

1883	Sir Joseph Dalton Hooker, K.C.S.I., M.D., LL.D., D.C.L., F.R.S., Corresp. Mem. Inst. of France,	*London.*
1884	Sir William Huggins, K.C.B., LL.D., D.C L., F.R.S., Corresp. Mem. Inst. of France,	*London.*
1908	Sir Alexander B. W. Kennedy, LL D., F.R.S , Past Pres. Inst. C.E.,	*London.*
1908	Sir Edwin Ray Lankester, K.C.B., LL.D., F.R.S.,	*London.*
1900	Archibald Liversidge, LL.D., F.R.S., Professor of Chemistry in the University of Sydney,	*Sydney.*
1908	Sir James A. H. Murray, LL.D., D.C.L , Editor of the Oxford English Dictionary,	*Oxford.*
1905	Sir William Ramsay, K.C.B., LL.D., F.R.S., Professor of Chemistry in the University College, London,	*London.*
1886	The Lord Rayleigh, D.C.L., LL.D., D.Sc Dub., F.R.S., Corresp. Mem. Inst. of France,	*London.*
1908	Charles S. Sherrington, M.A., M.D., LL.D., F.R.S., Holt Professor of Physiology in the University of Liverpool,	*Liverpool.*
1905	Joseph John Thomson, D.Sc., LL.D., F.R.S., Cavendish Professor of Experimental Physics, University of Cambridge,	*Cambridge.*
1900	Thomas Edward Thorpe, D.Sc., LL.D., F.R.S., Principal of the Government Laboratories, London,	*London.*
1895	Sir Charles Todd, K.C.M.G., F.R.S., Government Astronomer, South Australia,	*Adelaide.*

Total, 20.

ORDINARY FELLOWS OF THE SOCIETY ELECTED

During Session 1907-8.

(Arranged according to the Date of their Election.)

16*th December* 1907.

Lt.-Colonel John Campbell.

Walter Aubrey Kidd, M.D., F.Z.S.

3*rd February* 1908.

Archibald Hewat, F.F.A., F.I.A.

Theophilus Bulkeley Hyslop, M.D.

Thomas Francis Cavanagh, M.D.

17*th February* 1908.

James Ireland Craig, M.A.

William Davidge Page, F.C.S., F.G.S.

Wm. Dawson Henderson, M.A., Ph.D.

Linsdall Richardson, F.G.S., F.L.S.

Andrew Wm. Kerr, F.S.A. Scot.

Henry Charles Williamson, M.A., D.Sc.

George Cossar Pringle, M.A.

18*th May* 1908.

George Freeland Barbour Simpson, M.D.

Devendra Nath Mallik, B.A.

15*th June* 1908.

John Rennie, D.Sc.

Thomas Wood, M.D.

20*th July* 1908.

James H. Harvey Pirie, B.Sc., M.D.

Alexander Cameron Miller, M.D.

Harry Drinkwater, M.D.

ORDINARY FELLOWS DECEASED AND RESIGNED

During Session 1907-8.

DECEASED.

Professor Thomas Annandale, M.D.
John Archibald, M.D.
R. S. Fancourt Barnes, M.D.
Ludwik Bernstein, M.D.
W. J. Brock, M.B., D Sc.
Francis Chalmers Crawford.
James D. G. Dalrymple.
Herbert J. Gifford.
Sir James Hector, K.C.M.G., M.D.
The Rt. Hon. Lord Kelvin, G.C.V.O., etc., *Pres.*
Sir James D. Marwick, LL.D.
Col. Joshua A. Nunn, C.I.E., D.S.O., F.R.C.V.S.
William Sanderson.
Thomas Savage, M.D.
John J. Stevenson.
Major-General Sir J. H. M. Shaw Stewart.
Charles E. Underhill, B.A., M.B.
Sir Patrick Heron Watson, M.D.

RESIGNED.

Sir Thomas D. Gibson Carmichael, Bart.
Rev. S. M. Johnston.
Professor J. P. Kuenen.
James More.
George Romanes.

ABSTRACT

OF

THE ACCOUNTS OF JAMES CURRIE, ESQ.,

As Treasurer of the Royal Society of Edinburgh.

SESSION 1907-1908.

I. ACCOUNT OF THE GENERAL FUND.

CHARGE.

1. Arrears of Contributions at 1st October 1907		£221 11 0
2. Contributions for present Session :—		
1. 164 Fellows at £2, 2s. each	£344 8 0	
139 Fellows at £3, 3s. each	437 17 0	
	£782 5 0	
Less included in payments in lieu of future contributions	4 4 0	
	£778 1 0	
2. Fees of Admission and Contributions of eleven new Resident Fellows at £5, 5s. each	57 15 0	
3. Fees of Admission of eight new Non-Resident Fellows at £26, 5s. each	210 0 0	
4. Commutation Fees in lieu of Future Contributions of two Fellows	27 6 0	
		1073 2 0
3. Interest received—		
Interest, less Tax	£372 14 2	
Annuity from Edinburgh and District Water Trust, less Tax	49 17 6	
		422 11 8
4. Transactions and Proceedings sold		129 15 2
5. Annual Grant from Government		600 0 0
Amount of the Charge . .		**£2446 19 10**

DISCHARGE.

1. Rent of Society's Apartments for Year, less Tax		£285 0 0
2. INSURANCE, GAS, ELECTRIC LIGHT, COAL, WATER, ETC. :—		
Insurance	£10 13 0	
Gas	0 16 9	
Electric Light	2 12 9	
Coal	7 9 9	
Water	2 2 0	
Income Tax	15 0 0	
		38 14 3
3. SALARIES :—		
General Secretary	£100 0 0	
Librarian	150 0 0	
Assistant Librarian	55 0 0	
Office Keeper	35 0 0	
Doorkeeper	12 0 0	
Treasurer's Clerk	25 0 0	
		377 0 0
Carry forward		£700 14 3

Brought forward				£700 14 3
4. Expenses of Transactions:—				
Neill & Co., Ltd., Printers			£244 10 10	
Do. for illustrations			11 1 6	
Do. (Ben Nevis) . .		£105 11 9		
Less Contributions per Meteorological Society of Scotland—				
From British Association	£25 6 8			
,, R. T. Omond .	25 0 0			
		50 6 8		
			55 5 1	
M'Farlane & Erskine, Lithographers			33 0 3	
James Green, do.			24 17 0	
A. H. Searle, do.			10 0 0	
A. S. Huth, do.			5 16 0	
George Waterston & Sons, do.			10 0 0	
Hislop & Day, Engravers			15 7 7	
Orrock & Son, Bookbinders			105 6 6	
				515 4 9
5. Expenses of Proceedings:—				
Neill & Co., Ltd., Printers			£628 9 8	
Do. (for illustrations)			21 11 9	
M'Farlane & Erskine, Lithographers			9 10 0	
Werner & Winter, do.			6 13 0	
Hislop & Day, Engravers			7 18 6	
				674 2 11
6. Books, Periodicals, Newspapers, etc.:—				
Otto Schulze & Co., Booksellers			£142 6 4	
James Thin, do.			50 10 2	
R. Grant & Son, do.			3 18 2	
Kegan Paul & Co., do.			2 5 4	
Wm. Green & Sons, do.			0 15 6	
International Catalogue of Scientific Literature . . .			17 0 0	
Robertson & Scott, News Agents			6 12 9	
Orrock & Son, Bookbinders			45 19 6	
Egypt Exploration Funds Subscriptions			3 3 0	
Ray Society do.			1 1 0	
Palæontographical Society do.			1 1 0	
Journal de Conchyliologie			0 15 0	
				275 7 9
7. Other Payments:—				
Neill & Co., Ltd., Printers			£76 17 6	
R. Blair & Sons, Confectioners			36 4 0	
Lantern Exhibitions, etc., at Lectures			15 19 0	
Lindsay, Jamieson & Haldane			6 6 0	
National Telephone Co.			6 5 0	
Petty Expenses, Postages, Carriage, etc.			69 5 3	
				210 16 9
8. Irrecoverable Arrears of Contributions written off				10 10 0
9. Arrears of Contributions outstanding at 1st October 1908:—				
Present Session			£100 16 0	
Previous Sessions, ...			118 13 0	
				219 9 0
Amount of the Discharge . .				**£2606 5 5**

Amount of the Discharge	**£2606 5 5**
Amount of the Charge	**2446 19 10**
Excess of the Discharge	**£159 5 7**
Floating Balance due by the Society at 1st October 1907	£43 1 8
Add Excess of Discharge as above	159 5 7
Floating Balance due by the Society at 1st October 1908 . . .	**£202 7 3**

Being—

Balance due to Union Bank on Current Account	£206 2 3	
Less Balance due by Treasurer, ...	3 15 0	
		£202 7 3

II. ACCOUNT OF THE KEITH FUND

To 1st October 1908.

CHARGE.

1. Balance due by Union Bank at 1st October 1907			£56 10 6
2. Interest Received :—			
On £896, 19s.1d. North British Railway Company 3 per cent. Debenture Stock for year to Whitsunday 1908, less Tax		£25 11 4	
On £211, 4s. North British Railway Company 3 per cent. Lien Stock for year to Lammas 1908, less Tax		6 0 4	
			31 11 8
			£88 2 2

DISCHARGE.

1. Dr Alexander Bruce, money portion of Prize for 1905–1907		£47 3 4	
2. Alex. Kirkwood & Son, Engravers, for Gold Medal		16 0 0	
		£63 3 4	
3. Balance due by Union Bank at 1st October 1908		24 18 10	
			£88 2 2

III. ACCOUNT OF THE NEILL FUND

To 1st October 1908.

CHARGE.

1. Balance due by the Union Bank and in hand at 1st October 1907			£61 3 2
2. Interest Received :—			
On £355 London, Chatham and Dover Railway Company 4½ per cent. Arbitration Debenture Stock for year to 30th June 1908, less Tax			15 3 6
			£76 6 8

DISCHARGE.

1. Mr Frank J. Cole, money portion of Prize		£29 10 6	
2. Alex. Kirkwood & Son, Engravers, for Gold Medal		16 0 0	
		£45 10 6	
3. Balance—			
Uncashed Dividend Warrants	£37 18 9		
Less due to Union Bank at 1st October 1908	7 2 7		
		30 16 2	
			£76 6 8

IV. ACCOUNT OF THE MAKDOUGALL-BRISBANE FUND

To 1st October 1908.

CHARGE.

1. Balance due at 1st October 1907 :—		
By Union Bank of Scotland on Deposit Receipt	£135 0 0	
By Union Bank of Scotland on Current Account	30 3 3	
		£165 3 3
2. Interest Received on £365 Caledonian Railway Company 4 per cent. Consolidated Preference Stock No. 2 for year to 30th June 1908, less Tax	£13 17 4	
On Deposit Receipt with Union Bank	2 17 4	
		16 14 8
		£181 17 11

DISCHARGE.

Nil.

3. Balance due at 1st October 1908 :—		
By Union Bank of Scotland on Deposit Receipt	£135 0 0	
By Union Bank of Scotland on Current Account	46 17 11	
		£181 17 11

V. ACCOUNT OF THE MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND

To 1st October 1908.

CHARGE.

Sum on Deposit Receipt with the Union Bank of Scotland at 1st October 1907	£208 1 9
Interest on above	4 8 6
	£212 10 3

DISCHARGE.

Nil.

Above Sum on Deposit Receipt with the Union Bank of Scotland at 1st October 1908	£212 10 3

VI. ACCOUNT OF THE GUNNING-VICTORIA JUBILEE PRIZE FUND

To 1st October 1908.

(Instituted by Dr R. H. Gunning of Edinburgh and Rio de Janeiro.)

CHARGE.

1. Balance due by Union Bank at 1st October 1907	£74 7 10
2. Interest received on £1000 North British Railway Company 3 per cent. Consolidated Lien Stock for year to Lammas 1908, less Tax	28 10 0
	£102 17 10

DISCHARGE.

Nil.

Balance due by the Union Bank of Scotland on Current Account at 1st October 1908	£102 17 10

STATE OF THE FUNDS BELONGING TO THE ROYAL SOCIETY OF EDINBURGH

As at 1st October 1908.

1. GENERAL FUND—

1. £2090, 9s. 4d. three per cent. Lien Stock of the North British Railway Company at 84½ per cent., the selling price at 1st October 1908	£1766 9 5
2. £8519, 14s. 3d. three per cent. Debenture Stock of do. at 87¾ per cent., do.	7476 1 8
3. £52, 10s. Annuity of the Edinburgh and District Water Trust, equivalent to £875 at 170 per cent., do.	1487 10 0
4. £1811 four per cent. Debenture Stock of the Caledonian Railway Company at 115⅜ per cent., do.	2089 8 10
5. £35 four and a half per cent. Arbitration Debenture Stock of the London, Chatham and Dover Railway Company at 117 per cent., do.	40 19 0
6. Arrears of Contributions, as per preceding Abstract of Accounts	219 9 0
	£13,079 17 11
Deduct Floating Balance due by the Society, as per preceding Abstract of Accounts	202 7 3
Amount	£12,877 10 8

Exclusive of Library, Museum, Pictures, etc., Furniture of the Society's apartments at the Royal Institution.

2. KEITH FUND—

1. £896, 19s. 1d. three per cent. Debenture Stock of the North British Railway Company at 87¾ per cent., the selling price at 1st October 1908	£787 1 3
2. £211, 4s. three per cent. Lien Stock of do. at 84½ per cent., do.	178 9 3
3. Balance due by the Union Bank of Scotland	24 18 10
Amount	£990 9 4

STATE OF FUNDS—*continued.*

3. **NEILL FUND—**

1. £355 four and a half per cent. Arbitration Debenture Stock of the London, Chatham and Dover Railway Company at 117 per cent., the selling price at 1st October 1908		£415 7 0
2. Uncashed dividend warrants in hand	£37 18 9	
Less due to Union Bank of Scotland	7 2 7	
		30 16 2
AMOUNT		£446 3 2

4. **MAKDOUGALL-BRISBANE FUND—**

1. £365 four per cent. Consolidated Preference Stock No. 2 of the Caledonian Railway Company at 104½ per cent., the selling price at 1st October 1908	£381 8 6
2. Sum on Deposit Receipt with the Union Bank of Scotland	135 0 0
3. Balance due by do. on Current Account	46 17 11
AMOUNT	£563 6 5

5. **MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND—**

Sum on Deposit Receipt with the Union Bank of Scotland at 1st October 1908	£212 10 3

6. **GUNNING-VICTORIA JUBILEE PRIZE FUND**—Instituted by Dr Gunning of Edinburgh and Rio de Janeiro—

1. £1000 three per cent. Consolidated Lien Stock of the North British Railway Company at 84½ per cent., the selling price at 1st October 1908	£845 0 0
2. Balance due by the Union Bank of Scotland on Current Account	102 17 10
AMOUNT	£947 17 10

EDINBURGH, 21*st October* 1908.—We have examined the six preceding Accounts of the Treasurer of the Royal Society of Edinburgh for the Session 1907–1908, and have found them to be correct. The securities of the various Investments at 1st October 1908, as noted in the above Statement of Funds, have been exhibited to us.

LINDSAY, JAMIESON & HALDANE,
Auditors.

VIDIMUS of ESTIMATED INCOME of THE GENERAL FUND FOR SESSION 1908–1909.

I. INTEREST:—		
On £8519, 14s. 3d. Railway Debenture Stock at 3 per cent.		£255 11 10
On £2090, 9s. 4d. Railway Lien Stock at 3 per cent.		62 14 4
On £1811 Railway Debenture Stock at 4 per cent.		72 8 8
On £35 Railway Debenture Stock at 4½ per cent.		1 11 6
		£392 6 4
2. ANNUITY from the Edinburgh and District Water Trust		52 10 0
		£444 16 4
Deduct Income Tax at 1s. per £		22 4 8
		£422 11 8
3. ANNUAL CONTRIBUTIONS:—		
Of 162 Fellows at £2, 2s. each	£340 4 0	
Of 133 Fellows at £3, 3s. each	418 19 0	
		759 3 0
4. ANNUAL GRANT from Government		600 0 0
5. SALES of Society's Transactions		30 0 0
TOTAL ESTIMATED INCOME,		£1811 14 8

Exclusive of Fees of Admission and Contributions of New Fellows who may be admitted during the Year.

INDEX.

PRINTED BY NEILL AND CO., LTD., EDINBURGH.

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

SESSION 1907–8.

Part I.] VOL. XXVIII. [Pp. 1–64.

CONTENTS.

[*Continued on page* iv *of Cover.*

EDINBURGH:

PUBLISHED BY ROBERT GRANT & SON, 107 PRINCES STREET, AND
WILLIAMS & NORGATE, 14 HENRIETTA STREET, COVENT GARDEN, LONDON.

MDCCCCVIII.

Price Four Shillings and Sixpence.

REGULATIONS REGARDING THE PUBLICATION OF PAPERS IN THE PROCEEDINGS AND TRANSACTIONS OF THE SOCIETY.

THE Council beg to direct the attention of authors of communications to the Society to the following Regulations, which have been drawn up in order to accelerate the publication of the Proceedings and Transactions, and to utilise as widely and as fairly as possible the funds which the Society devotes to the publication of Scientific and Literary Researches.

1. MANUSCRIPT OF PAPERS.—As soon as any paper has been passed for publication, either in its original or in any altered form, and has been made ready for publication by the author, it is sent to the printer, whether it has been read or not.

The 'copy' should be written on large sheets of paper, on one side only, and the pages should be clearly numbered. The MS. must be easily legible, preferably typewritten, and must be absolutely in its final form for printing; so that corrections in proof shall be as few as possible, and shall not cause overrunning in the lines or pages of the proof. All tables of contents, references to plates, or illustrations in the text, etc., must be in their proper places, with the page numbers left blank; and spaces must be indicated for the insertion of illustrations that are to appear in the text.

2. ILLUSTRATIONS.—All illustrations must be drawn in a form immediately suitable for reproduction; and such illustrations as can be reproduced by photographic processes should, so far as possible, be preferred. Drawings to be reproduced as line blocks should be made with Indian ink (deadened with yellow if of bluish tone), preferably on fine white bristol board, free from folds or creases; smooth, clean lines, or sharp dots, but no washes or colours, should be used. If the drawings are done on a large scale, to be afterwards reduced by photography, any lettering or other legend must be on a corresponding scale.

If an author finds it inconvenient to furnish such drawings, the Society will have the figures re-drawn at his expense; but this will cause delay.

When the illustrations are to form plates, a scheme for the arrangement of the figures (in quarto plates for the Transactions, in octavo for the Proceedings) must be given, and numbering and lettering indicated.

3. PROOFS.—In general, a first proof and a revise of each paper will be sent to the author, whose address should be indicated on the MS. If further proofs are required, owing to corrections or alterations for which

[*Continued on page* iii *of Cover.*

the printer is not responsible, the expense of such proofs and corrections will be charged against the author.

All proofs must, if possible, be returned within one week, addressed to *The Secretary, Royal Society, Mound, Edinburgh,* and *not to the printer.* To prevent delay, authors residing abroad should appoint some one residing in this country to correct their proofs.

4. ADDITIONS TO A PAPER after it has been finally handed in for publication, if accepted by the Council, will be treated and dated as separate communications, and may, or may not, be printed immediately after the original paper.

5. BRIEF ABSTRACTS OF TRANSACTIONS PAPERS will be published in the Proceedings, provided they are sent along with the original paper.

6. SEPARATE ISSUE OF REPRINTS; AUTHOR'S FREE AND ADDITIONAL COPIES.—As soon as the final revise of a Transactions paper has been returned, or as soon as the sheet in which the last part of a Proceedings paper appears is ready for press, a certain number of separate copies or reprints, in covers bearing the title of the paper and the name of the author, are printed off and placed on sale. The date of such separate publication will be printed on each paper.

The author receives fifty of these reprints free, and may have any reasonable number of additional copies at a fixed scale of prices which will be furnished by the printer, who will charge him with the cost. To prevent disappointment, especially if the paper contains plates, the author should, immediately after receiving his first proof, notify *to the printer* the number of additional copies required.

7. INDEX SLIPS.—In order to facilitate the compilation of Subject Indices, and to secure that due attention to the important points in a paper shall be given in General Catalogues of Scientific Literature and in Abstracts by Periodicals, every author is requested to return to the Secretary, along with his final proof, a brief index (on the model given below), of the points in it which he considers new or important. These indices will be edited by the Secretary, and incorporated in Separate Index Slips, to be issued with each part of the Proceedings and Transactions.

MODEL INDEX.

Schäfer, E. A.—On the Existence within the Liver Cells of Channels which can be directly injected from the Blood-vessels. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

Cells, Liver,—Intra-cellular Canaliculi in.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

Liver,—Injection within Cells of.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

The Papers published in this Part of the PROCEEDINGS *may be had separately, on application to the Publishers, at the following prices:—*

No. XLI., . . .	Price	1s. 6d.
No. XLII., . . .	,,	6d.
No. XLIII., . . .	,,	6d.
No. XLIV., . . .	,,	6d.